ON THE DYNAMICAL BREAKING OF SU (3)

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I. INTRODUCTION

It has been suggested by several authors(1-5) that the various breakings of SU(3) may be of pure dynamical origin. A theory invariant under a group G, will be said to be dynamically broken when there is more than one type of invariant solutions, and those arising from a variational principle all belong to a special type. For instance, it has been proposed that stable solutions of an SU(3)-invariant problem might only be obtained in special directions in the octet space $\mathcal{E}_8$. This is indeed the case if stable solutions come from a variational principle on an SU(3)-invariant function on $\mathcal{E}_8$. The nature of these solutions does not depend upon the dynamical details of the theory but reflects the geometrical properties of $\mathcal{E}_8$. In order to show this (Sec. III) we will discuss in Sec. II the geometry of $\mathcal{E}_8$. In Sec. IV we will examine whether it is possible by using a similar variational problem to find the geometrical relations between the directions preferred by the three SU(3)-breaking interactions. We show that this is not possible if the function to be varied (the S-matrix for example) depends linearly on the external electromagnetic field, and on the lepton currents.

II. GEOMETRY OF THE OCTET

1. We denote by $\mathcal{E}_8$ the octet space, that is, the eight dimensional real vector space on which SU(3) acts linearly and irreducibly.

It can be realized by the set of traceless (tr x = 0) hermitian 3 x 3 matrices. The action of 3 x 3 unitary unimodular matrix $u \in SU(3)$ on $a \in \mathcal{E}_8$ is

$$a \rightsquigarrow uau^{-1} \quad . \tag{1}$$

2. One can define on $\mathcal{E}_8$ a Lie algebra structure :

$$x \wedge y = \frac{-i}{2} (xy - yx) \quad , \tag{2}$$

which is that of the SU(3) Lie algebra.

$\mathcal{E}_8$ can be given a euclidian space structure with the scalar product :

$$(x,y) = \frac{1}{2} \operatorname{tr} xy = (y,x) \quad , \tag{3}$$

which is the Cartan-Killing bilinear form of the SU(3) Lie algebra. Through the action (1), SU (3) is a group of automorphisms of the Lie algebra (2) and leaves invariant the scalar product (3),

$$u(x\wedge y)\,u^{-1} = (u\,x\,u^{-1})\wedge(u\,y\,u^{-1}) \quad , \qquad (4)$$

$$(u\,x\,u^{-1},\ u\,y\,u^{-1}) = (x,y) \quad . \qquad (5)$$

3. There is also on $\mathcal{E}_8$ a commutative (non-associative) algebra

$$x\vee y = \frac{1}{2}\ (xy + yx) - \frac{1}{3}\ \mathrm{tr}\ xy$$
$$= \frac{1}{2}\ (xy + yx) - \frac{2}{3}\,\mathbb{1}(x,y) \ , \qquad (6)$$

which has SU(3) as group of automorphisms.

4. A function of n vectors of $\mathcal{E}_8$ is invariant by SU(3) if for every $u \in SU(3)$

$$f(u\,x_1\,u^{-1},\dots,\ u\,x_n\,u^{-1}) = f(x_1,\dots,x_n) \ . \qquad (7)$$

The scalar product (3) is an example of a bilinear invariant function. We will use the trilinear invariant function[5]

$$(x,y,z) = (x\vee y,z) = (x,y\vee z)$$
$$= \frac{1}{4}\ (\mathrm{tr}\ xyz + \mathrm{tr}\ xzy) \qquad (8)$$

$$[x,u,z] = (x\wedge y,z) = (\ x,y\wedge z)$$
$$= \frac{-i}{4}\ (\mathrm{tr}\ xyz - \mathrm{tr}\ xzy) \quad . \qquad (9)$$

5. To continue our study of the action of SU(3) on $\mathcal{E}_8$ we need now a few general concepts concerning transformation groups.

Consider a group G acting on a set M. For every $a \in M$ we denote by G_a the little group of a, i.e. the subgroup of G which leaves a invariant.

An <u>orbit</u> of G on M is a subset of all points which are the transforms by G of a given point a. If a and b are on the same orbit of G there is a $u \in G$ such that $b = u(a)$ and the little group of a and b are conjugated in G :

$$G_b = u\, G_a\, u^{-1} \quad , \tag{10}$$

Thus the classification of types of orbit of a group G reduces to that of the subgroups of G, up to a conjugation.

A layer of G on M is a subset of all points of M whose little groups are conjugated in G. Thus the action of G on M divides it into layers. In turn, each layer is decomposed into orbits of the same nature.

We will be interested in the case where G is SU(3) and M a manifold on which G acts differentially. A layer with the same dimension as the manifold will be called general. In all examples we shall meet in this paper the general layer is dense in M. The other layers will be called singular.

Here we take $\mathcal{E}_8$ as the manifold M, and give directly the results in 6 and 7.

6. Let $x^3 - \gamma(x)\, x - \mu(x) = 0 \quad ,$ (11)

be the characteristic equation of $x \in \mathcal{E}_8$. One checks that

$$\gamma(x) = (x,x) = \frac{1}{2}\ \mathrm{tr}\ x^2 \quad ,$$
$$\mu(x) = \frac{2}{3}\ (x,x,x) = \frac{1}{3}\ \mathrm{tr}\ x^3 = \det x \quad . \tag{12}$$

The functions $\gamma(x)$ and $\mu(x)$ are invariant by SU(3), hence they are constant on every orbit. Since x is hermitian and tr x = 0 all roots of (11) are real, their sum is zero. It is easy to verify that the set of roots of equation (11) or, what is equivalent, the set of values of the pair $\gamma(x)$ and $\mu(x)$ classify completely the orbits of SU(3) on $\mathcal{E}_8$.

7. $\mathcal{E}_8$ is decomposed by SU(3) into 3 layers.

I. The origin, 0, whose little group is SU(3) itself.

II. A one parameter family of four dimensional orbits which correspond to a double root of (11) or equivalently

$$4(\gamma(x))^3 = 27(\mu(x))^2 \neq 0 \quad . \tag{13}$$

We call "charges"(6) the elements of this layer. The little groups are all the (four parameters) U(2) subgroups of SU(3). It is an exceptional layer.

III. The eight dimensional layer formed by a two parameter family

$$\gamma(x),\ \mu(x) \text{ with } 4(\gamma(x))^3 \neq 27\ (\mu(x))^2 \tag{13a}$$

of six dimensional orbits. The little groups are $U(1) \times U(1)$.

8. Among its three eigenvalues ξ, ξ, -2ξ a charge q has only two different ones so it satisfies a second degree equation :

$$0 = (q - \xi \mathbb{1})\ (q + 2\xi \mathbb{1}) \text{ which reads :}$$

$$q \vee q = - \xi\ q \quad . \tag{14}$$

This equation characterizes a charge.(6)

9. <u>Sub-algebra $\vee$ generated by an element</u>

If q is a charge, equation (14) shows that it generates a one dimensional sub-algebra $\vee$. If x is not a charge, since its minimal equation is of the third degree, x generates a two dimensional sub-algebra that we denote by C_x. We easily compute

$$x \vee (x \vee x) = x \vee x \vee x = \frac{1}{3} \gamma(x) \quad x, \tag{15}$$

$$(x \vee x) \vee (x \vee x) = \mu(x)x - \frac{1}{3} \gamma(x) \; x \vee x \quad . \tag{15a}$$

10. We introduce the name "special" for the elements of the orbits $\mu(x) = 0$ (the eigenvalues are λ, $-\lambda$, 0). Equation (15a) is of the type (14), i. e.

$$s \text{ special} \Rightarrow s \vee s \text{ is a charge.} \tag{16}$$

11. C_x defined in § 9, is also the Lie algebra of the little group of x, when x is not a charge. The little group of a charge q is a U(2), whose Lie algebra we denote $U(2)_q$. It is also a sub-algebra $\vee$.

It is easy to prove :

$$a \in U(2)_q \text{ and } a \wedge x = o \Rightarrow C_x \subset U(2)_q. \tag{17}$$

III. DYNAMICAL BREAKING

We now want to apply the formalism developed in the previous section to study the following problem.

Let f be a function which describes a system of strongly interacting particles. In particular f could be the S-matrix or the Lagrangian for the system. We shall assume that f is invariant under SU(3) and that it depends upon a vector $x \in \mathcal{E}_8$. This dependence represents the effect of the symmetry breaking.

We want to investigate if x has to satisfy any condition in order that f(x) be an extremum. This question is answered by the following mathematical theorem :

Let G be a Lie group acting on a manifold M and let its action be differentiable. Then any invariant function on the manifold (which is thus a constant on every orbit of the group) has extrema on the special layers.(7) In the case of the octet we have seen that besides the origin there is only one singular layer, that of the charges. Hence whatever the details of the dynamical theory, one can predict that at least a charge orbit will appear as a solution of every SU(3) invariant variational problem on the octet space.(8) Clearly the nature of the extremum, which determines the stability or instability of the solution, depends upon the detailed nature of the theory. This result can be easily proved for the action of SU(3) on the octet space $\mathcal{E}_8$. It is essentially contained in references (1-5), but here we shall prove it in a simpler and more direct way.

Let f(x) be an algebraic invariant function on $\mathcal{E}_8$, i.e., a function of x through the invariants $\gamma(x)$, $\mu(x)$(9); the gradient of f is

$$\frac{df}{dx} = \frac{\partial f}{\partial \gamma} x + \frac{\partial f}{\partial \mu} x \vee x \quad .$$

The extrema of f are defined by df/dx = 0, i.e., by one of the conditions :

(i) $x = 0$,

$$\text{(ii)} \quad \frac{\partial f}{\partial \gamma} = \frac{\partial f}{\partial \mu} = 0 \quad ,$$

$$\text{(iii)} \quad \frac{\partial f}{\partial \gamma} x + \frac{\partial f}{\partial \mu} x \vee x = 0, \text{ with} \qquad (18)$$

$$\frac{\partial f}{\partial \gamma} \text{ and } \frac{\partial f}{\partial \mu} \text{ different from zero} \quad .$$

The third condition is the same as Eq (14) of section II. Hence in this case the function f has an extremum when x is on a charge orbit.[10, 11] This shows that a dynamical breaking occurs along preferred directions of $\mathcal{E}_8$. Thus a dynamical breaking selects on $\mathcal{E}_8$ a four-dimensional submanifold which is a charge orbit. The removal of the remaining degeneracy by choosing one particular charge on the orbit requires a breaking of the symmetry (spontaneous or induced).

From the physical point of view it is interesting to consider invariant functions on the octet depending on more than one variable. This is the case if we want to take into account the breaking of SU(3) due to two interactions. In the manifold of pairs of vectors of $\mathcal{E}_8$ the general layer is the set of all orbits of dimensions eight. It contains all the pairs of non commuting vectors. If we forget the pairs of colinear vectors (which would correspond physically to consider only one interaction) the only other layer is made of pairs of commuting vectors. From the previous result we know that they are two charges. An extremum of an invariant function of two vectors should appear on a (six-dimensional) orbit of this layer.

The two known pairs of commuting charges like the electric charge and the hypercharge or the electric charge and the weak charge are examples of this situation.

IV. INDUCED DYNAMICAL BREAKING

Instead of considering the general problem of the breaking of SU(3) due to the three interactions, we shall consider the electromagnetic and weak interactions as given and investigate their influence on the semistrong breaking. More precisely we want to investigate if the breaking of SU(3) due to electromagnetic and weak interactions can induce a breaking along a preferred charge direction, the one of the hypercharge.

The electromagnetic interactions break SU(3) along the direction of a charge q of the octet $\mathcal{E}_8$. The breaking due to the semileptonic weak interactions occurs along two special vectors c_1, c_2. These vectors generate by the $\wedge$ and $\vee$ law a $U_2(z)$, where z is the charge $c_1 \vee c_1$ or $c_2 \vee c_2$ (see eq.16).

The electric charge q belongs to $U_2(z)$.

We assume that the breaking is such that the Lagrangian or the S-matrix is a function on the octet depending on q, c_1, c_2 which is invariant by a change of basis in $\mathcal{E}_8$ leading to new vectors q', c_1', c_2'. In other words we assume that the theory is described by a function $f(y;q,c_1,c_2)$ (the Lagrangian or the S-matrix for example) which is invariant for simultaneous SU(3)-transformations on the hadron fields and on the external vectors

q_1, c_1, c_2.[12]

We now ask whether, given $f(y;q,c_1,c_2)$, it is possible to define in a covariant way the direction y with the following properties :

(i) y is a charge .

(ii) $y \notin U_2\,(z)$. This is equivalent to requiring

$$y_\wedge z \neq 0 \quad .$$

(i) and (ii) are necessary properties for a vector in the direction of the hypercharge.

If the semistrong breaking in the y-direction is of a dynamical origin one could hope that the solution of this problem would arise from an equation of the form

$$\frac{df}{dy} = 0 \ ,$$

when q, c_1, c_2 are kept fixed.

We solve this equation in the case when f depends linearly on q, c_1, c_2. This is perhaps justified if the electromagnetic and weak interactions can be treated as first order perturbation. In this case the most general invariant function of the vectors y, q, c_1, c_2 is of the form

$$f = f_o + \alpha_o\,(q,y) + \alpha_1(c_1,y) + \alpha_2\,(c_2,y)$$
$$+ \alpha_3(q,y_\vee y) + \alpha_4(c_1,y_\vee y) + \alpha_5(c_2,y_\vee y) \quad ,$$

where f_o and the α's are invariant functions of y. The extrema of f are given by

$$0 = \frac{df}{dy} = \beta_1\,y + \beta_2\,y_\vee y + a_\vee y + b \quad , \qquad (19)$$

with a, b $\in U_2(z)$ and where β_1 and β_2 and the lengths (a,a) and (b,b) of a and b are scalar functions of y.

From equation(21)of the Appendix, we can deduce a necessary condition on y by operating on (19) with $y_\wedge$. We thus get

$$\frac{1}{2}(y_\vee y)_\wedge \; a + y_\wedge b = 0 \quad . \qquad (20)$$

In this form it is easy to see that if we require that the solution y be a charge then $y \in U_2(z)$. Indeed Eq.(20)becomes

$$y_\wedge \left(- \frac{1}{2\sqrt{3}} \; a + b \right) = 0 \quad .$$

which implies that y commutes with a vector of $U_2(z)$ and therefore belongs to it (see 17). Eq. (19) reduces to Eq. (24) of Cabibbo[4] if we set a = 0. In this case, the simplified equation means

$$b \in C_y \Leftrightarrow y \in C_b \subset U_2(z) \quad .$$

We conclude that it is impossible to satisfy (i) and (ii) in this way. An equivalent way to interprete this negative result is to say that the additional physical requirement on y,

$$y_\wedge q = 0,$$

is only compatible with the values 0 and $\pi/2$ for the Cabibbo angle.

We will only point out that the invariance properties of the purely hadronic weak interactions could be explained by the same method as due to a dynamical breaking induced by the semileptonic weak interactions.

V. CONCLUSION

The aim of this paper was to study what information about the properties of the SU(3) breaking interactions can be obtained from the geometrical structure of the internal symmetry space $\mathcal{E}_8$ of the hadrons. The following results have been obtained.

(i) Invariant functions on $\mathcal{E}_8$ like the S-matrix or the Lagrangian are minima (or maxima) on the charge orbits. This means that if S depends upon a vector of $\mathcal{E}_8$ this vector must be a charge if we require that S satisfies a variational principle. Thus the most stable case in the case of a minimum corresponds to an S-matrix invariant under a subgroup U_2 of SU(3).

(ii) If the S-matrix depends upon two vectors, an extremum of S is obtained when the two vectors are two commuting charges. This is the situation for the electromagnetic and semistrong interactions and for the electromagnetic and weak hadronic interactions.

(iii) There seems to be no hope to find the value of Cabibbo's angle from the solution of a variational principle, at least if we restrict our considerations to first order effects in the electromagnetic and weak interactions.

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VI. REFERENCES

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(6) These elements can equivalently be mathematically characterized by the condition : they are the only operators on $\mathcal{E}_8$: $a \rightsquigarrow q \wedge a$ whose proper values are the multiple of a fixed number by 1, 0, -1.

(7) We quote the theorem loosely without giving a precise mathematical reference because we would have to go into too much technical de-

tail. Strictly speaking the theorem is not as general as that. For instance, a condition of compacity should be introduced on M or on quotient space M/orbits. See footnote 8.

(8) Here the space of orbits is the half real plane $\mu, \gamma > 0$ and $\gamma = 0, \mu = 0$ which is not compact. However, if one is interested only in the "directions" of $\mathcal{E}_8$, (= rays = vectors up to a factors) they form P(7,R) the real projective seven dimensional space. It is compact. By action of SU(3) there are only two layers, the general one which is dense, the singular one which contains only one orbit of SU(3), that of the charges. Then any differentiable function on the directions (= rays) of $\mathcal{E}_8$, invariant by SU(3), has an extremum on the charge orbit.

(9) To the best of our knowledge this is also true for infinitely differentiable functions.

(10) This statement is not precise. But it is easy to be more precise. Since $\frac{\partial f}{\partial \gamma}$ and $\frac{\partial f}{\partial \mu}$ are functions of γ and μ, the charge orbit is given by the equation $\frac{\partial f}{\partial \gamma} \gamma + \frac{3}{2} \frac{\partial f}{\partial \mu} \mu = 0$, where $\gamma = 3(\mu/2)^{2/3}$. The statement is true if this equation in μ has at least one real solution. We believe this will always be the case in concrete physical problems ;

μ is then the strength of the breaking.

(11) Brout's letter (ref 3) is probably the first paper which says clearly that this known result is of a general mathematical nature. However, it is attributed to the property of SU(3) to be of rank > 1. This is not the cause. Indeed the rank one SO(3) can also divide the space on which it acts into general and singular ($\neq 0$) layers. (cf. the irreducible linear representation $j > 1$).

(12) This situation is similar to the breaking of rotational invariance by an external magnetic field B in the z direction; indeed the interaction hamiltonian $\underset{\sim}{B} \cdot \underset{\sim}{\mu}$ is a scalar for the rotation group.

VII. APPENDIX

We will publish in the Annales de I'Institut Henri Poincare a more detailed account of the geometry of the Octet. We will prove there the following formulae:

For every $a, x \in \mathcal{E}_8$ $\quad x_\wedge(x_\vee a) = x_\vee(x_\wedge a)$

$$= \frac{1}{2}\ (x_\vee x)_\wedge a \quad , \tag{21}$$

$$3x_\vee(x_\vee a) - x_\wedge(x_\wedge a) = a(x,x) \tag{22}$$

and for every charge q

$$3\ \gamma(q)\ q_\wedge a + 4\ q_\wedge(q_\wedge(q_\wedge a)) = 0 \quad , \tag{23}$$

$$2(q_\wedge)^2\ a = \sqrt{3\gamma\ (q)}\ q_\vee a + 2\ q(q,a) - \gamma(q)\ a. \tag{24}$$

With the help of these formulae, we can discuss an equation more general than (19); that is:

$$\phi y + y_\vee y + a_\vee y + b_\wedge y + c = 0 \quad , \tag{25}$$

where a, b, c $U(2)_z$. If y is a charge, it has to belong to $U(2)_z$. This can be proved by using (14), and operating on the equation with $2(y_\wedge)^2 + \sqrt{3\ \gamma\ (y)}\ y_\vee + 2\ \gamma(y)$ one obtains an equation similar to (19).