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Families of Transitive Primitive Maximal Simple Lie Subalgebras of diff_n

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ABSTRACT. Transitive primitive Lie algebras, corresponding to pairs (L, L_0), where L is a classical complex simple Lie algebra and L_0 is a maximal parabolic subalgebra of L, are constructed as subalgebras of diff_n for all finite values of n. All such realizations of $\mathrm{sl}(N, \mathbb{C})$ are shown to be maximal in diff_n. Mutual inclusions involving realizations of orthogonal and symplectic Lie algebras are pointed out.

1. Introduction

The purpose of this article is expressed in its title, namely we construct, in a systematic manner, certain infinite series of finite dimensional classical simple Lie algebras that are maximal subalgebras of diff_n, the algebra of vector fields on $\mathbb{C}^n$. In other words we construct certain Lie algebras that are realized by vector fields of the form:

$$\widehat{X}_i = \sum_{\mu=1}^{n} f_i^{\mu}(z)\frac{\partial}{\partial z^{\mu}}, \quad 1 \le i \le r < \infty, \quad z \in \mathbb{C}^n, \quad f_i \in \mathbb{C}^n. \tag{1.1}$$

We are thus addressing two related problems going back to Sophus Lie [**11, 12, 13**]. They can be stated as follow:

PROBLEM A. Classify all finite dimensional subalgebras of diff_n into conjugacy classes under the action of the group Diff_n of local diffeomorphisms of $\mathbb{C}^n$ and construct a representative of each class.

PROBLEM B. Construct all dynamical systems

$$\frac{dz_{\mu}}{dt} = \eta^{\mu}(\vec{z}, t), \quad t \in \mathbb{R}, \quad \vec{z} \in \mathbb{C}^n, \quad 1 \le \mu \le n, \tag{1.2}$$

allowing a fundamental set of solutions, i.e. having a superposition formula

$$\vec{z}(t) = \vec{F}\big(\vec{z}_1(t), \ldots, \vec{z}_m(t), c_1, \ldots, c_n\big), \tag{1.3}$$

expressing the general solution $\vec{z}(t)$ in terms of a finite number m of particular solutions $\vec{z}_i(t)$ and n significant constants $c_1, \ldots, c_n$, specifying the initial conditions.

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This is the final form of the paper.

The direct relation between Problems A and B is expressed in a theorem, proven by Sophus Lie [**13**]. Namely, a necessary and sufficient condition for a fundamental set of solutions to exist is that the right hand side of (1.2) should lie in a finite dimensional Lie algebra, i.e. we must have (with $1 \leq k, \ell, m \leq r < \infty$):

$$\eta^{\mu}(z,t) = \sum_{k=1}^{r} Z_k(t)\xi_k^{\mu}(z), \quad \widehat{X}_k = \sum_{\mu=1}^{n} \xi_k^{\mu}(z)\partial_{z^{\mu}}, \quad [\widehat{X}_k, \widehat{X}_\ell] = \sum_m f_{k\ell}^m \widehat{X}_m. \tag{1.4}$$

Lie's study of differential equations [**11**] led him to give a realization of all classes of finite dimensional subalgebras of diff_n for $n = 1$ and 2. He summed up the results, when studying the case $n = 3$, in Abtheilung II of a book with Engel [**12**].

For $n = 1$ the result is very simple: the only finite dimensional subalgebras of diff_1 are $\mathrm{sl}(2)$ and its subalgebras, aff_1, and $\mathbb{C}$, of dimensions 3,2, and 1, respectively. For $n = 2$ the situation is already quite different. The algebra diff_2 has infinitely many different, mutually nonconjugate finite dimensional subalgebras and their dimension r can be arbitrary. Lie classified all of them into a not too large number of different types.

Among the infinite series of subalgebras of diff_2 we mention as a type those that are contained in the Abelian infinite dimensional algebra $A_\infty^{(1)}$ consisting of all vector fields $f(x)\partial_y$ (we denote temporarily x, y the two variables of diff_2, instead of z_1, z_2) where $f(x)$ are all (e.g. analytic) functions of x. Notice that $A_\infty^{(1)}$ has finite dimensional subalgebras of any dimension; none of them is maximal: indeed we can always continue indefinitely any increasing chain of $A_\infty^{(1)}$ subalgebras $a_1 \subset a_2 \subset a_3 \subset \ldots$.

Another type corresponds to an interesting infinite series of subalgebras of diff_2; it contains the symmetry algebras of the equations $y^{(m)}(x) = 0$ for $m \geq 3$. Let us denote them by W_m; their dimension is $\dim W_m = m+4$ and a basis can be chosen to be (with $0 \leq k \leq m-1$):

$$\begin{gathered} \widehat{X}_1 = \partial_x, \quad \widehat{X}_2 = x\partial_x + \frac{m-1}{2} y\partial_y, \\ \widehat{X}_3 = x^2\partial_x + (m-1)xy\partial_y, \quad \widehat{X}_4 = y\partial_y, \quad \widehat{Y}_k = x^k\partial_y. \end{gathered} \tag{1.5}$$

Each subalgebra W_m is actually a maximal finite dimensional subalgebra of diff_2, which has only two more maximal finite dimensional subalgebras: $\mathrm{sl}(3)$ and $\mathrm{sl}(2) \oplus \mathrm{sl}(2)$ (see Section 3 and [**10**] for more details).

The number of different types of finite subalgebras of diff_n increases rapidly with n and it seems hopeless to try and proceed beyond $n = 3$ by dimension. We address a more limited problem, namely the following one:

PROBLEM C. Classify all maximal simple or semisimple finite dimensional subalgebras of diff_n into conjugacy classes under the action of Diff_n and choose a representative of each class.

This problem is also out of reach of the present mathematical methods although we know that for each value of n the number of solutions is finite. There is a natural grouping of these algebras into infinite families. Those we will construct are defined by the title whose technical meaning will be explained in Section 3.

Lie himself already knew two families of maximal simple finite dimensional subalgebras of diff_n (each with one representative for each n). One family contains O_{n+2} realized in Eq. (5.23) as the Lie algebra of all local conformal transformations;

the other family contains $\mathrm{sl}(n+1)$ realized as the Lie algebra of all local projective transformations of $\mathbb{C}^n$. This series was used in the context of Problem B to construct and solve systems of projective Riccati equations [**1, 2**].

It was pointed out [**15, 16**] in the context of Problem B that it is important to first construct "indecomposable" systems of ordinary differential equations with superposition formulas. These are systems of n equations from which it is not possible to split of $n_1 < n$ equations that themselves have a superposition formula. Decomposability occurs if the variables z can be split into two subsets

$$z = (x_1, x_2, \dots, x_{n_1}, y_1, y_2, \dots, y_{n_2}), \quad n_1 + n_2 = n, \tag{1.6}$$

such that all infinitesimal operators $\widehat{X}_k$ satisfy:

$$\widehat{X}_k = \sum_{a=1}^{n_1} \eta_k^a(x)\partial_{x^a} + \sum_{b=1}^{n_2} \xi_k^b(x,y)\partial_{y^b}, \tag{1.7}$$

i.e. the coefficients of ∂_{x^a} depend only on the variables x.

It was shown [**15**] that the system of equations (1.2) is indecomposable, if and only if the algebra (1.4) is transitive, primitive and effective, i.e. it is defined by a pair of Lie algebras $L_0 \subset L$ where L_0 is maximal subalgebra of L and does not contain a proper ideal of L. Ref. [**15**] was devoted to constructing indecomposable systems of ordinary differential equations with superposition formulas. This has a significant relation to Problem C, formulated above.

From Eq. (1.7) we see why constructing the transitive primitive Lie algebras is a crucial step towards solving Problem C. Indeed, assume that we have already introduced the coordinates (1.6) and that no further subset of the coordinates $(x_1, \dots x_{n_1})$ can be split off. Then the "truncated" vector fields

$$\widehat{X}_k^{\top} = \sum_{a=1}^{n_1} \eta_k^a(x)\partial_{x^a}, \tag{1.8}$$

realize the same Lie algebra L as do the fields (1.7), but in $n_1 < n$ dimensions. In this lower dimensional case we have a transitive primitive Lie algebra (L, L_1), where $L_0 \subset L_1 \subset L$ and L_1 is maximal in L. This provides a method for constructing the imprimitive Lie algebras, once the primitive ones are found. Indeed, we start from the primitive ones in dimension n_1, then add the second sum in Eq. (1.7) for the chosen value of n_2. The coefficients $\eta_k^a(x)$ are to be determined from the commutation relations: a lower dimensional and simpler task.

In [**15**] families of classical simple Lie algebras were constructed as subalgebras of diff_n. The present paper can be viewed as a continuation of Ref. [**15**], in that we integrate the results into a systematic solution of Problem C. Moreover we address a question, not posed earlier [**15**], namely that of maximality of the algebra within diff_n and possible mutual inclusions and equivalences among various finite simple subalgebras of diff_n.

The problem of finding all maximal simple subalgebras of diff_n is of considerable mathematical interest as it stands. As far as physical motivation is concerned, it comes from several directions. One is the above mentioned study of ordinary differential equations with superposition formulas (Problem B). Another is the fact that in studies of quantum mechanical systems with degeneracies, Lie algebras usually occur as algebras of differential operators. They may of course have a more complicated form than given in (1.1), i.e. involve nonderivative terms and higher

derivatives. Solving Problem C is however definitely an important ingredient in the study of algebras of quantum mechanical operators.

Throughout this article we consider all Lie algebras and Lie groups over $\mathbb{C}$.

2. The Lie Algebra diff_n of Vector Fields on $\mathbb{C}^n$ and Some of its Subalgebras

The aim of this section is to establish the notations and recall some definitions and some basic properties of diff_n.

A point $z \in \mathbb{C}^n$ is given by the set of its n coordinates z_i, $1 \leq i \leq n$. We denote by T_a the tangent plane to $\mathbb{C}^n$ at the point a. All functions we shall consider are analytic. We distinguish between the maps (e.g. φ) from $\mathbb{C}^n$ to itself and those from $\mathbb{C}^n$ to $\mathbb{C}$ (e.g. f):

$$\mathbb{C}^n \xrightarrow{\varphi} \mathbb{C}^n, \quad \mathbb{C}^n \xrightarrow{f} \mathbb{C}. \tag{2.1}$$

The coordinates of $\varphi(z)$ are denoted by $\varphi_i(z)$. The differential $d_a\varphi$ of the map φ at a point a is a linear operator:

$$T_a \xrightarrow{d_a\varphi} T_{\varphi(a)} : \frac{\partial \varphi_i}{\partial z_j}(a), \tag{2.2}$$

whose matrix is called the Jacobian of φ at a. If $d_a\varphi$ is invertible, that is if $\det(\partial_i\varphi/\partial z_j)(a) \neq 0$, then φ is called a local diffeomorphism at a. We have a diffeomorphisms on $\mathbb{C}^n$ (respectively on the open set $\mathcal{O} \subset \mathbb{C}^n$) when $d_a\varphi$ is invertible at every point of $\mathbb{C}^n$ (respectively of $\mathcal{O}$). We emphasize that in this paper we generally consider *local diffeomorphism*, i.e. defined on the neighbourhood of a regular point. The diffeomorphisms on the same domain form a local group. The action of the diffeomorphism φ on the function f is

$$\varphi \cdot f = f \circ \varphi^{-1} \iff \big(\varphi \cdot f\big)(z) = f(\varphi^{-1}(z)); \quad \varphi\big(\varphi^{-1}(z)\big) = z. \tag{2.3}$$

By multiplication (of their values) the complex valued functions on $\mathbb{C}^n$ form an associative and commutative algebra. The derivations of that algebra are called vector fields and are denoted by $\hat{v} \equiv \sum_i v_i(z)\partial_i$ where ∂_i is short for $\partial/\partial z_i$:

$$\hat{v}f \equiv \sum_i v_i \partial_i f, \quad \hat{v}(fg) = (\hat{v}f)g + f\hat{v}g. \tag{2.4}$$

One shows that the Lie algebra formed by these derivations:

$$[\hat{v}, \hat{w}] = \sum_{ij} (v_i \partial_i w_j - w_i \partial_i v_j)\partial_j, \tag{2.5}$$

is diff_n, the Lie algebra of the group of diffeomorphisms. A linear combination of vector fields $\sum_i \lambda_i \hat{v}^{(i)}$, $\lambda_i \in \mathbb{C}$ is a vector field. An m-dimensional vector space V_m of vector fields defines an m-dimensional Lie algebra when $\hat{v}, \hat{w} \in V_m \implies [\hat{v}, \hat{w}] \in V_m$; for example:

$$\mathcal{A}_m^{(1)} = \{z_1^{k-1}\partial_2\}, \quad 1 \leq k \leq m, \tag{2.6}$$

is an m-dimensional Abelian Lie algebra, which is a finite dimensional (nonmaximal) subalgebra of $\mathcal{A}_\infty^{(1)}$ introduced in Section 1. Notice that if $\hat{v}$ is a vector field and f a function, $f\hat{v}$ is a vector field.

Consider a set of m vector fields $\hat{v}^{(k)} = \sum_i v_i^{(k)}\partial_i$, $1 \leq k \leq m$, depending on n variables $1 \leq i \leq n$.

DEFINITION 2.1. *The* functional rank r_f *of the set of* m *vector fields* $v^{(k)}$ *is equal to the rank of the matrix of the coefficients of these fields*:

$$r_f = rank\ \{v_i^{(k)}\}, \quad 1 \leq i \leq n,\ 1 \leq k \leq m. \tag{2.7}$$

The functional rank can be defined at a point a, *in a neighbourhood of* a, *on some open domain* $\mathcal{O} \subset \mathbb{C}^n$, *or on* $\mathbb{C}^n$ *itself. For a given* n, *the functional rank satisfies* $r_f \leq n$. *For instance, the* m *dimensional Lie algebra* $\mathcal{A}_m^{(1)}$ *defined in* (2.6) *has functional rank* 1 (*that explains the upper index*).

The action $\varphi \cdot \hat{v}$ of a diffeomorphism $y = \varphi(z)$ on a vector field $\hat{v}$ is given explicitly by

$$\varphi \cdot \left(\sum_i v_i \partial_{y_i}\right) = \sum_{ij} \left(v_i(\varphi^{-1}(y)) \frac{\partial \varphi_j}{\partial z_i}\right) \partial_{y_j}. \tag{2.8}$$

One can check the properties:

$$\begin{gathered} \varphi \cdot (\lambda_1 \hat{v}^{(1)} + \lambda_2 \hat{v}^{(2)}) = \lambda_1 \varphi \cdot \hat{v}^{(1)} + \lambda_2 \varphi \cdot \hat{v}^{(2)}, \quad \varphi \cdot (f\hat{v}) = (\varphi \cdot f)(\varphi \cdot \hat{v}), \\ \varphi \cdot [\hat{v}, \hat{w}] = [\varphi \cdot \hat{v}, \varphi \cdot \hat{w}], \end{gathered} \tag{2.9}$$

and prove from them that a diffeomorphism transforms a subalgebra of diff_n into an isomorphic subalgebra with same dimension and same functional rank.

It is well know that every nonsingular vector field can be transformed by a diffeomorphism into ∂_{z_1}. In other words all dimension 1 Abelian subalgebras are equivalent. This can be extended to:

LEMMA 2.1. *The* k*-dimensional Abelian subalgebras of functional rank* k, $k \leq n$, *form one equivalence class* $[\mathcal{A}_k^{(k)}]$ *under the diffeomorphisms of* Diff_n.

For instance they can be transformed into $\{\partial_{z_1}, \partial_{z_2}, \ldots, \partial_{z_k}\}$ or into the set $\{z_1\partial_{z_1}, z_2\partial_{z_2}, \ldots, z_k\partial_{z_k}\}$ when $z_i \neq 0$, $1 \leq i \leq k$.

Let $\mathcal{L}$ be a Lie algebra and $\mathcal{A}, \mathcal{B}$ two vector subspaces of $\mathcal{L}$. We denote by $[\mathcal{A}, \mathcal{B}]$ the vector space:

$$[\mathcal{A}, \mathcal{B}] = \big\{[a, b], a \in \mathcal{A}, b \in \mathcal{B}\big\}. \tag{2.10}$$

For example: $[\mathcal{B}, \mathcal{B}] \subseteq \mathcal{B}$ means: $\mathcal{B}$ is a subalgebra of $\mathcal{L}$ and $[\mathcal{K}, \mathcal{L}] \subseteq \mathcal{K}$ means: $\mathcal{K}$ is an ideal of $\mathcal{L}$. Let us recall the following definitions:

DEFINITION 2.2. $C_{\mathcal{L}}(\mathcal{B})$, *the* centralizer *of* $\mathcal{B}$ *in* $\mathcal{L}$, *is the largest subalgebra* $\mathcal{H} \subseteq \mathcal{L}$ *such that* $[\mathcal{B}, \mathcal{H}] = 0$; *e.g.* $C_{\mathcal{L}}(\mathcal{L})$ *is the center of* $\mathcal{L}$.

DEFINITION 2.3. $N_{\mathcal{L}}(\mathcal{B})$, *the* normalizer *of* $\mathcal{B}$ *in* $\mathcal{L}$, *is the largest subalgebra* $\mathcal{H} \subseteq \mathcal{L}$ *such that* $\mathcal{B}$ *is an ideal of* $\mathcal{H}$.

If a family of $\mathcal{L}$ subalgebras forms one orbit for the group G, then the same is true of its centralizers and its normalizers.

For an Abelian subalgebra $\mathcal{A} \subseteq \mathcal{L}$, one has $\mathcal{A} \subseteq C_{\mathcal{L}}(\mathcal{A})$; when the equality holds, $\mathcal{A}$ is a *maximal Abelian subalgebra* of $\mathcal{L}$. Obviously the n-dimensional, functional rank n subalgebra $\{\partial_k,\ 1 \leq k \leq n\}$ is a maximal Abelian subalgebra of diff_n; this extends to all its conjugates, hence:

LEMMA 2.2. *Every* n*-dimensional, functional rank* n *algebra* $\mathcal{A}_n^{(n)}$ *is a maximal Abelian subalgebra of* diff_n.

Similarly:

$$1 \leq k \leq n;$$
$$C_{\mathrm{diff}_n}(\mathcal{A}_k^{(k)}) = \mathcal{A}_k^{(k)} \oplus \mathrm{diff}_{n-k} \iff C_{\mathrm{diff}_n}(\mathrm{diff}_{n-k}) = \mathrm{diff}_k\,. \tag{2.11}$$

In this paper we shall restrict the qualification of simple and semisimple to *finite dimensional* Lie algebras. By definition a simple Lie algebra has no nontrivial ideals (i.e. different from itself and from 0) and a semisimple Lie algebra is a direct sum of simple Lie algebras. We have also to recall the existence and properties of some of their remarkable subalgebras.

On the N dimensional vector space of a Lie algebra $\mathcal{L}$ there is a natural linear representation, called the *adjoint representation* and denoted by $\mathrm{ad}\mathcal{L}$. The representant of $a \in \mathcal{L}$ is the linear operator traditionally written as

$$a \mapsto \mathrm{ad}(a), \quad \forall x \in \mathcal{L}, \quad \mathrm{ad}(a)x = [a, x]. \tag{2.12}$$

The Killing form of a finite dimensional Lie algebra is defined by:

$$(a, b) = \mathrm{tr}\,\mathrm{ad}(a)\mathrm{ad}(b). \tag{2.13}$$

We have

$$\forall x \in \mathcal{L}, \quad \big([x, a], b\big) = \big(a, [x, b]\big). \tag{2.13'}$$

This property of the Killing form is the infinitesimal expression of its invariance under the group G. For any semisimple algebra $\mathcal{S}$, its adjoint representation is faithful and its Killing form is nondegenerate.

DEFINITION 2.4. *A* Cartan subalgebra *of a semisimple Lie algebra $\mathcal{S}$ is a maximal Abelian subalgebra $\mathcal{H}$ consisting of nonnilpotent elements (i.e. elements h such that there is no positive integer m such that* $\mathrm{ad}(h)^m = 0$*). All Cartan subalgebras form one orbit of $\mathcal{S}$ (over $\mathbb{C}$). Equivalently, a Cartan subalgebra is a maximal Abelian selfnormalizing subalgebra of $\mathcal{S}$. The rank of $\mathcal{S}$ is the dimension of its Cartan subalgebra*: $r(\mathcal{S}) = \dim \mathcal{H}$.

There exists a set of common eigenvectors of all $\mathrm{ad}(h)$; they span the space $\mathcal{L}$. Let $\{e_\alpha\}$ be a set of $(\dim \mathcal{S} - r(\mathcal{S}))$ eigenvectors forming a basis of the eigenspace with nonvanishing eigenvalues (i.e. complementary to $\mathcal{H}$). Each e_α defines a linear form on $\mathcal{H}$ that we can write (using the Killing form) as $h \mapsto (h, r_\alpha)$. We have

$$\mathrm{ad}(h)e_\alpha \equiv [h, e_\alpha] = (h, r_\alpha)e_\alpha. \tag{2.14}$$

These elements $r_\alpha \in \mathcal{H}$ are called the *roots* of $\mathcal{H}$. It is easy to prove that $-r\alpha$ (that we shall also denote by $r_{-\alpha}$) is also a root. By choosing a normalization of the e_α's we have

$$[e_\alpha, e_{-\alpha}] = r_\alpha. \tag{2.15}$$

The last two equations show that in $\mathcal{S}$ each root r_α, or each eigenvector e_α, defines a unique $\mathrm{sl}(2)_\alpha$ subalgebra with basis $e_\alpha, e_{-\alpha}, r_\alpha$.

Let us choose in the Cartan subalgebra H a hyperplane which contains no roots. We denote by R_+ the set of roots on one side of the hyperplane and R_- the roots on the other side. Each set $R_\pm$ contains $\big(d(\mathcal{S}) - r(\mathcal{S})\big)/2$ roots ($d(\mathcal{S})$ is short for $\dim(\mathcal{S})$). To each set $R_\pm$ corresponds a set of eigenvectors which generate a subspace $\mathcal{N}_\pm$ of $\mathcal{S}$. Each space contains only nilpotent elements and forms a maximal nilpotent subalgebra of $\mathcal{S}$. Their normalizers $\mathcal{B}_\pm = \mathcal{H} \oplus \mathcal{N}_\pm$ are maximal solvable subalgebras of $\mathcal{S}$. Such algebras are called Borel subalgebras. A complex

semi-simple Lie algebra $\mathcal{S}$ has precisely one Borel subalgebra up to a conjugacy in $\mathcal{S}$.

We can prove now a theorem directly related to our problem:

THEOREM 2.1. *The functional rank $r_f(\mathcal{H})$ of a Cartan subalgebra of a semisimple algebra $\mathcal{S} \subset \mathrm{diff}_n$ is equal to rank $(\mathcal{S})$.*

PROOF. Consider a semisimple Lie algebra $\mathcal{S}$ of rank r. We choose a root $r \in \mathcal{H}$, a Cartan subalgebra. We take a basis in $\mathcal{H}$: $r, h_1, \ldots, h_{r-1}$, such that $(r, h_i) = 0$, $1 \leq i \leq r-1$. We need to consider only the $\mathrm{sl}_2(e_\pm, r)$ subalgebra corresponding to the chosen root. From equation (2.14), (2.15) we have the partial set of commutation relations in $\mathcal{L}$

$$\begin{gathered} 1 \leq i, \quad j \leq r-1, \\ [h_i, h_j] = [r, h_i] = [h_i, e_\pm] = 0, \quad [r, e_\pm] = \pm(r,r)e_\pm, [e_+, e_-] = r. \end{gathered} \tag{2.16}$$

Assume that $\mathcal{S}$ is represented by a subalgebra of diff_n and that the vector fields $\widehat{R}$, $\widehat{H}_i$, representing basis elements of $\mathcal{H}$ are functionally dependent i.e.,

$$\widehat{R} = \sum_k \varphi_k \widehat{H}_k, \text{ with } [\widehat{H}_i, \widehat{H}_j] = 0, \tag{2.17}$$

where the $\varphi(x)$'s are functions (and not constants!). Then

$$0 = [\widehat{R}, \widehat{H}_i] = -\sum_k (\widehat{H}_i \varphi_k)\widehat{H}_k = 0. \tag{2.18}$$

The other commutation relations are represented by:

$$[\widehat{H}_i, \widehat{E}_\pm] = 0, \quad \pm(r,r)\widehat{E}_\pm = [\widehat{R}, \widehat{E}\pm] = -\sum_k (\widehat{E}_\pm \varphi_k)\widehat{H}_k, \tag{2.19}$$

and the last commutation relation in (2.16) yields

$$\widehat{R} = [\widehat{E}_+, \widehat{E}_-] = -(r,r)^{-2} \sum_{k,\ell} \big[(\widehat{E}_+ \varphi_k)\widehat{H}_k, (\widehat{E}_- \varphi_\ell)\widehat{H}_\ell\big]. \tag{2.20}$$

With a possible change of labeling of mute indices and the further use of the first relation in (2.19), we obtain

$$(r,r)^2 \widehat{R} = \sum_{k,\ell} \big((\widehat{E}_- \varphi_k)(\widehat{E}_+ \widehat{H}_k \varphi_\ell) - (\widehat{E}_+ \varphi_k)(\widehat{E}_- \widehat{H}_k \varphi_\ell)\big)\widehat{H}_\ell. \tag{2.21}$$

Equation (2.18) shows that this expression vanishes, so $\widehat{R} = 0$, which is a contradiction. Hence Eq. (2.17) is contradictory. This ends the proof of the theorem. □

COROLLARY 2.1. *Every semisimple subalgebra of* diff_n *has a rank $r \leq n$.*

We shall show at the end of this section that the equality $r = n$ can be reached for $\mathrm{sl}(n+1)$. In the conclusion of the paper we show that it is not reached for any other simple Lie subalgebra of diff_n.

For convenience, let us relate the Cartan notation for the classical simple Lie algebras of rank r to the standard notation used for the special linear $\mathrm{sl}(n)$, orthogonal $o(n)$, symplectic $\mathrm{sp}(n)$ algebras in dimension n:

$$\begin{gathered} r \geq 1, \quad A_r = \mathrm{sl}(r+1); \qquad r \geq 2, \quad B_r = o(2r+1); \\ r \geq 3, \quad C_r = \mathrm{sp}(2r); \qquad r \geq 4, \quad D_r = o(2r). \end{gathered} \tag{2.22}$$

TABLE 1. Lower bounds on n for which the Lie algebra L can be realized as a subalgebra of diff_n. The rank of L is denoted r.

L	$\mathrm{sl}(N)$	$o(N); N \neq 4,6$	$o(6)$	$\mathrm{sp}(2N)$	G_2	F_4	E_6	E_7	E_8
n	$N-1$	$N-2$	3	$2N-1$	5	15	16	27	57
r	$N-1$	$\left[\frac{N}{2}\right]$	3	N	2	4	6	7	8

There exist the following isomorphisms:

$$o(3) \sim \mathrm{sl}(2) \sim \mathrm{sp}(2); \quad o(4) \sim \mathrm{sl}(2) \oplus \mathrm{sl}(2); \quad o(5) \sim \mathrm{sp}(4); \quad o(6) \sim \mathrm{sl}(4). \tag{2.23}$$

Notice that $O(2)$ is a 1-dimensional Abelian Lie algebra and $o(4)$ is not simple. Among the simple Lie algebras over $\mathbb{C}$, the only ones which have a simple subalgebra of the same rank are:

$$\begin{gathered} r \geq 3, \quad o(2r+1) \supset o(2r); \\ G_2 \supset \mathrm{sl}(3), \quad F_4 \supset o(8), \quad E_7 \supset \mathrm{sl}(8), \quad E_8 \supset \mathrm{sl}(9), \quad E_8 \supset o(16). \end{gathered} \tag{2.24}$$

There is another limitation for the minimum value of n such that the simple Lie algebra G can be realised a a subalgebra of diff_n. Indeed in this realization it acts locally on a manifold of dimension n and the minimum n value corresponds to a (locally) transitive action. This can occur only if L has a subalgebra L_0 of codimension $n = \dim L - \dim L_0$. For simple Lie algebras L over the field $\mathbb{C}$ the algebra L_0 of maximal dimension is always one of the maximal parabolic subalgebras of L.

In Table 1 we give a list of the minimal values of n of the homogeneous spaces G/G_0, together with the rank r of L, for all complex simple Lie algebras. From the Table we see that for $\mathrm{sl}(n)$ (including $o(3) \sim \mathrm{sl}(2)$, $o(6) \sim \mathrm{sl}(4)$) we have $n = r$. In all other cases we have $n > r$.

The dimension n is obtained according to the formula $n = \dim L - \dim P$ where P is the highest dimensional maximal parabolic subalgebra of L. The dimension of a maximal parabolic subalgebra is obtained as $\dim P = r + n_1 + n_+$ where n_+ is the number of positive roots of L, n_1 the number of positive roots of a regular maximal simple subalgebra S of L and r is the rank of L. The numbers n_1 and n_+ can be read off from tables [**4, 5**].

We end this section by giving for every n the realization of the simple Lie algebra $\mathrm{sl}(n+1)$ of rank n as a subalgebra of diff_n.

We recall first that the $n \times n$ complex matrices form the Lie algebra $gl(n, \mathbb{C})$ when the Lie bracket of two elements is their commutator $[a, b] = ab - ba$. Let us denote by e_{ij}, $i \neq j$ the matrices whose all elements are vanishing except the one of the i-th line and j-th column which is equal to 1. In this basis for the vector space of $n \times n$ matrices the Lie algebra law is:

$$[e_{ij}, e_{k\ell}] = \delta_{jk} e_{i\ell} - \delta_{i\ell} e_{kj}. \tag{2.25}$$

The multiples of I_n form the center of $\mathrm{gl}(n)$. So the algebra $\mathrm{sl}(n)$ is faithfully represented by the $n \times n$ traceless matrices. The diagonal matrices represent a Cartan subalgebra $\mathcal{H}$: indeed they are nonnilpotent matrices and they form an $(n\text{-}1)$-dimensional maximal Abelian subalgebra. Eq. (2.25) shows that the $e_{k\ell}$'s, $k \neq \ell$ are the eigenvectors of $\mathrm{ad}(h)$, $h \in \mathcal{H}$ and the corresponding root is $e_{kk} - e_{\ell\ell}$.

The upper (respectively lower) triangular matrices form the Borel subalgebra $\mathcal{B}_+$ (respectively $\mathcal{B}_-$). The antisymmetrical matrices represent the subalgebra $o(n)$ which is a simple algebra when $n = 3$ and $n \geq 5$. Notice that for $n > 2$ this representation of $o(n)$ is irreducible. The identity $-[a,b]^\top = [-a^\top, -b^\top]$ shows that $a \mapsto -a^\top$ represents an automorphism of sl(n) which reduces to the identity on the subalgebra $o(n)$. So this automorphism of sl(n) is an outer automorphism for $n > 2$ and an inner one for $n = 2$ (conjugation by the Pauli matrix σ_2).

From the commutation rule of vector fields:

$$[z_i\partial_j, z_k\partial_\ell] = \delta_{jk} z_i\partial_\ell - \delta_{i\ell} z_k\partial_j, \tag{2.26}$$

and the comparison with (2.25) one sees that the map $e_{ij} \mapsto z_i\partial_j$ gives a realization of the Lie algebra gl(n) as subalgebra of diff$_n$. This realization can be extended to the rank n simple algebra sl$(n+1)$, with $1 \leq i, j, k, \ell \leq n$:

$$\begin{gathered} \mathcal{D} = \sum_i z_i\partial_i, \quad e_{ij} \mapsto z_i\partial_j, \\ e_{n+1,j} \mapsto \partial_j, \quad e_{i,n+1} \mapsto -z_i\mathcal{D}, \quad e_{n+1,n+1} \mapsto -\mathcal{D}. \end{gathered} \tag{2.27}$$

This example was known to Lie for any n. It shows that the equality can be reached in Corollary 2.1.

We have shown above that $a \mapsto -a^\top$ is an outer automorphism of any sl(m) algebra. That automorphism transforms into itself the sl(n) of (2.26) as a subalgebra of sl$(n+1)$ realized in (2.27) and it exchanges the two n-dimensional Abelian subalgebras $\mathcal{A}_n^n(\partial_j)$, $\mathcal{A}_n^1(z_i\hat{d})$, For $n > 1$, these two algebra are not conjugate in diff$_n$ since their functional ranks are respectively n and 1.

It is easy to verify that the normalizer of $\mathcal{A}_n^{(n)}(\partial_i)$ is an affine algebra:

$$\mathrm{aff}_n^- \sim N_{\mathrm{diff}_n}(\mathcal{A}_n^{(n)}) = \mathcal{A}_n^{(n)} \rtimes \mathrm{gl}(n), \quad gl(n) = \{z_i\partial_j\}. \tag{2.28}$$

Since the n-dimensional, functional rank n Abelian subalgebras of diff$_n$ form a unique orbit of the group Diff$_n$, that is also true of their normalizers in diff$_n$. We denote this orbit of affine algebras in dimension n by $[\mathrm{aff}_n^-]$. while the other orbit of n dimensional affine algebras is that of

$$\mathrm{aff}_n^+ = \mathcal{A}_n^{(1)}(z_i\hat{d}) \rtimes \mathrm{gl}(n), \tag{2.29}$$

i.e. that of the normalizer of $\mathcal{A}_n^{(1)}(z_i\mathcal{D})$. The sl$(n+1)$ algebra is generated by the two subalgebras $\mathrm{aff}_n^\pm$ satisfying $\mathrm{aff}_n^+ \cap \mathrm{aff}_n^- = \mathrm{gl}(n)$. Since each type of affine algebras forms one orbit of Diff$_n$ that is also true of the algebras sl$(n+1) \subset$ diff$_n$.

3. Construction of the Homogeneous Spaces and Algebras of Vector Fields

The method to be applied in this article is closely related to one used earlier to construct systems of nonlinear ordinary differential equations with superposition formulas [**15**].

We are constructing finite dimensional subalgebras L of diff$_n$ so that elements of L have the form (1.1). The algebra can be integrated to obtain a local Lie group G of local diffeomorphisms that act on a manifold $M \sim \mathbb{C}^n$. We make the restriction that G acts regularly, that is that it can be stratified into orbits of orbit type G/G_0 where G_0 is a Lie subgroup of G. Its Lie algebra L_0 is realized by vector fields

vanishing at the origin. Abusing notation, we shall assume that this stratification has already been performed and we shall consider M to be a homogeneous space

$$M \sim G/G_0. \tag{3.1}$$

Let us assume that G_0 is not a maximal subgroup of G; let G_1 be a maximal subgroup of G containing G_0. The orbit g/g_0 is a disjoint union of imprimitivity cells which are permuted by the action of G. There exists a G-equivariant surjective map σ between the orbits: $G/G_0 \xrightarrow{\sigma} G/G_1$ which projects the imprimitivity cells of G/G_0 on the points of G/G_1. Each cell can be considered as an orbit G_1/G_0. In the case of Lie groups, the imprimitivity cells are leaves of a foliation. Here we restrict all considerations to local ones; G_0 is the stabilizer (= isotropy group) of the origin of coordinates. Transitive group actions have been studied in a geometric context [**6, 7, 8, 9**]. In particular a decomposition of the type indicated in eqs. (1.6)–(1.7) was shown to correspond to the existence of an invariant foliation on M. Locally this occurs if the pair of Lie algebras $L_0 \subset L$ does *not* define a transitive primitive and effective Lie algebra. We recall their definition:

DEFINITION 3.1. *The pair of Lie algebras $L_0 \subset L$ defines a transitive primitive and effective Lie algebra if L_0 is a maximal subalgebra of L and does not contain a proper ideal of L.*

DEFINITION 3.2. *The transitive primitive effective pair of Lie algebras is nonlinear if there exists a nontrivial subalgebra $L_1 \subset L_0$ such that $[L_1, L] \subseteq L_0$.*

DEFINITION 3.3. *The transitive primitive effective nonlinear pair of Lie algebras is* irreducible *if the only subspaces S satisfying $L_0 \subseteq S \subseteq L$ and $[L_0, S] \subseteq S$ are $S = L_0$ and $S = L$. Otherwise the pair $L_0 \subset L$ is* reducible.

DEFINITION 3.4. *A parabolic subalgebra of a simple Lie algebra L is any subalgebra containing a Borel subalgebra (the unique, up to a conjugation, maximal solvable subalgebra). A maximal parabolic subalgebra is not properly contained in any other subalgebra of L.*

DEFINITION 3.5. *A reductive Lie algebra is a direct sum of simple and Abelian Lie algebras containing at least one simple Lie algebra (but not necessarily any Abelian ones).*

All transitive primitive pairs of Lie algebras over $\mathbb{C}$ have been classified [**6, 7, 8, 9**]. Let us state, without proof, the classification theorem, summing up results due to these authors.

THEOREM 3.1. *Precisely five types of transitive primitive effective pairs $L_0 \subset L$ of Lie algebras over $\mathbb{C}$ exist. They are distinguished by the nature of L_0 and L.*

1. *L is simple and L_0 is a maximal parabolic irreducible subalgebra.*
2. *L is simple and L_0 is a maximal parabolic irreducible subalgebra.*
3. *L is simple and L_0 is a maximal reductive subalgebra.*
4. *L is semi-simple and has the form $L = K \oplus K$, $L_0 = K_D$, where K is simple and L_0, isomorphic to K, is the diagonal subalgebra of $K \oplus K$.*
5. *L is an affine Lie algebra, either* aff_n *or an affine subalgebra of it. Then L_0 is reductive and acts faithfully and irreducibly on an Abelian ideal A:*

$$L = A \rtimes L_0, \quad [A, A] = 0, \quad [A, L_0] = A, \quad [L_0, L_0] \subseteq L_0. \tag{3.2}$$

With L simple, the types 1,2,3 are effective. This is also true of the other types when they are realized as subalgebras of diff_n and all are transitive. So from now on we will use the shorter expression "primitive Lie algebras".

REMARK 1. A primitive Lie algebra has no center. In Case 5, that is implied by the faithful action of L_0 on A; for the other case it is implied by the definition of simple or semisimple Lie algebras.

The list of the primitive subalgebras of diff_2 and diff_3 has been given by S. Lie in [**12**]. There are 3 of them for dimension $n = 2$ (Theorem 6, p. 71): $\mathrm{sl}(3)$, type 1, maximal finite dimensional subalgebra; aff_2^- and it subalgebra saff_2^-, both of type 5 and nonmaximal. Notice that $\mathrm{sl}(2) \oplus \mathrm{sl}(2) \equiv o(4)$ and the infinite family of subalgebras W_m, $m \geq 3$ (given in (1.5)) are maximal finite dimensional subalgebras but they are not primitive. For $n = 3$ (Chap. 7, Theorem 9, p. 139), there are 8 primitive subalgebras but only 2 of them are maximal: they are of type 1 ($\mathrm{sl}(4)$ and $o(5)$); the other primitive subalgebras are: 1 of type 2 (another $o(5) \subset \mathrm{sl}(4) \sim o(6)$), one of type 4 ($\mathrm{sl}(2) \oplus \mathrm{sl}(2) \subset \mathrm{sl}(4) \sim o(6)$), four of type 5: aff_3^- and three of its subalgebras. The primitive algebra ([2] p. 134) is interesting; its two factors $K \sim \mathrm{sl}(2)$ are realized by (with $\hat{d} = x\partial_x + y\partial_y + z\partial_z$):

$$\hat{a}_1 = \partial_y + x\partial_z, \quad \hat{a}_2 = y\partial_y + z\partial_z, \quad \hat{a}_3 = -z\partial_x + y\hat{d}, \tag{3.3}$$

$$\hat{b}_1 = \partial_x + y\partial_z, \quad \hat{b}_2 = x\partial_x + z\partial_z, \quad \hat{b}_3 = -z\partial_y + x\hat{d}. \tag{3.3'}$$

Notice that the functional rank of each $\mathrm{sl}(2)$ is $r_f = 3$ for $xy - z \neq 0$.

More generally the minimum value of n for the realization of the primitive algebra (L, L_0) as a subalgebra of diff_n is

$$n = \dim M = \dim L - \dim L_0, \tag{3.4}$$

where M is the homogeneous space of the transitive L action. Of course we are interested only in the minimal n realizations.

REMARK 2. In the Case 5 of Theorem 3.1 we have, $n = \dim M = \dim A$. As we have shown as the end of Section 2, there are only two realizations of aff_n in diff_n and both are subalgebras of the projective $\mathrm{sl}(n+1)$. So Case 5 does not yield maximal subalgebras.

LEMMA 3.1. *Let (L, L_0) be a transitive primitive Lie algebra realized as a subalgebra of* diff_n *(with $n = \dim L - \dim L_0$). Let K be another finite dimensional Lie algebra, satisfying $L \subset K \subset \mathrm{diff}_n$. Then there exists an algebra $K_0 \subset K$ such that (K, K_0) is a transitive primitive Lie algebra and we have $L_0 = L \cap K_0$, $n = \dim K - \dim K_0$.*

PROOF. Consider the groups G_0 and G, corresponding to the Lie algebras L_0 and L. Locally in the neighbourhood of the origin in $\mathbb{C}^n$, we have $M \sim G_0/G$, the manifold on which G acts transitively. We denote $K_0 \subset K$ the subalgebra of K realized by vector fields vanishing at the origin, and H_0, H the Lie groups corresponding to K_0 and K. Since G acts transitively on M and $G \subset H$, H also acts transitively. Hence we must also have $M \sim H/H_0$. If (K, K_0) did not define a primitive Lie algebra, then the local action of H on M would allow an invariant foliation. Then so would any subgroup of H, in particular G. Then (L, L_0) would not be primitive either and we obtain a contradiction.

In other words, if (L, L_0) is primitive and L is properly contained in $K \subset \mathrm{diff}_n$, the (K, K_0) is also primitive. □

COROLLARY 3.1. *Assume that* (L, L_0) *is a primitive algebra realized as a subalgebra of* diff_n *with* n *satisfying* (3.4); *then the centralizer* $C_{\mathrm{diff}_n}(L)$ *is trivial.*

Assume the contrary. Since L has no center (Remark 1) $L \cap C_{\mathrm{diff}_n}(L) = 0$. Let $\hat{a}$ be a vector field of the centralizer; $L' = L \oplus \mathbb{C}\hat{a}$ is an algebra containing L. From the preceding lemma, it is primitive and from Theorem 3.1 it is not. End of proof per absurdum.

COROLLARY 3.2. $\mathrm{sl}(n+1)$ *is a maximal finite subalgebra of* diff_n.

Let us assume the contrary: there exists H such that $\mathrm{sl}(n+1) \subset H \subset \mathrm{diff}_n$. From the preceding lemma, H is primitive. From Remark 2 and the end of Section 2, Case 5 of Theorem 3.1 is excluded. So H is semisimple and from Corollary 2.1, rank H = rank $\mathrm{sl}(n+1) = n$. Case 4 of Theorem 3.1 is excluded: the rank of such an H has to be larger than n. From (2.24) there are 3 values $n = 2, 7, 8$ for which there exists a simple H of rank n containing $\mathrm{sl}(n+1)$, but Table 1 shows that these simple algebras cannot be realised in diff_n. So cases 1,2,3 of Theorem 3.1 are also ruled out and the corollary is proven.

LEMMA 3.2. *Assume that* (L, L_0) *is a primitive Lie algebra realized as a subalgebra of* diff_n *with* n *satisfying* (3.4), *and that* L *is a proper subalgebra* $L \subset H \subset \mathrm{diff}_n$ *of the finite dimensional* **maximal** *subalgebra* H *of* diff_n. *Then* H *is simple.*

PROOF. From the previous lemma we know that H corresponds to a primitive algebra. Since it is maximal we know from Remark 2, that it cannot belong to type 5. We prove in three steps that it does not belong to type 4, i.e. $H = S \oplus S$, S simple. We have $L_0 \subset L \subset H \subset \mathrm{diff}_n$. First assume that L is simple (type 1,2,3). If L were a subalgebra of one of the factors S of H the other factor S would be in the centralizer of L; that contradicts the Corollary 3.1. So L has to be a subalgebra of the diagonal $S^d \subset S \oplus S$. Then $n = \dim M = \dim S \geq \dim L > \dim M$ which is self contradictory. Now assume that L is semisimple, i.e. $L = S' \oplus S'$ of type 4, then $n = \dim M = \dim S = \dim S'$ which implies $S = S'$, hence $H = L$. Finally, if L is of type 5, we have shown $L \subsetneq \mathrm{aff}_n \subset \mathrm{sl}(n+1)$ and we have shown in Section 2 that aff_n is a maximal subalgebra of $\mathrm{sl}(n+1)$ and Corollary 3.1 has shown that this simple algebra is maximal in diff_n; That concludes the proof of the lemma. □

This article is devoted to classical simple Lie algebras, so we will only encounter the first three types of transitive primitive Lie algebras listed in Theorem 3.1. The coordinates in the first two cases are easy to construct. For both of them the homogeneous space M is a Grassmanian of r-planes. If the group G acting on M has an invariant metric (i.e. G is $O(N)$ or $\mathrm{SP}(2N)$), then we shall deal with a Grassmanian of null planes.

(a) If the maximal parabolic subalgebra L_0 is irreducible, then coordinates can be so chosen that the coefficients of the vector fields (1.1) are second order polynomials.

(b) If L_0 is a maximal parabolic reducible subalgebra,then the coefficients of the vector fields can again be chosen to be polynomials but their degree may be higher than 2. For the classical simple Lie algebras and for g_2 [**3**] the degree of the polynomials is at most 4.

(c) If L_0 is a maximal reductive subalgebra, the coefficients of the vector fields are not necessarily polynomials (in any coordinate system). The construction of the homogeneous spaces M is less uniform than for the case of maximal parabolic subalgebras.

The following sections are devoted to the specific construction of the homogeneous spaces M, corresponding to the two first cases of Theorem 3.1. We construct convenient coordinate patches in the neighbourhood of the origin. Then we obtain realizations of $\mathrm{sl}(N)$ in Section 4, of $o(N)$ in Section 5, of $\mathrm{sp}(2N)$ in Section 6. In Section 7 we find the list of those which may be, up to an equivalence, subalgebras of others. Excluding them, we obtain several large families of nonequivalent and independent realizations of these classical simple Lie algebras by vector fields.

4. The Homogeneous Spaces $\mathbf{M} = \mathrm{SL}(\mathbf{N}, \mathbb{C})/\,\mathrm{P}(\mathbf{r}, \mathbf{s})$

In order to present explicitly the complete list of maximal parabolic subalgebras of $\mathrm{sl}(N)$, we realize this Lie algebra of the Lie group $\mathrm{SL}(N)$ by the matrices

$$W = \left\{ \begin{pmatrix} C & A \\ -D & -B \end{pmatrix} \right\}, \quad r+s=N, \quad 1 \le s \le r \le N-1, \tag{4.1}$$

$$C \in \mathbb{C}^{r\times r}, \quad A \in \mathbb{C}^{r\times s}, \quad D \in \mathbb{C}^{s\times r}, \quad B \in \mathbb{C}^{s\times s}, \quad \mathrm{tr}\, B = \mathrm{tr}\, C, \tag{4.2}$$

where A, B, C, D represent all possible matrices satisfying conditions (4.2).

For each different possible set of values of r, s, the elements of W with $A = 0$ form a maximal parabolic subalgebra $L_0 \sim p(r,s) \subset \mathrm{sl}(N)$; they are all irreducible. For $N > 2$ there are two inequivalent N-dimensional representations of $\mathrm{sl}(N)$ which are transformed into each other by the outer automorphism $W \leftrightarrow -W^\top$. We notice that this outer automorphism exchanges r and s in the decomposition of W. However, these two inequivalent representations have the same image; since the construction which follows depends only on the representation image, the subalgebras $p(r,s)$ and $p(s,r)$ lead to equivalent results; that justifies the convention $s \le r$ made in (4.1). We realize the homogeneous space

$$M = \mathrm{SL}(N)/\,\mathrm{P}(r,s), \quad \dim M = rs, \tag{4.3}$$

as the Grassmannian of r-planes in $\mathbb{C}^N$.

Following Ref. [**15**] we first introduce (redundant) homogeneous coordinates as the components of a matrix in $\mathbb{C}^{N\times s}$in which the action of $G = GL(N)$ is linear:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad \begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} XG_0 \\ YG_0 \end{pmatrix}, \quad G_0 = GL(s). \tag{4.4}$$

The origin is the point $(0, I)^\top$. In the neighbourhood of the origin we remove the redundancy inherent in homogeneous coordinates by introducing (complex) affine coordinates $Z = XY^{-1}$ in which the action of $\mathrm{SL}(N)$ is a fractional linear one:

$$Z = XY^{-1}, \quad Z' = (G_{11}Z + G_{12})(G_{21}Z + G_{22})^{-1}. \tag{4.5}$$

The corresponding infinitesimal action is represented by vector fields $\hat{L}$:

$$\hat{L} = \left\{ \mathrm{tr}(A + ZB + CZ + ZDZ)\nabla^\top \right\}, \text{ where } (\nabla)_{a\alpha} = \partial_{a\alpha}, \tag{4.6}$$

and where A, B, C, D are the matrices figuring in (4.1)–(4.2). Eq. (4.6) is a realization of

$$\mathrm{sl}(N) \subset \mathrm{diff}_n, \quad N = r+s, \quad 1 \le s \le r \le N-1, \quad n = rs. \tag{4.7}$$

As basis is given by the vector fields

$$\hat{a}_{a\alpha} = \partial_{z_{a\alpha}}, \quad \hat{b}_{\alpha\beta} = z_{c\alpha}\partial_{z_{c\beta}}, \quad \hat{c}_{ab} = z_{b\gamma}\partial_{z_{a\gamma}}, \quad \hat{d}_{\alpha a} = z_{c\alpha}z_{a\gamma}\partial_{z_{c\gamma}}. \tag{4.8}$$

There is a linear relation among the fields $\hat{b}$'s and $\hat{c}$'s:

$$\sum_a \hat{c}_{aa} = \sum_{a,\alpha} z_{a\alpha}\partial_{z_{a\alpha}} = \sum_\alpha \hat{b}_{\alpha\alpha}, \tag{4.9}$$

which is related to the last equality of (4.2).

The sets of vector fields denoted $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ in (4.8) generate four Lie subalgebras of sl(N) that we denote $\hat{A}$, $\widehat{B}$, $\widehat{C}$, $\widehat{D}$, respectively. Evidently $\hat{A}$ is Abelian; this is also the case of $\widehat{D}$. Moreover

$$[\hat{A}, \widehat{B}] \subset \hat{A}, \quad [\hat{A}, \widehat{C}] \subset \hat{A}, \quad [\widehat{D}, \widehat{B}] \subset \widehat{D}, \quad [\widehat{D}, \widehat{C}] \subset \widehat{D}. \tag{4.10}$$

We also check that $\widehat{B} \cup \widehat{C}$ is also a subalgebra; from (2.26) we know that it is the subalgebra $\mathrm{sl}(r) \oplus \mathrm{sl}(s) \oplus \mathbb{C}$.

The Abelian Lie algebra $\hat{A}$ has functional rank n. Since $s \leq r$ (see (4.7), we note that the Abelian algebra $\widehat{D}$ is of functional rank s^2; indeed all its $n = rs$ basic vector fields are different linear combinations with coefficients linear in $z_{a\gamma}$ of s^2 different fields $z_{c\alpha}\partial_{z_{c\gamma}}$ that are linearly independent at each generic point $z_{c\alpha}$. To summarize:

$$\hat{A} \subset [\mathcal{A}_n^{(n)}], \quad \text{functional rank } (\widehat{D}) = s^2 \leq n. \tag{4.11}$$

The equality holds in the last expression only when $r = s = N/2$.

When $s = 1$, $N = n + 1$: we obtain the realization of sl($n + 1$) in diff_n already given in (2.27). When n is not prime, it can be written in different manners as the product of two integers. We then obtain the realization of more than one sl(N) Lie algebra in diff_n. We need to study if they are essentially distinct or if one of them, sl(N_1) can be conjugate to a subalgebra of another one, sl(N_2). That will be done in Section 7.

5. The Homogeneous Spaces $\mathbf{M} = \mathbf{O(N)}/\mathrm{P}(\lambda, \mu, \lambda)$

Following Ref. [**15**] we realize the Lie algebra $o(N)$ of the orthogonal group $O(N)$ by the set of matrices W with the division into blocks

$$W = \left\{ \begin{pmatrix} A & -B^\top & C \\ D & E & B \\ F & -D^\top & -A^\top \end{pmatrix} \right\} \quad C^\top = -C; \quad E^\top = -E; \quad F^\top = -F, \tag{5.1}$$

where A, B, C, D, E, F represent all possible matrices satisfying conditions (5.1) with the dimensions:

$$A, C, F \in \mathbb{C}^{\lambda\times\lambda}, \quad B, D \in \mathbb{C}^{\mu\times\lambda}, \quad E \in \mathbb{C}^{\mu\times\mu}. \tag{5.2}$$

The invariant quadratic form is

$$K_N = \begin{pmatrix} 0 & 0 & I_\lambda \\ 0 & I_\mu & 0 \\ I_\lambda & 0 & 0 \end{pmatrix}, \quad WK_N + K_N W^\top = 0, \quad 1 \leq \lambda \leq \left[\frac{N}{2}\right], \tag{5.3}$$

$$0 \leq \mu \leq N - 2, \quad 2\lambda + \mu = N.$$

As we shall see, to study the maximality of parabolic algebras, it is easier to use an equivalent matrix representation of $o(N)$, corresponding to the invariant quadratic

form represented by the matrix K'_N with 1's along the second diagonal and 0's elsewhere; i.e.

$$m>0,\ (K'_m)_{ij}=\delta_{i,m+1-j};\quad K'_N=\begin{pmatrix}0&0&K'_\lambda\\0&K'_\mu&0\\K'_\lambda&0&0\end{pmatrix}. \tag{5.4}$$

Then the set W' of matrices of $o(N)$ are the $N\times N$ matrices antisymmetrical with respect to the second diagonal; indeed

$$W'K'+K'W'^{\top}=0\iff W'^{\top}=-K'W'K'. \tag{5.5}$$

The block division of W' corresponding to λ,μ with $2\lambda+\mu=N$ is:

$$W'=\begin{pmatrix}A'&B''&C'\\D'&E'&B'\\F'&D''&A''\end{pmatrix}, \tag{5.6}$$

where the block matrices A',B',C',D',E',F' have the dimensions of the corresponding unprimed matrices given in (5.2) and satisfy the conditions:

$$\begin{aligned}&A''=-K'A'^{\top}K',\quad B''=-K'B'^{\top}K',\quad D''=-K'D'^{\top}K',\\&C'=-K'C'^{\top}K',\quad E'=-K'E'^{\top}K',\quad F'=-K'F'^{\top}K'.\end{aligned} \tag{5.7}$$

The parabolic subalgebras, candidates for maximality, $L_0=p(\lambda,\mu,\lambda)$ are obtained by setting $B'=0=B''^{\top}$, $C'=0$ in (5.7) (correspondingly $B=0$, $C=0$ in (5.1). We recall the dimension of $o(N)$ and that of the homogeneous space $M=O(N)/P(\lambda,\mu,\lambda)$ as functions of λ and μ:

$$\dim o(N)=\binom{N}{2}=\lambda(2\lambda+2\mu-1)+\binom{\mu}{2};\quad \dim(M)=n=\lambda\mu+\binom{\lambda}{2}. \tag{5.8}$$

Let us compare two different parabolic subalgebras of $o(N)$ for N fixed, namely $p_2\equiv p(\lambda+1,\mu-2,\lambda+1)$ and $p_1\equiv p(\lambda,\mu,\lambda)$. The matrix W' of (5.6) for p_2 is obtained from that of p_1 by adding λ new elements to the last column of the old matrix A' and eliminating $\mu-2$ elements from the first row of the old matrix E'. We hence have $\dim p_2-\dim p_1=\lambda-\mu+2$. Now consider the case $\mu=2$. We obtain $p(\lambda,2,\lambda)\subset p(\lambda+1,0,\lambda+1)$ since in this case, and only this case, the smaller parabolic subalgebra is obtained from the larger one by setting $a_{1,\lambda+1},\dots,a_\lambda$ equal to zero, *without* adding any new elements in the matrix E. Indeed, we have $e_{11}(\text{new})=a_{\lambda+1,\lambda+1}(\text{old})$, $e_{12}(\text{new})=0$ (because of the second relation in (5.7)).

When the exceptional case $\mu=2$ is excluded one has the lemma:

LEMMA 5.1. *The parabolic subalgebras $p(\lambda,\mu,\lambda)$ with $\mu\neq 2$ are maximal subalgebras of $o(N)$.*

However, most of the general formulas we will obtain are also valid for $\mu=2$. When this is not true, we shall state it explicitly.

The matrices W's with $C=0$, $B\neq 0$ form a subspace $\mathcal{S}$ satisfying:

$$L_0=p(\lambda,\mu,\lambda),\quad L_0\subseteq\mathcal{S}\subseteq o(N),\quad [L_0,\mathcal{S}]\subseteq\mathcal{S},\quad [\mathcal{S},\mathcal{S}]=o(N). \tag{5.9}$$

From Definition 3.3, the pair $p(\lambda,\mu,\lambda)\subset o(N)$ determines a transitive primitive *reducible* Lie algebra unless we have

(a) $S=o(N)$, i.e. $\lambda=1$ so $C=-C^{\top}=0$;
(b) $S=L_0$, i.e. $\mu=0$ so $N=2\lambda$, $B=D=0, E=0$.

In these two exceptional cases (L, L_0) is an irreducible transitive primitive subalgebra.

In all cases one can realize the homogeneous space $O(N)/P(\lambda, \mu, \lambda)$ as a Grassmannian of isotropic planes in $\mathbb{C}^N$. Following Ref. [**15**], Eqs. (3.72)–(3.73) we first introduce homogeneous coordinates

$$U_1, U_3 \in \mathbb{C}^{\lambda\times\lambda}, \quad U_2 \in \mathbb{C}^{\mu\times\lambda}, \quad U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}, \quad U^\top K U = 0, \tag{5.10}$$

which satisfy

$$\forall g \in GL(\lambda), \quad U' = \begin{pmatrix} U_1 g \\ U_2 g \\ U_3 g \end{pmatrix} \sim U, \tag{5.11}$$

i.e. U' and U represent the same point of M. To remove the redundancy represented by the arbitrary matrix g, we introduced affine coordinates (in the neighbourhood of the origin with $U_3 = I$, $U_1 = 0$, $U_2 = 0$),

$$Y' = U_1 U_3^{-1}, \quad Z = U_2 U_3^{-1}. \tag{5.12}$$

In these coordinates the last equality of (5.10) becomes

$$2X + Z^\top Z = 0 \text{ with } X = \frac{1}{2}(Y' + Y'^\top), \quad Y = \frac{1}{2}(Y' - Y'^\top). \tag{5.13}$$

That means that X, the symmetric part of Y' does not represent independent coordinates; so we will eliminate X in the following computations. The coordinates on M are thus provided by the matrix elements of the matrices:

$$Z \in \mathbb{C}^{\mu\times\lambda}, \quad Y = -Y^\top \in \mathbb{C}^{\lambda\times\lambda}. \tag{5.14}$$

Introducing the symbolic matrices:

$$\begin{gathered} 1 \le a, b \le \mu, \quad 1 \le \alpha, \beta \le \lambda, \\ (\nabla_Y)_{\alpha\beta} = \partial_{y_{\alpha\beta}}, \quad y_{\beta\alpha} = -y_{\alpha\beta}, \quad (\nabla_Z)_{a\alpha} = \partial_{z_{a\alpha}}, \end{gathered} \tag{5.15}$$

we obtain directly the formula corresponding to (3.64) of Ref. [**15**], that is the realization of $o(N)$ as subalgebra of diff_n:

$$\begin{aligned} \hat{L} = \mathrm{tr}\Big(& C + AY + YA^\top + \frac{1}{2}(-B^\top Z + Z^\top B) - YFY \\ & + \frac{1}{2}(YD^\top Z + Z^\top DY) + \frac{1}{4} Z^\top (DZ^\top - ZD^\top) Z \\ & - \frac{1}{4} Z^\top Z F Z^\top Z \Big) \nabla_Y^\top \\ + \mathrm{tr}\Big(& B + DY + EZ + ZA^\top - \frac{1}{2} DZ^\top Z \\ & + ZD^\top Z - ZFY + \frac{1}{2} ZFZ^\top Z \Big) \nabla_Z^\top, \end{aligned} \tag{5.16}$$

where the matrices A, B, C, D, E, F are arbitrary complex matrices satisfying the conditions (5.1) and (5.2). That symbolic notation simply means that a basis for the representation of $o(N)$ in diff_n is given by the following vector fields (summation

over identical indices is assumed):

$$\hat{c}_{\alpha\beta} = \partial_{y_{\alpha\beta}}, \quad \hat{a}_{\alpha\beta} = 2y_{\beta\mu}\partial_{y_{\alpha\mu}} + z_{a\beta}\partial_{z_{a\alpha}}, \tag{5.17}$$

$$\hat{f}_{\alpha\beta} = y_{\alpha\mu}y_{\beta\nu}\partial_{y_{\mu\nu}} - \frac{1}{4}z_{a\mu}z_{a\alpha}z_{b\beta}z_{b\nu}\partial_{y_{\mu\nu}} - z_{a\alpha}y_{\beta\mu}\partial_{z_{a\mu}} + \frac{1}{2}z_{a\alpha}z_{b\beta}z_{b\mu}\partial_{z_{a\mu}}, \tag{5.18}$$

$$\hat{d}_{a\alpha} = z_{a\mu}y_{\alpha\nu}\partial_{y_{\mu\nu}} + z_{a\mu}z_{b\alpha}z_{b\nu}\partial_{y_{\mu\nu}} + y_{\alpha\mu}\partial_{z_{a\mu}} + z_{b\alpha}z_{a\mu}\partial_{z_{b\mu}} - \frac{1}{2}z_{b\alpha}z_{b\mu}\partial_{z_{a\mu}}, \tag{5.19}$$

$$\hat{b}_{a\alpha} = -z_{a\beta}\partial_{y_{\alpha\beta}} + \partial_{z_{a\alpha}}, \quad \hat{e}_{ab} = z_{b\mu}\partial_{z_{a\mu}} - z_{a\mu}\partial_{z_{b\mu}}. \tag{5.20}$$

As we did in the previous section, we can denote by $\hat{A}, \widehat{B}, \hat{C}, \ldots$ the vector spaces generated by the vector fields $\hat{a}, \hat{b}, \hat{c}, \ldots$. Some of them are subalgebras, e.g; $\hat{A}, \widehat{C}, \widehat{E}$ but not $\widehat{B}$. The sum of those which are of degree 0 or 1 in the variables $y_{\alpha\beta}$, $z_{a\alpha}$ form the subalgebra:

$$\begin{gathered} \widehat{C} \oplus \widehat{B} \oplus \hat{A} \oplus \widehat{E}, \quad [\widehat{C}, \widehat{C}] = [\widehat{C}, \widehat{B}] = [\widehat{C}, \widehat{E}] = 0, \quad [\widehat{E}\hat{A}] = 0, \\ [\widehat{C}, \hat{A}] \subset \widehat{C}, \quad [\widehat{B}, \widehat{B}] \subset \widehat{C}, \quad [\widehat{B}, \widehat{E}] \subset \widehat{B}, \\ [\widehat{B}, \hat{A}] \subset \widehat{B}, \quad [\widehat{E}, \widehat{E}] \subset \widehat{E}, \quad [\hat{A}, \hat{A}] \subset \hat{A}. \end{gathered} \tag{5.21}$$

The centralizer in diff_n of the subalgebra $\widehat{C} \oplus \widehat{B}$ is:

$$C_{\mathrm{diff}_n}(\widehat{C} \oplus \widehat{B}) = \widehat{C}. \tag{5.22}$$

So the centralizer of $\hat{L} = o(n)$ is trivial (verification of Corollary 3.1).

We study now the two cases of transitive primitive irreducible Lie algebras we have already announced and also a reducible case, for which the vector fields have quadratic coefficients.

CASE 1 ($\lambda = 1$, so $n = \mu$, $N = \mu + 2$). Then $C = 0$ from (5.1), $Y = 0$ from (5.14): there are no y variables and Z is a one column matrix; so we denote simply by z_i the $n = \mu = N - 2$ variables (see (5.8)). Then we obtain, as a particular case of (5.17)–(5.20), the basic vector fields ($1 \leq i, j \leq n = \mu$):

$$\begin{gathered} b_i = \partial_{z_i}, \quad a = -\sum_j z_j\partial_{z_j} = \mathcal{D}, \quad e_{ij} = z_j\partial_{z_i} - z_i\partial_{z_j}, \\ d_i = z_i\mathcal{D} - \frac{1}{2}\Big(\sum_j z_j^2\Big)\partial_{z_i} \end{gathered} \tag{5.23}$$

defines the realization of the algebra $o(n+2) \subset \mathrm{diff}_n$, which is the conformal algebra on M. Lie found it first for $o(5)$ ($n = 3$) and quoted Liouville for recognizing it as the conformal algebra. Later, in [**12**], he wrote it for arbitrary n. There is a grading of the conformal algebras by the degree of the polynomial fields:

$$\mathcal{L} = L^{(0)} \oplus L^{(1)} \oplus L^{(2)}, \quad L^{(0)} = \widehat{B}, \quad L^{(1)} = \hat{A} \oplus \widehat{E}, \quad L^{(2)} = \widehat{D}, \tag{5.24}$$

with (summation on identical indices is implied)

$$\widehat{B} = B_i\hat{b}_i, \quad \hat{A} = A\hat{a}, \quad \widehat{E} = E_{ij}\hat{e}_{ij}, \quad \widehat{D} = D_i\hat{d}_i. \tag{5.24'}$$

Notice that $\hat{A}$ represents the dilations on $M = \mathbb{C}^n$, $\widehat{E}$ the algebra $o(n)$ and $L^{(0)} = \widehat{B}$, $L^{(2)} = \widehat{D}$ are two n-dimensional Abelian algebras of functional rank n, so they are maximal Abelian subalgebras of diff_n (i.e. of the class $[\mathcal{A}_n^n]$). They are respectively ideals of $L^{(0)} \oplus L^{(1)}$, $L^{(2)} \oplus L^{(1)}$, which are both isomorphic to the *similitude algebra*

in dimension n (i.e. the Euclidean algebra with the dilations). The subalgebras $L^{(0)}$ and $L^{(2)}$ generate the conformal algebra. To summarize:

$$\widehat{B} \sim \widehat{D} \sim A_n^{(n)}, \quad \widehat{E} \sim o(n), \quad \hat{A} = \mathbb{C}\mathcal{D} \sim \mathcal{A}_1. \tag{5.25}$$

Remark that $\dim \mathrm{sl}(n+1) \geq \dim o(n+2)$ with the equality holding for $n = 1$. In this case:

$n = 1$**:** The conformal algebra $o(3)$ is identical to $\mathrm{sl}(2)$, the projective Lie algebra on the line: $o(3) \sim \mathrm{sl}(2) = \{\partial_z, z\partial_z, z^2\partial_z\}$.

$n > 1$**:** The conformal algebra cannot be a subalgebra of the projective algebra. Indeed, from (2.27) and (5.25) we note that their intersection is the similitude algebra in n dimensions. Only linear transformations preserve this intersection. Then they also preserve the functional rank, r_f, of the n-dimensional Abelian subalgebras formed by the quadratic fields: these r_f's are respectively n and 1, so they are different for $n > 1$.

Let us discuss some low dimensional cases.

$n = 2$**:** We have the conformal algebra $o(4)$ of the plane. Notice that $\hat{a}$ and $\hat{e}$ are 1 dimensional subalgebras. The six basic fields can be combined into $\hat{h}_\pm = -\hat{a} \pm i\hat{e}$, $\hat{b}_\pm = \hat{b}_1 \pm i\hat{b}_2$, $\hat{d}_\pm = \hat{d}_1 \pm i\hat{d}_2$ satisfying $[\hat{h}_\pm, \hat{d}_\pm] = 2\hat{d}_\pm$, $[\hat{h}_\pm, \hat{b}_\pm] = -2\hat{b}_\pm$, $[\hat{d}_\pm, \hat{b}_\pm] = 2\hat{h}_\pm$ and any field with $+$ index commutes with any with—index. This corresponds to the known isomorphism: $o(4) \sim \mathrm{sl}(2) \oplus \mathrm{sl}(2)$; indeed $o(4)$ is semi-simple but not simple.

$n = 3$**:** This case was studied by Lie who showed that $o(5)$ is a maximal algebra and quoted Liouville for recognizing it as the conformal algebra. Later in his book [**12**], the conformal algebra $o(n+2)$ series was given for arbitrary n.

$n = 4$**:** There is another primitive Lie algebra in this case: $S = \big(\mathrm{sl}(4), p(2,2)\big)$: its Abelian subgroup formed by the quadratic fields has maximal functional rank $r_f = n = 4$. We verify that the conformal algebra $Q = o(6)$ is identical (in agreement with the known isomorphism recalled in (2.23)). The two algebras S, Q have the same constant fields: $L_0 \sim [\mathcal{A}_4^{(4)}]$. Their subalgebras $L^{(2)}$ belong to the same class $[\mathcal{A}_4^{(4)}]$ and can be transformed into each other by a linear transformation that we do not compute explicitly. We only remark that the linear fields in S form the algebra $\widehat{B} \oplus \widehat{C}$, $[\widehat{B}, \widehat{B}] \subset \widehat{B}$, $[\widehat{C}, \widehat{C}] \subset \widehat{C}$, $[\widehat{B}, \widehat{C}] = 0$, with the two 2×2 matrices B, C satisfying $\mathrm{tr}\, B = \mathrm{tr}\, C$; this algebra is isomorphic to $\mathrm{sl}(2) \oplus \mathrm{sl}(2) \oplus \mathcal{A}_1 \sim o(4) \oplus \mathcal{A}_1$. So the subalgebras $L^{(0)} \oplus L^{(1)}$ of S and Q are isomorphic to the similitude algebra in dimension 4.

Case 2 ($\mu = 0$). Then $B = D = E = 0$, $Z = 0$. The only variables are

$$y_{\alpha\beta} = -y_{\beta\alpha}, \quad n = \frac{\lambda(\lambda-1)}{2}, \quad N = 2\lambda, \quad \lambda \geq 3. \tag{5.26}$$

We have added the last condition because for $\lambda \leq 2$, $o(2\lambda)$ is not primitive and not simple. The realization of $o(2\lambda)$ is given by:

$$\begin{gathered} o(2\lambda)\colon C_{\alpha\beta}\partial_{y_{\alpha\beta}} + 2A_{\alpha\mu}y_{\mu\beta}\partial_{y_{\alpha\beta}} + y_{\alpha\mu}F_{\mu\nu}y_{\nu\beta}\partial_{y_{\alpha\beta}}, \\ C = -C^\top, \ F = -F^\top, \end{gathered} \tag{5.27}$$

or, with matrix notation:

$$o(2\lambda) : \operatorname{tr}(C + 2AY - YFY)\nabla_Y. \tag{5.27'}$$

We first show that for $\lambda = 3, 4$ we have already obtained these algebras.

$\lambda = 3 = n$: Any 3×3 antisymmetric matrix Y can be written as $Y_{ij} = \varepsilon_{ijk} y_k$ were ε_{ijk} is the completely antisymmetric representation of the permutation of the three indices and the components of y_k (which transform as those of a vector for SL_3) are the three independent matrix elements of Y. Then

$$\begin{aligned} -(YFY)_{ij} &= -\sum_{abrst} \varepsilon_{iar}\varepsilon_{abs}\varepsilon_{bjt} y_r f_s y_t \\ &= \sum_{abrst} (\delta_{ib}\delta_{rs} - \delta_{is}\delta_{br})\varepsilon_{bjt} y_r f_s y_t = (y, f) Y_{ij}. \end{aligned} \tag{5.28}$$

Hence $-\operatorname{tr} YFY\nabla^\top = (f, y)(y_j \partial_{y_j}) = f_k y_k \mathcal{D}$. Thus we have shown that the quadratic terms $L^{(2)}$ of (5.27) are identical to those of (2.27). The $L^{(0)} = \{\partial_{y_i}\}$ terms are also identical. Since $L^{(0)}$ and $L^{(2)}$ generate the whole algebra we have proven the identity between the two algebras; that is a verification of the well known isomorphism $\mathrm{sl}(4) \sim o(6)$.

$\lambda = 4$, $n = 6$: We show that the obtained algebra o(8) is the conformal one. Indeed an explicit computation of the quadratic terms yields:

$$1 \le i \le 6, \quad L^{(2)} = \{ y_j \mathcal{D} - \mu(y)\partial_{y_j} \}, \text{ with } \mu(y) = y_1 y_4 + y_2 y_5 + y_3 y_6 = \frac{1}{2}\left(y, \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} y \right). \tag{5.29}$$

The linear transformation of variables

$$S = \frac{1+i}{\sqrt{2}} \begin{pmatrix} I_3 & -iI_3 \\ -iI_3 & I_3 \end{pmatrix} \text{ satisfies } S^\top = S, \quad SS^\top = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}, \tag{5.30}$$

so it transforms the quadratic form I_6 (which appears in the last quadratic form of (5.23) for $n = 6$) into $SS^\top$ and therefore $1/2(y, y)$ into $\mu(y)$. So the conformal algebra for $n = 6$ is identified with the O_8 algebra defined by (5.27) with $n = 6$. We recall the triality principle of E. Cartan. The outer automorphism group of O_8 is $\mathcal{S}_3$ (the permutation group of the three branches of the Coxeter diagram of D_4); it permutes the 3 inequivalent (the vector one and the two spinors) 8-dimensional linear irreducible representations of O_8. This leads to three inequivalent monomorphism $o_8 \to \mathrm{diff}_6$; however, their images are equivalent subalgebras of diff_6.

Case 3 ($\mu = 1$). Then $N = 2\lambda + 1$ and $n = \lambda(\lambda + 1)/2$. That shows that the dimension n is identical to that of the previous family (that with $\mu = 0$) with λ replaced by $\lambda' = \lambda + 1$. The N of $O(N)$ in this case satisfies $N = N' - 1$, where $o(N')$ was obtained for the same n, but $\mu = 0$ (the previous family). We shall show that this $o(2\lambda + 1)$ algebra is a subalgebra of $o(2\lambda')$ obtained in the previous family with $\mu = 0$, $\lambda' = \lambda + 1$.

With $\mu = 1$, in the general equation (5.1) we have to introduce the following modifications: $E = 0$, $B^\top$ and $D^\top$ become one column matrices that we replace by

the "vectors" b, d. Then (5.1) reads:

$$W_{\lambda,1,\lambda} = \begin{pmatrix} A & -b & C \\ d^\top & 0 & b^\top \\ F & -d & -A^\top \end{pmatrix}, \text{ with } C^\top = -C, \quad F^\top = -F. \tag{5.31}$$

It is straightforward to write the corresponding realization of $o(2\lambda+1)$ as a particular case of the equations (5.16). Indeed, we replace $Z^\top$ and $\nabla^\top$ by the λ component column vectors z and ∂; for instance $-B^\top Z$ becomes a rank one $\lambda \times \lambda$ matrix that we denote by $bz^\top$ while the matrix $DZ^\top$ becomes the number ($= 1 \times 1$ matrix) $d^\top z = (d, z)$:

$$\begin{aligned} \operatorname{tr}\Big(C + AY + YA^\top + \frac{1}{2}(bz^\top - zb^\top) - YFY + \frac{1}{2}\big(Y(dz^\top) - (zd^\top)Y\big)\Big)\nabla_Y^\top \\ + \Big(b^\top + z^\top A^\top + d^\top Y + (z^\top d)z^\top - \frac{1}{2}(d^\top z)z^\top - z^\top FY, \partial_z\Big), \end{aligned} \tag{5.32}$$

where we have $(ab^\top) \in \mathbb{C}^{\lambda\times\lambda}$, $(a^\top b) \in \mathbb{C}$.

LEMMA 5.2. *The algebra $o(2\lambda+1)$ constructed in* (5.32) *is not maximal in* diff_n *but is a subalgebra of $o(2\lambda+2)$ constructed in* (5.27):

$$\begin{gathered} o(2\lambda+1) \subset o(2\lambda+2) \subset \operatorname{diff}_n, \quad n = \frac{\lambda(\lambda+1)}{2} \\ O(2\lambda+1)/\mathrm{P}(\lambda,1,\lambda) \sim O(2\lambda+2)/\mathrm{P}(\lambda+1,0,\lambda+1). \end{gathered} \tag{5.33}$$

PROOF. We consider the algebra $o(2\lambda+2)$ realized in the space $M \sim O(2\lambda+2)/\mathrm{P}(\lambda+1,0,\lambda+1)$ in the realization (5.27). We relabel all the $(\lambda+1)\times(\lambda+1)$ matrices involved as follows:

$$\tilde{A} = \begin{pmatrix} A & a \\ r^\top & \alpha \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C & c \\ -c^\top & 0 \end{pmatrix}, \quad \tilde{F} = \begin{pmatrix} F & f \\ -f^\top & 0 \end{pmatrix}, \tag{5.34}$$

$$\tilde{Y} = \begin{pmatrix} Y & z \\ -z^\top & 0 \end{pmatrix}, \quad \tilde{\nabla} Y = \begin{pmatrix} \nabla_Y & \partial_z \\ -\partial_z^\top & 0 \end{pmatrix}, \tag{5.35}$$

$$\begin{gathered} A, C, F, Y, \nabla_Y \in \mathbb{C}^{\lambda\times\lambda}, \quad a, c, f, z, \partial_z \in \mathbb{C}^{\lambda\times 1}, \\ C + C^\top = 0, \quad F + F^\top = 0, \quad Y + Y^\top = 0. \end{gathered} \tag{5.36}$$

Rewriting Eq. (5.27) for $o(2\lambda+2)$, using the notations (5.34)–(5.36) we see that (5.32) is a subcase of the obtained equations with the identification

$$2a = c = -b, \quad 2f = -r = d, \quad \alpha = 0. \tag{5.37}$$

This proves the assertions in Eq. (5.33). □

Finally, let us consider an example when the algebra $\{L, L_0\}$ is not primitive, namely $L \sim o(N, \mathbb{C})$, $L_0 \sim p(\lambda, 2, \lambda)$, $N = 2\lambda + 2$. More specifically, we consider $N = 6$, $\lambda = \mu = 2$. The construction (5.16) is valid, just as in the primitive cases. However, for $\mu = 2$ we can transform from the coordinates $\{y_{\alpha\beta}, z_{a\alpha}\}$ to new coordinates (u, v) in which the invariant foliation becomes manifest. Indeed, consider the case of the algebra $o(6)$ with coordinates $\{y_{12} \equiv y, z_{11}, z_{12}, z_{21}, z_{22}\}$ and put

$$\begin{gathered} u_1 = y + \frac{i}{2}(z_{11}z_{22} - z_{12}z_{21}), \quad u_2 = z_{21} + iz_{11}, \\ u_3 = z_{12} - iz_{22}, \quad w_1 = z_{12} + iz_{22}, \quad w_2 = z_{21} - iz_{11}. \end{gathered} \tag{5.38}$$

The basis vector fields (5.17), ..., (5.20) (for $N = 6$, $\lambda = \mu = 2$) are transformed into

$$
\begin{aligned}
&\hat{c}_{12} \equiv \hat{c} = \partial_{u_1}, \quad \hat{b}_{11} + i\hat{b}_{21} = 2i\partial_{u_2}, \quad b_{22} - ib_{12} = -2i\partial_{u_3},\\
&\hat{b}_{11} - i\hat{b}_{21} = -u_3\partial_{u_1} - 2i\partial_{w_2}, \quad \hat{b}_{22} + i\hat{b}_{12} = u_2\partial_{u_1} + 2i\partial_{w_1},\\
&\hat{a}_{11} = u_1\partial_{u_1} + u_2\partial_{u_2} + w_2\partial_{w_2}, \quad \hat{a}_{12} = i(u_3\partial_{u_2} - w_1\partial_{w_2}),\\
&\hat{a}_{21} = i(-u_2\partial_{u_3} + w_2\partial_{w_1}), \qquad \hat{a}_{22} = u_1\partial_{u_1} + u_3\partial_{u_3} + w_1\partial_{w_1},\\
&\hat{e}_{12} \equiv \hat{e} = i(u_2\partial_{u_2} + u_3\partial_{u_3} - w_1\partial_{w_1} - w_2\partial_{w_2}),\\
&\hat{d}_{11} + i\hat{d}_{21} = 2u_1\partial_{u_3} + iw_2(w_1\partial_{w_1} + w_2\partial_{w_2}),\\
&-\hat{d}_{22} + i\hat{d}_{12} = 2u_1\partial_{u_2} + iw_1(w_1\partial_{w_1} + w_2\partial_{w_2}),\\
&i\hat{d}_{11} + \hat{d}_{21} = u_2(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}) + [2iu_1 - u_2w_1 - u_3w_2]\partial_{w_1},\\
&-i\hat{d}_{22} + \hat{d}_{12} = u_3(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}) + [2iu_1 - u_2w_1 - u_3w_3]\partial_{w_2},\\
&\hat{f}_{12} \equiv \hat{f} = -u_1(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}) + w_1\left[u_1 - \frac{i}{2}(w_1u_2 + w_2u_3)\right]\partial_{w_1}\\
&\qquad\qquad + w_2\left[u_1 - \frac{i}{2}(w_1u_2 + w_2u_3)\right]\partial_{w_2}.
\end{aligned}
\tag{5.39}
$$

The point of the above exercise is that in Eq. (5.39) the coefficients of ∂_{u_1}, ∂_{u_2} and ∂_{u_3} depend only on (u_1, u_2, u_3). Thus, we could have started out from the transitive primitive case $o(6) \supset p(3,0,3)$, constructed the $n = 3$ realization (with $y_{12} = u_3$, $y_{23} = u_1$, $y_{31} = u_2$) and then extended it to the higher dimension $n = 5$ by adding further coordinates w_1, w_2, labeling leaves in an invariant foliation. This corresponds to the procedure described in the Introduction for constructing nonprimitive algebras from primitive ones.

SUMMARY OF THE RESULTS OF THIS SECTION. We have constructed a double series, $\lambda \geq 1$, $\mu \geq 0$ of algebras $o(N)$, $N = 2\lambda + \mu$, as subalgebras of diff_n, $n = \lambda\mu + \lambda(\lambda - 1)/2$. The series $\lambda = 1$, which constructs the conformal algebras $o(n+2)$ (with $n = \mu$), was known to S. Lie. For $n = 1$ and $n = 4$ they have been already obtained in Section 4, due to the respective isomorphisms $\mathrm{sl}(2) \sim o(3)$ and $\mathrm{sl}(4) \sim o(6)$; for $n = 2$, $o(4)$ is not a simple algebra (it is maximal in diff_2). The series $\mu = 0$ starts at $\lambda = 3$; the corresponding $o(6)$ is identical to the projective $\mathrm{sl}(4)$ and for $\lambda = 4$ the corresponding $o(8)$ is identical to the conformal $o(8)$. The series $\mu = 1$ gives only subalgebras of the series $\mu = 0$ and the series $\mu = 2$ does not yield primitive algebras. The vector fields in general have polynomial coefficients of order up to four. They are of second order if the primitive algebra is irreducible ($\lambda = 1$, or $\mu = 0$) or if it is reducible, but not maximal, ($\mu = 1$) and contained in a larger irreducible subalgebra $\big(o(2\lambda+1) \subset o(2\lambda+2) \subset \mathrm{diff}_n\big)$. □

6. The Homogeneous Spaces $\mathbf{M} = \mathrm{Sp}(2\mathbf{N})/\,\mathrm{P}(\lambda, 2\boldsymbol{\mu}, \lambda)$

The symplectic group $\mathrm{Sp}(2N)$ is the subgroup of $\mathrm{SL}(2N)$ which leaves invariant the quadratic form

$$
J_N = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \text{ satisfying } J_N^{\top} = -J_N. \tag{6.1}
$$

In order to introduce maximal parabolic subalgebras, we will use a more refined block decomposition $\lambda, 2\mu, \lambda$:

$$N = \lambda + \mu, \quad J_N = \begin{pmatrix} 0 & 0 & I_\lambda \\ 0 & J_\mu & 0 \\ -I_\lambda & 0 & 0 \end{pmatrix}, \quad J_\mu^\top = -J_\mu, \tag{6.2}$$

which leads to the realization the Lie algebra $\mathrm{sp}(2N)$ of the symplectic group by the set of matrices W:

$$W = \left\{ \begin{pmatrix} A & B^\top & C \\ D & E & J_\mu B \\ F & -D^\top J_\mu & -A^\top \end{pmatrix} \right\} \; C^\top = C; \quad F^\top = F; \quad E^\top = J_\mu E J_\mu, \tag{6.3}$$

where A, B, C, D, E, F represent all possible matrices with the dimensions:

$$A, C, F \in \mathbb{C}^{\lambda\times\lambda}, \quad B, D \in \mathbb{C}^{2\mu\times\lambda}, \quad E \in \mathbb{C}^{2\mu\times 2\mu}, \tag{6.4}$$

satisfying conditions given in (6.3).

As in the previous section, following Ref. [**15**] we realize the homogeneous space $M = \mathrm{SP}(2N)/\mathrm{P}(\lambda, 2\mu, \lambda)$ as a Grassmannian of isotropic planes in $\mathbb{C}^{2N}$ (where $\mathrm{P}(\lambda, 2\mu, \lambda)$ is the maximal parabolic subgroup corresponding to the maximal parabolic subalgebra obtained by putting $B = 0$, $C = 0$ in (6.3)). To eliminate the redundancy of homogeneous coordinates we transform to affine coordinates namely elements of the matrices

$$Z \in \mathbb{C}^{2\mu\times\lambda}, \quad Y' \in \mathbb{C}^{\lambda\times\lambda}, \tag{6.5}$$

verifying the relation:

$$2X = Z^\top J_\mu Z \text{ with } X = \frac{1}{2}(Y' - Y'^\top), \quad Y = \frac{1}{2}(Y' + Y'^\top). \tag{6.6}$$

That means that X, the antisymmetric part of Y, does not represent independent coordinates; so we shall eliminate it. That yields for $n = \dim M$:

$$n = 2\lambda\mu + \frac{1}{2}\lambda(\lambda+1), \quad 1 \le \lambda, \quad 0 \le \mu, \quad N = \lambda + \mu. \tag{6.7}$$

Introducing the symbolic matrices:

$$\begin{gathered} 1 \le a, \; b \le 2\mu, \; 1 \le \alpha, \beta \le \lambda, \\ (\nabla_Y)_{\alpha\beta} = \partial_{y_{\alpha\beta}}, \quad y_{\beta\alpha} = y_{\alpha\beta}, \quad (\nabla_Z)_{a\alpha} = \partial_{z_{a\alpha}}. \end{gathered} \tag{6.8}$$

We obtain the formula corresponding to (3.76) of Ref. [**15**], that is the realization of $\mathrm{sp}(2N)$ as subalgebra of diff_n:

$$\begin{aligned} \hat{L} = \mathrm{tr}\Big(& C + AY + YA^\top + \frac{1}{2}(B^\top Z + Z^\top B) - YFY \\ & + \frac{1}{2}(YD^\top J_\mu Z - Z^\top J_\mu DY) \\ & + \frac{1}{4}Z^\top J_\mu (DZ^\top + ZD^\top) J_\mu Z - \frac{1}{4}Z^\top J_\mu Z F Z^\top J_\mu Z \Big) \nabla_Y^\top \\ + \mathrm{tr}\Big(& J_\mu B + DY + EZ + ZA^\top + \frac{1}{2}DZ^\top J_\mu Z + ZD^\top J_\mu Z \\ & - ZFY - \frac{1}{2}ZFZ^\top J_\mu Z \Big) \nabla_Z^\top, \end{aligned} \tag{6.9}$$

where the matrices A, B, C, D, E, F are arbitrary complex matrices satisfying the conditions (6.3)–(6.4)).

Let us now study the two cases when the quartic vector fields in Eq. (6.9) reduce to quadratic ones.

CASE 1 ($\mu = 0$). Then $B = D = E = 0$, $Z = 0$ and this corresponds to a transitive primitive irreducible Lie algebra. The only variables are

$$y_{\alpha\beta} = y_{\beta\alpha}, \quad n = \frac{1}{4}\lambda(\lambda+1), \quad 2N = 2\lambda. \tag{6.10}$$

The corresponding realization of $\mathrm{sp}(2\lambda)$ is given by:

$$\mathrm{sp}(2\lambda)\colon\ \mathrm{tr}(C + AY + YA^\top - YFY)\nabla_Y, \quad C = C^\top, \quad F = F^\top. \tag{6.11}$$

We notice that

$$r_f(\mathrm{tr}\, YFY\nabla_Y^\top) = n. \tag{6.12}$$

For $\lambda = 1$, $n = 1$ we simply verify the known isomorphism $\mathrm{sl}(2) = \mathrm{sp}(2)$. For $\lambda = 2$, $n = 3$; since $\mathrm{sp}(4) \sim o(5)$, we have obtained one of the two $o(5)$'s found in Section 5 and known to Lie. From the value of the functional rank of the quadratic fields, this realization of $\mathrm{sp}(4)$ is the conformal $o(5)$.

There is another series of realizations of symplectic Lie algebras with polynomial vector fields of degree ≤ 2; it corresponds to $\lambda = 1$. Remark that it is obtained (for $\mu > 0$) from reducible primitive transitive algebras:

CASE 2 ($\lambda = 1$). $2N = 2\mu + 2$, $n = 2\mu + 1$, so $2N = n + 1$. The elements of the W matrix can be decomposed into 3 numbers: $(\alpha, \gamma, \varphi)$, 4 vectors with μ components $(b, b', d, d'$ represented by a μ lines, 1 column matrix), and three $\mu \times \mu$ matrices: (R, $S = S^\top$, $T = T^\top$). Explicitly:

$$W = \begin{pmatrix} \alpha & b^\top & b'^\top & \gamma \\ d & R & S & b' \\ d' & T & -R^\top & -b \\ \varphi & d'^\top & -d^\top & -\alpha. \end{pmatrix}. \tag{6.13}$$

The $n = 2\mu + 1$ variables are η, z, z'. We denote by $(b.d) = b^\top d$ the scalar product of two vectors. The derivative operators are ∂_η and two symbolic vectors: $\nabla_z, \nabla_{z'}$. With this structure for W one finds that the algebra $\hat{L}$ of (6.9), when decomposed according to the degree of the vector fields, has only polynomials of degree $0, 1, 2$ (the vanishing of higher degree terms is due to the vanishing of $Z^\top J_\mu Z = 2X$). The set of quadratic terms is:

$$\big(-\varphi\eta - (d' \cdot z) + (d \cdot z')\big)\mathcal{D}, \text{ with } \mathcal{D} = \eta\partial_\eta + z \cdot \nabla_z + z' \cdot \nabla_{z'}, \tag{6.14}$$

where $\mathcal{D}$ is the dilation operator. This set of vector fields has functional rank $r_f = 1$. So the realization in diff_n of this algebra $\mathrm{sp}(2\mu+2) \equiv \mathrm{sp}(n+1)$ contains only linear combinations with constant coefficients of the basis vector fields of $\mathrm{sl}(n+1)$. We have arrived at the following result.

LEMMA 6.1. *The* $\mathrm{sp}(n+1)$ *algebra* (n *odd*) *constructed in* (6.14) *as a subalgebra of* diff_n *is not maximal. It is contained in the projective* $\mathrm{sl}(n+1)$ *realized in* (2.27) *for the same* n:

$$\begin{gathered} \mathrm{sp}(n+1) \subset \mathrm{sl}(n+1) \subset \mathrm{diff}_n, \\ \mathrm{SP}(n+1)/\,\mathrm{P}(1n-11) \sim \mathrm{SL}(n+1)/P(n,1). \end{gathered} \tag{6.15}$$

As in the case of $o(N)$ we see that the coefficients of vector fields are at most second order polynomials, if the parabolic subalgebra is irreducible ($\mu = 0$) or if it is reducible, but the algebra is not maximal. Then it is contained in an irreducible one.

The other symplectic algebras described by (6.9), with $\lambda \neq 1$, $\mu \neq 0$ are represented by polynomial fields of degree up to 4. The smallest values of $2N$ and n for these algebras are for $\lambda = 2$, $\mu = 1$; they define $\mathrm{sp}(6) \subset \mathrm{diff}_7$.

7. Summary and Conclusions

We have explicitly constructed all transitive primitive Lie algebras (L, L_0), where L is a classical complex simple Lie algebra and L_0 is one of its maximal parabolic subalgebras. They have all been realized as subalgebras of diff_n for some value of n. We thus have a list of all such subalgebras of diff_n for all values of n.

The dimension of the corresponding homogeneous spaces are:

$$
\begin{aligned}
&\mathrm{SL}(N)/\mathrm{P}(r,s)\colon n = rs, && N = r+s,\\
&O(N)/\mathrm{P}(\lambda,\mu,\lambda)\colon n = \lambda\left(\mu + \frac{\lambda-1}{2}\right), && N = 2\lambda+\mu, \quad \mu \neq 2,\\
&\mathrm{Sp}(2N)/\mathrm{P}(\lambda,2\mu,\lambda)\colon n = \lambda\left(2\mu + \frac{\lambda+1}{2}\right), && N = \lambda+\mu.
\end{aligned}
\tag{7.1}
$$

All such pairs (L, L_0) are summed up in Table 2 for $1 \leq n \leq 20$.

We have already seen that some seemingly different realizations are actually equivalent under local diffeomorphisms. Moreover, not all of the constructed subalgebras of diff_n are maximal.

Let us discuss the question of mutual inclusions amongst the constructed algebras somewhat further.

The following lemma is of use in this analysis.

LEMMA 7.1. *Let L and S be two classical simple complex Lie algebras and let us have $L \subset S$. Then L can be a maximal subalgebra of S in only two cases*:

1. *If L has an irreducible representation of dimension N where S is* $\mathrm{sl}(N), o(N)$, *or* $\mathrm{sp}(N)$ (N *even in the last case*).
2. *If S is $o(N)$ and L is $o(N-1)$.*

PROOF. The simple subalgebra $L \subset S$ can be imbedded in the defining representation of S (and any other representation of S), either reducibly, or irreducibly. If it is imbedded irreducibly, it leaves no nontrivial subspace of the representation space invariant. In this case L is maximal in S if it has no centralizer in S.

If L is imbedded in S reducibly, it leaves a nontrivial subspace V of the representation space $\mathbb{C}^N$ invariant. If S is $\mathrm{sl}(N)$ the invariant subspace are completely characterized by their dimension. Moreover, all maximal reducibly imbedded subalgebras are parabolic ones. If S is $o(N)$, or $\mathrm{sp}(N)$ then any invariant subspace is characterized by its dimension and the degree of its degeneracy, i.e. the number of zero length vectors in an orthogonal basis (with respect to the corresponding orthogonal, or symplectic invariant form). If the invariant subspace is degenerate, the corresponding maximal subalgebra will again be parabolic.

TABLE 2. Primitive subalgebras (L, L_0) of diff_n for $1 \leq n \leq 20$ with L simple classical and L_0 maximal parabolic

$\mathrm{sl}(N)/\mathrm{P}(r,s)$				$o(N)/\mathrm{P}(\lambda,\mu,\lambda)$			$\mathrm{Sp}(2N)/\mathrm{P}(\lambda,2\mu,\lambda)$		
n	N	r	s	N	λ	μ	N	λ	μ
1	2	1	1	3	1	1	1	1	0
2	3	2	1						
3	4	3	1	6	3	0	2	1	1
				5	1	3	2	2	0
				5	2	1			
4	5	4	1	6	1	4			
	4	2	2						
5	6	5	1	7	1	5	3	1	2
6	7	6	1	8	1	6	3	3	0
	5	3	2	8	4	0			
				7	3	1			
7	8	7	1	9	1	7	4	1	3
				7	2	3	3	2	1
8	9	8	1	10	1	8			
	6	4	2						
9	10	9	1	11	1	9	5	1	4
	6	3	3	8	2	4			
10	11	10	1	12	1	10	4	4	0
	7	5	2	10	5	0			
				9	4	1			
11	12	11	1	13	1	11	6	1	5
				9	2	5	4	2	2
12	13	12	1	14	1	12	4	3	1
	8	6	2	9	3	3			
	7	4	3						
13	14	13	1	15	1	13	7	1	6
				10	2	6			
14	15	14	1	16	1	14			
15	16	15	1	17	1	15	8	1	7
	8	5	3	12	6	0	5	2	3
				11	5	1	5	5	0
				11	2	7			
				10	3	4			
16	17	16	1	18	1	16			
	10	8	2						
	8	4	4						
17	18	17	1	19	1	17	9	1	8
				12	2	8			
18	19	18	1	20	1	18	5	3	2
	11	9	2	11	4	3	5	4	1
	9	6	3	11	3	5			
19	20	19	1	21	1	19	10	1	9
				13	2	9	6	2	4
20	21	10	1	22	1	20			
	12	10	2						
	9	5	4						

Let $V \subset \mathbb{C}^N$ be nondegenerate. Then the orthogonal complement $V^\perp$ is also invariant. The maximal subalgebra leaving V and $V^\perp$ invariant is in general semisimple, namely $o(N_1) \oplus o(N_2)$, $N_1 + N_2 = N$ for $S \sim o(N)$, or $\mathrm{sp}(N_1) \oplus \mathrm{sp}(N_2)$ for $S \sim \mathrm{sp}(N)$ (N_1, N_2 and N even).

The sole exception occurs for $N_1 = N-1$, $N_2 = 1$ when the reducibly imbedded maximal subalgebra of $o(N)$ is $o(N-1)$. The reason for the exception is that the complement $o(1)$ is the Lie algebra consisting of only the null element.

This completes the proof of Lemma 7.1. □

COROLLARY 7.1. *Let (L, L_0) and (S, S_0) be two transitive primitive Lie algebras with*

$$n = \dim L - dim L_0 = \dim S - \dim S_0, \quad \dim L < \dim S. \tag{7.2}$$

We can have $L \subset S$ only if L has an irreducible linear representation of dimension N, the dimension of the defining representation of S, or if we have $L \sim o(N-1)$, $S \sim o(N)$.

PROOF. Any subalgebra of S is contained in at least one maximal subalgebra of L. The maximal subalgebra must be primitive, if L is to be primitive. According to Lemma 7.1 this is only possible in the two cases covered by the corollary. □

Lemma 7.1, together with dimensional considerations, help us to establish, or in the contrary, to rule out many possible mutual inclusions. Other cases need a detailed analysis of the form of the vector fields. This we postpone to a future publication.

We have seen that for all Lie algebras considered in this article the vector fields have polynomial coefficients. The order of these polynomials for the primitive Lie algebras is at most four.

The polynomials are at most quadratic in the following cases.

1. L_0 is a maximal parabolic irreducible subalgebra of L. This covers the cases $\big[(\mathrm{sl}(N), p(r,s)\big]$, $\big[o(2,k), p(k,0,k)\big]$, $\big[o(N), p(1,N-2,1)\big]$, $\big[\mathrm{sp}(2N), p(N,o,N)\big]$.

2. L_0 is a maximal parabolic reducible subalgebra of L, L is not maximal in diff_n, but we have

$$L \subset S \subset \mathrm{diff}_n, \quad \dim L - \dim L_0 = \dim S - \dim S_0, \tag{7.3}$$

and S_0 is a maximal parabolic irreducible subalgebra of S. The only such cases correspond to the spaces

$$\begin{aligned} O(2k+1)/\mathrm{P}(k,1,k) &\sim O(2k+1)/\mathrm{P}(k+1,0,k+1), \\ \mathrm{Sp}(2N)/\mathrm{P}(1,2N-2,1) &\sim \mathrm{SL}(2N)/\mathrm{P}(2N-1,1). \end{aligned} \tag{7.4}$$

Notice that both cases of Lemma 7.1 and its corollary are represented here. Indeed $o(2k+1)$ is imbedded reducibly in $o(2k)$, $\mathrm{sp}(2N)$ is imbedded irreducibly in $\mathrm{sl}(2N)$.

The situation is particularly simple for the $\mathrm{sl}(N)$ subalgebras of diff_n.

THEOREM 7.1. *The primitive Lie algebras $\mathrm{sl}(N)$ constructed in Section 4 are maximal among the simple subalgebras of diff_n for all values of n.*

PROOF. Let $\mathrm{sl}(N') \subset \mathrm{sl}(N) \subset \mathrm{diff}_n$, $N' = \lambda' + \mu' \leq N = \lambda + \mu$ and $\lambda'\mu' = \lambda\mu = n$. We can exclude the case $N' = N$ because that implies $\lambda' = \lambda$ and $\mu' = \mu$, i.e. the two algebra would be identical. So n must be the product of two different pairs of integers; that requires $n \geq 4$. From Section 4 we know that $N \leq n+1$. From the theory of irreducible representations of the $\mathrm{sl}(N)$'s, it is known that N is the smallest dimension of a nontrivial irreducible representation and that the next smallest dimension greater than N for an irreducible representation is the antisymmetric tensor square of the defining (= N-dimensional) representation; its

dimension is $N(N-1)/2$. So we must have $d = N'(N'-1)/2 \leq N \leq n+1$. Using $n = \lambda'\mu'$, an explicit computation gives:

$$(7.5)\quad d = \frac{1}{2}(\lambda'+\mu')(\lambda'+\mu'-1) = \frac{1}{2}\lambda'(\lambda'-1) + \frac{1}{2}\mu'(\mu'-1) + n \leq n+1$$
$$\iff \lambda'(\lambda'-1) + \mu'(\mu'-1) \leq 2.$$

The inequality is incompatible with $n = \lambda'\mu' \geq 4$. □

The smallest orthogonal representation of $\mathrm{sl}(N')$ is the adjoint representation, of dimension $d = N'^2 - 1$ except for $\mathrm{sl}(4) \sim o(6)$ (special case that we have already studied). The largest value of N for $o(N) \subset \mathrm{diff}_n$ is $n+2$ (except for $n=3$: it is $N=6$; case that we have studied). So we have to study when the inequality $d \leq n+2$ can be satisfied with $N' = \lambda' + \nu'$ and $n = \lambda'\mu'$. We obtain

$$(7.6)\qquad d = \lambda'^2 + \mu'^2 + 2n - 1 \leq n+2 \iff \lambda'^2 + \mu'^2 \leq 3 - n \iff n \leq 1.$$

That shows that no $\mathrm{sl}(N')$ of Section 4 is a subalgebra of $o(N)$ of Section 5 except for the two isomorphisms already quoted.

The smallest symplectic linear representation of $\mathrm{sl}(N')$ has a larger dimension than the adjoint one except for $\mathrm{sl}(6) \subset \mathrm{sp}(20)$. Relation (7.6) applies a fortiori in the general case and the special case is directly ruled out.

Let us now run through the low dimensional cases of diff_n following Table 2 and identify all maximal subalgebras, all equivalences and all inclusions. The notation $L/L_0 \sim L'/L'_0$ means that the two realizations are equivalent $L/L_0 \subset S/S_0$ means that $L \subset S$ and the realization of L is not maximal in diff_n.

$n=1$ $\mathrm{sl}(2)/p(1,1) \sim o(3)/p(1,1,1) \sim \mathrm{sp}(2)/p(1,0,1)$
Maximal in diff_n.

$n=2$ $\mathrm{sl}(3)/p(2,1)$
Maximal in diff_2.

$n=3$ $\mathrm{sl}(4)/p(3,1) \sim o(6)/p(3,0,3)$
$o(5)/p(1,3,1) \sim \mathrm{sp}(4)/p(2,0,2)$
$o(5)/p(2,1,2) \sim \mathrm{sp}(4)/p(1,2,1) \subset \mathrm{sl}(4)/p(3,1)$
The first two are maximal in diff_3.

$n=4$ $\mathrm{sl}(5)/p(4,1)$
$\mathrm{sl}(4)/p(2,2) \sim o(6)/p(1,4,1)$
Both are maximal in diff_4.

$n=5$ $\mathrm{sl}(6)/p(5,1)$
$o(7)/p(1,5,1)$
$\mathrm{sp}(6)/p(1,4,1) \subset \mathrm{sl}(6)/p(5,1)$
The first two are maximal.

$n=6$ $\mathrm{sl}(7)/p(6,1)$
$\mathrm{sl}(5)/p(3,2)$
$o(8)/p(1,6,1) \sim o(8)/p(4,0,4)$
$\mathrm{sp}(6)/p(3,0,3)$
$o(7)/p(3,1,3) \subset o(8)/p(1,6,1)$

	The first four are maximal in diff_6.
$n=7$	$\mathrm{sl}(8)/p(7,1)$
	$o(9)/p(1,7,1)$
	$\mathrm{sp}(6)/p(2,2,2)$
	$\mathrm{sp}(8)/p(1,6,1) \subset \mathrm{sl}(8)/p(7,1)$
	The first three are maximal. The $\mathrm{sp}(6)/p(2,2,2)$ case is the first primitive one with quartic coefficients.
$n=8$	All three algebras in Table 2 are maximal.
$n=9$	$\mathrm{sl}(10)/p(1,9)$
	$o(11)/p(1,9,1)$
	$o(8)/p(2,4,2)$
	$\mathrm{sl}(6)/p(3,3)$
	$\mathrm{sp}(10)/p(1,4,1) \subset \mathrm{sl}\,\mathrm{sl}(10)/p(9,1)$
	The first four are maximal.
$n=10$	All algebras are maximal except $o(9)/p(4,1,4) \subset o(10)/p(5,0,5)$.
$n=11$	All algebras are maximal except $\mathrm{sp}(12)/p(1,10,1) \subset \mathrm{sl}(12)/p(11,1)$.
$n=12,13,14$	All algebras are maximal.
$n=15$	$\mathrm{sl}(16)/p(15,1)$
	$\mathrm{sl}(8)/p(5,3)$
	$o(17)/p(1,15,1)$
	$o(12)/p(6,0,6)$
	$\mathrm{sp}(10)/p(5,0,5)$
	$\mathrm{sp}(10)/p(2,6,2)$
	$o(11)/p(5,1,5) \subset o(12)/p(6,0,6)$
	$\mathrm{sp}(16)/p(1,14,1) \subset \mathrm{sl}(16)/p(15,1)$
	The first six are maximal; $\mathrm{sp}(10)/p(2,6,2)$ has quartic coefficients.

For $n = 15$ we encounter the first case when our simple criteria based on dimensions and ranks of algebras, dimensions of representations and on the order of the polynomials involved, are not sufficient to decide upon mutual inclusions. Indeed, consider the pair $o(10)/p(3,4,3)$ and $o(11)/p(2,7,2)$. As we know, $o(10)$ can be imbedded reducibly in $o(11)$, without having a centralizer. For both algebras, the coefficients of the vector fields involve quartic polynomials. A different study is needed to decide whether this $o(10)$ can be transformed into the $o(10)$ contained reducibly in $o(11)$.

This problem is typical of the ones that we are postponing to a future article and that occur for higher values of n. For a given value of n we can have two different realizations of $o(N)$, corresponding to two different pairs (λ_1, μ_1) and (λ_2, μ_2). Similarly, we can have two different realizations of $\mathrm{sp}(2N)$. Pairs of algebras $o(N)$, $o(N+1)$ occur for many values of n. An analysis of inclusions and equivalences among such algebras is in preparation.

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