II. ELEMENTARY STUDY OF FERMI COUPLINGS.

(In this chapter, we assume that only particle energy momenta are observed. No polarization effect are included).

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1. TRANSITION RATE FOR SPONTANEOUS DECAY

1.1 Relativistic Phase Space :

The differential transition rate $d\lambda$ is proportional to the phasespace volume element for the transition. For one particle, this phase-space volume element is the covariant volume element $d\Omega_m$ of the hyperboloid of mass m.

An easy way to obtain it is through the use of Dirac δ-function. Indeed, the positive mass sheet of the hyperboloid

$$(p_0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = E^2 - \vec{p}^{\,2} = p^2 = m^2 \qquad .. (1)$$

is given by

$$\theta(p)\,\delta(p^2 - m^2) \neq 0 \qquad .. (2)$$

where

$$\theta(p) = 1 \text{ if } p^2 \geq 0 \text{ and } p^0 > 0 \qquad .. (3)$$

$$= 0 \text{ . otherwise}$$

In the unit $\hbar = c = 1$, the tradition is to normalize this covariant hyperboloid volume element to

$$d\Omega_m = \frac{1}{(2\pi)^3} \int 2\,\theta(p)\,\delta(p^2 - m^2)\,d^4p \qquad .. (4)$$

$$= \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{\theta(p)}{\sqrt{\vec{p}^{\,2}+m^2}} \left[\delta\left(p^0 - \sqrt{\vec{p}^{\,2}+m^2}\right) + \delta\left(p^0 + \sqrt{\vec{p}^{\,2}+m^2}\right)\right] dp^0$$

$$\text{i.e. } d\Omega_m = \frac{1}{(2\pi)^3} \frac{d^3\vec{p}}{\sqrt{\vec{p}^{\,2}+m^2}} \quad \text{when } m \geq 0 \qquad .. (4')$$

(we have used, see e.g. Dirac, Foundation of Quantum Mechanics, chap.)

$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a) \qquad .. (5)$$

and

$$\delta(x^2 - a^2) = \frac{1}{2|a|}\left[\delta(x-a) + \delta(x+a)\right] \qquad .. (5')$$

1.2 General Formula.

So for the spontaneous decay of a particle of energy momentum $\underline{P}=(E,\vec{p})$ into n -particles,

$$d\lambda = \mathcal{M}\,(2\pi)^4\,\delta\Big(\underline{P}-\sum_{i=1}^{n}\underline{p}_i\Big)\prod_{i=1}^{n} d\Omega_{m_i} \qquad ..(6)$$

where the invariant coefficient $\mathcal{M}$ is the microscopic transition probability (for given $\underline{P}$ and $\underline{p}_i$'s) and the factor $(2\pi)^4\,\delta(\underline{P}-\sum_{i=1}\underline{p}_i)$ takes care of the energy momentum conservation.

The lifetime is not an invariant, but transforms as the time component of a vector. It is E/τ which is invariant for a decaying particle, so the lifetime of this spontaneous decay is,

$$\frac{1}{\tau} = \frac{1}{E}\int d\lambda = \frac{(2\pi)^{4-3n}}{E}\int \mathcal{M}\,\delta\Big(\underline{P}-\sum_{i=1}^{n}\underline{p}_i\Big)\prod_{i=1}^{n}(\vec{p}_i^{\,2}+m_i^2)^{-1/2}d^3\vec{p}_i \;..(7)$$

The dimension of $\mathcal{M}$ is given by

$$M = M^{-1}\,(\dim \mathcal{M})\,M^{-4}\,M^{2n} \qquad ..(8)$$
$$\dim \mathcal{M} = M^{6-2n}$$

1.3 Two Body Decay.

Let us consider a two body decay : $\underline{P}=\underline{p}_1+\underline{p}_2$. Since $\mathcal{M}$ is an invariant, it is a function of the invariants

$$\underline{p}_1^2 = m_1^2,\quad \underline{p}_2^2 = m_2^2,\quad \underline{p}_1\cdot\underline{p}_2 = \tfrac{1}{2}(\underline{P}^2-\underline{p}_1^2-\underline{p}_2^2) = \tfrac{1}{2}(M^2-m_1^2-m_2^2)$$

In the c.m. (centre of mass) frame

$$\frac{1}{\tau} = \frac{\mathcal{M}}{4\pi^2 M}\int \frac{1}{E_1E_2}\,\delta(M-E_1-E_2)\,\delta(\vec{p}_1+\vec{p}_2)\,d^3\vec{p}_1\,d^3\vec{p}_2 \qquad ..(9)$$

where E_i is a short-hand for $(\vec{p}_i^{\,2}+m_i^2)^{1/2}$. Finally

$$\frac{1}{\tau} = \frac{\mathcal{M}\,p}{\pi M^2} \qquad ..(10)$$

where $\quad p = |\vec{p}_1|_{c.m.} = |\vec{p}_2|_{c.m.} = \dfrac{\sqrt{-\Delta}}{2M}$

For the covariant minded physicist, one can introduot the vector

$$q = \frac{1}{2M^2}\left[(M^2 - m_1^2 + m_2^2)\, p_1 - (M^2 + m_1^2 - m_2^2)\, p_2 \right] \qquad ..(11)$$

it satisfies

$$(p_1 + p_2)\cdot q = P\cdot q = 0 \qquad ..(12)$$

and

$$q_{c.m.} = (0, \vec{q}_{c.m.}) \quad \text{with} \quad \vec{q}_{c.m.} = \vec{p}_{1\,c.m.} = -\vec{p}_{2\,c.m.}$$

Since

$$d^3\vec{p}_1\, d^3\vec{p}_2 = d^3\vec{P}\, d^3\vec{q} \qquad ..(13)$$

(indeed $|D(\vec{P},\vec{q})/D(\vec{p}_1,\vec{p}_2)| = 1$)

$$\frac{1}{\tau} = \frac{\mathcal{M}_0}{4\pi^2}\int_0^\infty \frac{\delta\left(M - \sqrt{\vec{q}^{\,2}+m_1^2} - \sqrt{\vec{q}^{\,2}+m_2^2}\right)}{\sqrt{\vec{q}^{\,2}+m_1^2}\,\sqrt{\vec{q}^{\,2}+m_2^2}}\, 4\pi q^2\, dq$$

and from

$$\delta(f(x))\,dx = \delta(y)\left|\frac{dy}{dx}\right|^{-1} dy \qquad ..(14)$$

where

$$y = f(x)$$

one obtains (10) or in terms of masses

$$\frac{1}{\tau} = \frac{\mathcal{M}_0\sqrt{-\Delta}}{2\pi M^3} \qquad ..(15)$$

where $\Delta = (M + m_1 + m_2)(-M + m_1 + m_2)(M - m_1 + m_2)(M + m_1 - m_2)$.. (16)

Table (4) gives for all measured two body weak decay, the experimental value of the pure number

$$\frac{\mathcal{M}_0}{M^2} = \frac{2\pi M}{\tau\sqrt{-\Delta}} .$$

Table 4

Pure number $\frac{\mathcal{M}}{M^2} = \frac{2\pi b M}{\tau\sqrt{-\Delta}} = \frac{\pi b}{p\tau}$ for two body decays.

$\mathcal{M}$ = microscopic transition rate

M, τ = mass and life time of the initial particle.

$\Delta = (M + m_1 + m_2)(-M + m_1 + m_2)(M - m_1 + m_2)(M + m_1 - m_2)$

$p = \frac{\sqrt{-\Delta}}{2M}$ = the momentum of the secondary particles in centre of mass.

b = branching ratio of decay mode

Decay mode	$\sqrt{-\Delta}$ in $(\text{MeV})^2$	p in MeV	$\mathcal{M} M^{-2}$
$\pi^+ \to \mu^+ + \nu'$	8.32×10^3	29.81	4.13×10^6
$\to e^+ + \nu$	1.95×10^4	69.8	2.42×10^2
$k^+ \to \mu^+ + \nu'$	2.33×10^5	235.65	6.32×10^5
$\to \pi^+ + \pi^0$	2.03×10^5	205.25	3.20×10^5
$k_1^0 \to \pi^+ + \pi^-$	2.05×10^5	206.05	1.05×10^8
$\to \pi^0 + \pi^0$	2.08×10^5	209.05	4.65×10^7
$\Lambda^0 \to p + \pi^-$	2.23×10^5	100.17	8.00×10^7
$\to n + \pi^0$	2.31×10^5	103.64	4.36×10^7
$\Sigma^+ \to p + \pi^0$	4.50×10^5	189.15	1.025×10^8
$\to n + \pi^+$	4.40×10^5	185.09	1.50×10^8
$\Sigma^- \to n + \pi^-$	4.59×10^5	191.77	1.02×10^8
$\Xi^- \to \Lambda^0 + \pi^-$	3.58×10^5	135.81	1.21×10^8
$\Xi^0 \to \Lambda^0 + \pi^0$	3.43×10^5	130.85	6.16×10^7

Note that $\mathcal{M} M^{-2}$ is resonably constant within a factor of 2 for hyperon, and k^0 decays. It is lower for k^+ and π^+ decay. This will have to be explained.

1.4 Three Body Decay.

The energy momentum conservation is given by $\underline{P} = \underline{p}_1 + \underline{p}_2 + \underline{p}_3$. Since we do not observe polarization, the dimensionless invariant $\mathcal{M}$ is function of the masses and of three linearly dependent invariants, that, for obvious reasons, we denote by E_1, E_2, E_3.

They satisfy

$$M = E_1' + E_2' + E_3' \qquad ..(17)$$

$$2ME_i' = 2\underline{P}\cdot\underline{p}_i = M^2 + m_i^2 - m_j^2 - m_k^2 - 2\underline{p}_j\cdot\underline{p}_k \qquad ..(17')$$

where i, j, k are a permentation of 1, 2, 3.

Then from (7)

$$\frac{1}{\tau} = \frac{1}{(2\pi)^5 M}\int \mathcal{M}\, \frac{d^3\vec{p}_1}{E_1}\cdot\frac{d^3\vec{p}_2}{E_2}\cdot\frac{d^3\vec{p}_3}{E_3}\,\delta(M-E_1-E_2-E_3)\times\cdots$$

$$\times\,\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3).$$

$$= \frac{1}{(2\pi)^5 M}\int \mathcal{M}\,\frac{d^3\vec{p}_1\, d^3\vec{p}_2}{E_1 E_2 E_3}\,\delta(M-E_1-E_2-E_3) \qquad (18).$$

Where E_3 is a short hand for

$$E_3 = (\vec{p}_1^{\,2} + \vec{p}_2^{\,2} + p_1p_2\cos\theta + m_3^2)^{1/2} \qquad ..(18')$$

with

$$\vec{p}_1\cdot\vec{p}_2 = p_1p_2\cos\theta.$$

We have

$$d^3\vec{p} = d\phi\, d(\cos\theta)\, p^2 dp = d\phi\, d(\cos\theta)\, pE\,dE$$

So

$$\frac{1}{\tau} = \frac{1}{(2\pi)^5 M}\int \mathcal{M}\,\frac{4\pi p_1 dE_1}{E_3}\, 2\pi\, d(\cos\theta)\, p_2\, dE_2 \times \qquad ..(18'')$$

$$\times\,\delta(M-E_1-E_2-E_3)$$

Using (14) with $x = \cos\theta$, $y = E_3$, we get

$$\frac{1}{\tau} = \frac{1}{\ldots}\left(\mathcal{M}(E_1, E_2, E_3)\right) dE_1\, dE_2\, dE_3\, \delta(M-E_1-E_2-E_3). \qquad ..(19)$$

To my knowledge, this elegant formula was first written by O. Kofoed-Hansen, Phil.Mag. 42, 1412 (1951). The dependence of $\mathcal{M}$ on E_1, E_2, E_3 only is essential. For weak decay, $\mathcal{M}$ is proportional to g^2 and $\mathcal{M}g^{-2}$ has dimension M^4.

1.5 The Invariant $\mathcal{M}$ for Fermi Coupling.

For a Fermi coupling, the four particles are on the same footing. So the simplest possible form for $\mathcal{M}g^{-2}$ of dimension M^4 is

$$\begin{aligned}\mathcal{M}g^{-2} = {} & A(\underline{p}_1\cdot\underline{p}_2)(\underline{p}_3\cdot\underline{p}_4) + A'(\underline{p}_1\cdot\underline{p}_3)(\underline{p}_2\cdot\underline{p}_4) \\ & + A''(\underline{p}_1\cdot\underline{p}_4)(\underline{p}_2\cdot\underline{p}_3) + B_1(m_1m_2)(\underline{p}_3\cdot\underline{p}_4) \\ & + B_1'(m_1m_3)(\underline{p}_2\cdot\underline{p}_4) + B_1''(m_1m_4)(\underline{p}_2\cdot\underline{p}_3) \\ & + B_2(\underline{p}_1\cdot\underline{p}_2)(m_3m_4) + B_2'(\underline{p}_1\cdot\underline{p}_3)(m_2m_4) \\ & + B_2''(\underline{p}_1\cdot\underline{p}_4)(m_2m_3) + C\,m_1m_2m_3m_4 .\end{aligned} \tag{20}$$

The ten constants A,B,C are pure numbers which depends on the detailed nature of the Fermi interaction.

As we shall see this formula is exact at the first order of perturbation in quantum field theory. Its multilinearity in $\underline{p}_i$ or m_i just expresses that the Fermi interaction is "punctual" (i.e. a product of 4 fields at the same point). So each particle is emitted in S-state, which, for Dirac particle means S-wave for large component (hence the constant m) or P-wave for small component (hence the term linear in $\underline{p}$). In V the value of g^2A, g^2A', ..., g^2C will be given in turms of the different possible punctual four fermion interactions. To make such a comparison, we assume that $\mathcal{M}$ represents a probability transition with all spin-summations done. Hence one has to divide by 2 for the decay of unpolarized particle and more generally by 2^l for spin average over the l initial particles. What are the "crossing relations" for $\mathcal{M}$? They are obtained by the following rules.

The m in formula (20) should be regarded as the mass for emitted particle, and the opposite of the mass for absorbed particles. This will be proved in V. (see also L. Michel, Proc. Royal.Soc.Lond. 63, 514, 1950). Restrictions on the A, B, C, are given by the condition

$$\mathcal{M}_0 \geq 0$$

for $p_i^2 = m_i^2 \geq 0$. Question ! Compute them .

Partial answer : Necessary conditions are

$$A + A' \geq 0 , \quad A + A'' \geq 0 ; \quad A' + A'' \geq 0 .$$

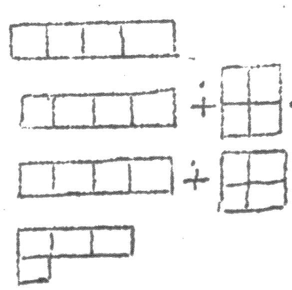
The order in which we write the particle in (20) is arbitrary. To change this order, is equivalent to a relabelling of the A, B, . . constants, the new constants being linear combination of the old one while C is left invariant. In plain words the A_0' , the B_0' and C form the basis of three linear representation of the permutation group of four objects. We leave to group addicts to compute equation (21) (It is extremely simple : just for a permutation in each class compute the number of invariant coefficients, this is the corresponding character . Then use the character table (table 5) given in the appendix of this chapter.) We write (21) for the sake of the readers of young diagram hieroglyphs

C is invariant : representation

A, A', A'' form the basis of the representation ..(21)

B_+, B_+', B_+'' " " "

B_-, B_-', B_-'' " " "



where $B_\pm = \frac{1}{2}(B_1 \pm B_2)$

There is a remarkable interaction that which yields results covariant for every permutation :

$$A = A' = A'' , \quad B_+ = B_+' = B_+'' , \quad B_- = B_-' = B_-'' = 0 \qquad ..(22)$$

Such interaction correspond to the particular Fermi interaction proposed

by C.L. Critchfield and E.P. Wigner, Phys. Rev. 60, 414 (1941)
(see also C.L. Critchfield, Phys. Rev. 63 , 417 (1943).

2. The μ -decay : $\mu \rightarrow e + \nu + \nu'$

The life time is given by (19). Since there are two neutrinos ($m_\nu = m_{\nu'} = 0$), a natural change of variable, symmetrical for the two neutrinos is

$$E' + E'' = v, \quad E' - E'' = u$$

From $\left| \frac{D(u,v)}{D(E',E'')} \right| = 2$ and the triangular relation from momentum conservation in the c.m. frame yields

$$|u| = |E' - E''| \leq p = (E^2 - m^2)^{1/2}$$

so the life time is given by

$$\frac{1}{\tau} = \int P(E)\, dE \qquad ..(23)$$

where $P(E)$, the electron energy spectrum is given by

$$P(E) = \frac{1}{8\pi^3 \mu} \int_{-p}^{+p} \mathcal{M}\, du. \qquad ..(24)$$

μ = mass of μ-meson

A good approximation (about 1%) is to neglect m_e/E_e compared to 1, so $\mathcal{M}$ reduces to (from (20) and (17) and the factor 1/2 for spin-average over the initial μ-particle):

$$\mathcal{M} = \frac{g^2 \mu^2}{4} \left[A E(\mu - 2E) + A' E'(\mu - 2E') + A'' E''(\mu - 2E'') \right]. \qquad (25)$$

The lifetime depends only on the symmetrical part of $\mathcal{M}$, so it depends on the A' only through Q_μ, where

$$Q_\mu = A + A' + A''. \qquad ..(26)$$

In the same way, the electron energy spectrum will depend only on the even part of $\mathcal{M}$ by exchange of E' and E'' (i.e. the even terms in u). Hence $P(E)$ depends only on the parameter Q_μ and another parameter which is traditionally chosen by

$$A_2 + A_3 = \frac{4}{3}\rho Q_\mu \qquad ..(27)$$

Hence

$$P(E) = \frac{g^2 \mu Q_\mu}{32\pi^3}\int_{-E}^{+E}\Big[E(\mu-2E) + \frac{2}{3}\rho(\mu E - E^2 - u^2) + \frac{A''-A'}{2}u(\mu-2E)\Big]du$$

or

$$P(E) \doteq \frac{g^2\mu Q_\mu E^2}{8\pi^3}\Big[\Big(\frac{\mu}{2}-E\Big) + \frac{2}{9}\rho\Big(4E - \frac{3\mu}{2}\Big)\Big] \qquad (28)$$

and

$$\frac{1}{\tau} = \int_0^{\mu/2} P(E)\,dE = \frac{g^2\mu^5 Q_\mu}{2^9 \cdot 3\pi^3} = \frac{G^2\mu^5}{192\pi^3}(1-0\cdot0044) \qquad ..(29)$$

where $0\cdot0044$ is the centre of mass correction.

In equation (28) $P(E) \geqq 0$, $0 \leqq E \leqq \mu/2$ imply

$$0 \leqq \rho \leqq 3/2 \qquad ..(28')$$

Equation (28) is a good approximation of the experimental spectrum for $\rho = 3/4$. This rules out the Critchfield-Wigner interaction ($A_1 = A_2 = A_3$ implies $\rho = 1/2$). The value of $\rho = 3/4$ implies

$$A_1 = 0 \qquad ..(30)$$

Equation (29) yields for the value

$$g Q_\mu^{1/2} = M_p^{-2} \times 2\cdot84 \times 10^{-5} \qquad ..(31)$$

Note that $g^2 Q_\mu^{1/2}$ is invariant by permutation of the order of particles.

3. Neutron Decay and Nuclear β-radioactivity.

The kinetic energy of the final particles in neutron decay is

$$W - m_e = m_N - m_p - m_e = 0 \cdot 783 \ \text{MeV}, \qquad ..(32)$$

It means for the proton a kinetic energy $\leqq \frac{1^2}{2 \times 10^3} < 10^{-3}$ MeV. So we can neglet it.

The microscopic transition probability $\mathcal{M}$ can be written with a good approximation for neutron decay. (M = nucleon mass, order, n, p, e, ν, $m_1 = M = m_2$, $p_1 = p_2 = (M, \vec{0})$; do not forget 1/2 for neutron spin average).

$$\mathcal{M} = \frac{1}{2} g^2 Q_\beta M^2 E_\nu E_e \left(1 + \beta \frac{m_e}{E_e} + \alpha \frac{p_e \cos\theta}{E_e}\right) \qquad ..(33)$$

where θ is the angle between p_e and p_ν and we have used the shorthand

$$Q_\beta = A + A' + A'' - B, \quad Q_\beta \beta = B_2'' - B_1', \quad Q_\beta \alpha = B_1 - A \qquad ..(33')$$

Formulae (18'') and (33) yield for neutron decay,

$$\frac{1}{\tau} = \frac{g^2 Q_\beta}{8\pi^3} \int p_e E_e E_\nu^2 \, dE_e \, dE_\nu \, d(\cos\theta) \, \delta(W - E_e - E_\nu) \times \qquad ..(34)$$
$$\times \left(1 + \beta \frac{m_e}{E_e} + \alpha \frac{p_e \cos\theta}{E_e}\right)$$

The term in α do not contribute to the integral. So the electron energy spectrum is

$$\frac{1}{\tau} = \frac{g^2 Q_\beta}{4\pi^3} \int_{m_e}^{W} (E^2 - m_e^2)^{1/2} E (W - E)^2 \left(1 + \beta \frac{m_e}{E}\right) dE \qquad ..(34')$$

With $\beta = 0$, this spectrum, first computed by Fermi in 1934, described well the electron energy spectrum for neutron decay and many light nuclei $\beta^{\pm}$ decay: the so called allowed decay (For heavier nuclei the coulomb interaction between the electron and the final nucleus cannot be neglected). So we conclude

$$\beta = 0 \qquad \text{.. (35)}$$

The term in β was first computed by Fierz, Z.Phys. 104, 553 (1937) and was generally called Fierz term in the literature.

The term in α has been measured for neutron decay (J.M.Robson, Phys. Rev. 100, 933, 1955)

$$\alpha = 0.09 \pm 0.11 \qquad \text{..(36)}$$

Equation (34) is also valid for the "allowed decay" of light nuclei, but with the parameters Q_β and α changing from nuclei to nuclei. This will be studied in VI.

Let us call

$$f(W) = \int_{m_e}^{W} (E^2 - m^2)^{1/2} E (W-E)^2 dE . \qquad \text{..(37)}$$

(when $m_e \ll W$, an approximation of $f(W)$ is

$$f(W) = \frac{W^5}{30} \qquad \text{..(37')}$$

The quantity $\tau f(W)$ is listed in nuclear physics tables for β-decay (see VI). We use here only the neutron data

$$f(W) = 1.632\, m_e^5 , \quad \tau = (1.013 \pm 0.029) 10^3 \sim 10^3 \text{ sec.}$$

so

$$g Q_\beta^{1/2} = M_p^{-2} \times 4.26 \times 10^{-5} \qquad \text{..(38)}$$

This is just 1.5 the value found in (31) for $g Q_\mu^{1/2}$, but from (33'), we expect Q_β to be different from Q_μ.

4. Leptonic Decay of Hyperon.

Our study of neutron decay can be transposed to:

$\Lambda^0 \longrightarrow p + e^- + \nu$: observed branching ratio $2 \cdot 10^{-3}$ (see table 2).

$\Sigma^+ \longrightarrow n + e^+ + \nu$

$\Sigma^- \longrightarrow n + e^- + \nu$

$\Xi^- \longrightarrow \Lambda + e^- + \nu$: 1 event is observed for 353.

(P.R.L. 10 381, 1963, D.D. Carmony and G.H. Pjerron).

The decay $\Xi^- \longrightarrow n + e^- + \nu$ and $\Xi^0 \longrightarrow p + e^- + \nu$ have not been seen.

Neglecting again the kinetic energy of the nucleon, we predict for the transition rate

$$\frac{1}{\tau} = \frac{g^2 Q_\beta}{4\pi^3} \cdot \frac{W^5}{30} = \frac{(4.26 \times 10^{-5})^2\,(177.15)^5}{6.58 \times 10^{-22}\, 4\pi^3\,(938.21)^4} = 5 \times 10^9 \text{ sec}^{-1}.$$

This is about 600 times too large.

Some decays $\Lambda^0 \rightarrow p + \mu^- + \nu'$ and also two cases of $\Sigma \rightarrow n + \mu + \nu'$ have been observed. Physicists do believe in the universality of Fermi interactions of leptonic decays of baryon, but decays which change strangeness are slower, so the universality must have some not yet understood sophistication.

Some Σ-decays are expected without change of strangeness: This will be the case of

$$\Sigma^\pm \longrightarrow \Lambda^0 + e^\pm + \nu.$$

with an expected rate of

$$\frac{1}{\tau} = \frac{g^2 Q_\beta}{4\pi^3} \frac{W^5}{30} = 5 \times 10^8 \ \text{sec}^{-1} \text{ for } \Sigma^+$$
$$7.6 \times 10^8 \ \text{sec}^{-1} \text{ for } \Sigma^-$$

Since $W = 74.0$ MeV for Σ^+ and 80.6 for Σ^-.

It seems also that these decays are slower. Their experimental study may be made now and they probably will teach us more on the fundamental nature of the phenomenological Fermi interaction.

5. PLAYING WITH THE GOLDEN RULES.

5.1 Nuclear β-decay.

It is very useful for us, physicists to be able to compute orders of magnitude as we did it in 1.4 by dimensional analysis. We may also want crude but a trifle more precise techniques to get results within a factor 2 or 3. The majority of books and lecture notes use for the transition rate

$$\frac{1}{\tau} = 2\pi \, |H|^2 \rho_E \qquad ..(39)$$

where ρ_E, for n final particles is

$$\rho_E = \frac{d}{dE} \prod_{i=1}^{n-1} \frac{d^3\vec{p}_i}{(2\pi)^3} = \frac{1}{(2\pi)^{3(n-1)}} \int \delta\left(E - \sum_{i=1}^{n} E_i\right) \delta\left(\vec{P} - \sum_{i=1}^{n} \vec{p}_i\right) \times \prod_{i=1}^{n} d^3\vec{p}_i \qquad ..(40)$$

With the relation

$$\mathcal{M} = |H|^2 E_0 E_1 E_2 \cdots \cdots E_n \qquad ..(41)$$

Formula (39) is nothing else than equation (7), when explicit relativistic covariance has been destroyed by the decomposition in factors

$|H|^2$ and ρ_E. However, many physicists still prefer to use the interaction Hamiltonian matrix element H. For instance, for the nuclear β-decay Fermi interaction

$$H = g\int \psi_n \psi_p \psi_e \psi_\nu \, d^3\vec{x} \,, \qquad ..(42)$$

Let us do very rough but astonishingly efficient computation. We quantize wave functions ψ in a large convenient box (the lab., the solar system, what you like); of volume V; so

$$\int_V |\psi|^2 d^3\vec{x} = 1 \qquad ..(43)$$

For quantized plane waves for instance $|\psi|^2 = V^{-1}$. Plane waves are even unnecessarily . for us. The simplest wave function which satisfies (43) is

$$\psi = V^{-1/2} \qquad ..(44)$$

The dimension of g is given by

$$M = L^{-1} = \dim g \times V^{-3/2 \times 4} L^3$$

so $\dim g = L^2 = M^{-2}$ as we know.

For nuclear β-decay, the neutron or proton are inside the nucleus of volume V_N, so their wave function is

$$\psi_n = \psi_p = V_N^{-1/2} \quad \text{inside the nucleus}$$
$$= 0 \quad , \quad \text{outside the nucleus.}$$

So the matrix element of the interaction hamiltonian (42) is for a β-decay

$$H = g\int_{V_N} V_N^{-1} V_e^{-1/2} V_\nu^{-1/2} d^3\vec{x} = \frac{g}{\sqrt{V_e V_\nu}}$$

The phase space volume element is $(2\pi)^{-3} V d^3\vec{p}$. So we obtain immediately for a nuclear β-decay

$$\frac{1}{\tau} = 2\pi \int \frac{g^2}{V_e V_\nu} \frac{V_e d^3\vec{p}_e}{(2\pi)^3} \frac{V_\nu d^3\vec{p}_\nu}{(2\pi)^3 dE}$$

$$= \frac{g^2}{(2\pi)^3} \int p_e E_e (W - E_e)^2 dE_e$$

since $\quad p_\nu = E_\nu = W - E_e$.

This is just equation (34') when $Q_\beta = \frac{1}{2}$ and $\beta = 0$, (as we have seen actually $\beta = 0$).

5.2 <u>Electron Capture.</u>

In 1934, shortly after Fermi's paper, his student Wick predicted the capture of an atomic electron by a proton of the nucleus :

$$e^- + p \longrightarrow n + \nu$$

The most probable captive is that of a K electron. Indeed the nucleus is small compared to the Bohr radius of electron.

r = radius of nucleus $\sim 1.4 \times 10^{-13} \sim \frac{\alpha}{2} \frac{\hbar}{m_e c} A^{1/3}$

R = Bohr radius $= \frac{1}{2\alpha} \frac{\hbar}{mc} = \frac{a}{Z}$

where $\quad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$, the fine structure constant;

so $\quad \frac{r}{R} \sim \alpha^2 Z A^{-1/3} << 1$ ..(45)

It is a good approximation to take the wave function of the electron inside the nucleus to be constant and equal to $\psi(0)$. In other words, the capture probability of electron with orbital momentum $l > 0$ (p, f, . . . electron) is negligible.

For s electrons the wave function at the origin is

for K- electrons : $n = 1 \quad \psi_e = \frac{1}{\sqrt{2\pi}} R^{-3/2} = \frac{1}{\sqrt{2\pi}} \left(\frac{Z}{a}\right)^{3/2} = \frac{(Z \alpha m_e)^{3/2}}{\sqrt{2\pi}}$

One obtains the right order of magnitude by the relation

$$\psi^2 \frac{4}{3} \pi n^3 R^3 = 1.$$

Since the captured electrons are K-electrons, this phenomena is generally called K-capture. Then

$$H = g\int \psi_n \psi_p \psi_e \psi_\nu \, d^3\vec{x} = \frac{g}{\sqrt{2\pi}} (Z\alpha m_e)^{3/2} \frac{1}{\sqrt{V_\nu}}$$

..(46)

and

$$\frac{1}{\tau} = 2\pi |H|^2 \frac{p_\nu^2 4\pi V_\nu}{(2\pi)^3} \times 2 = \frac{1}{\pi^2} g^2 (Z\alpha)^3 m_e^3 p_\nu^2 .$$

The last factor 2 accounts for the two electrons in K-shell.

We have $p_\nu = W - \frac{1}{2} m_e (Z\alpha)^2 = W -$ (the binding energy of the electron).

If $W > m_e$, β^+ decay competes with K-capture. Since for large W its rate is proportional to W^5, β^+ decay is the principal transition mode when W is large enough.

5.3 μ-Capture.

μ-capture is quite similar to electron capture. The Bohr orbit of the μ is $(Z\alpha\mu)^{-1}$, i.e. $\frac{\mu}{m_e}$ times smaller than that of electrons. So μ is not disturbed by the atomic electrons. So in equation (46), we have to replace $(Z\alpha m_e)^3$ by $(Z\alpha\mu)^3$. Since there is only one μ^-, we devide by 2. Here $p_\nu = \mu\left(1 - \frac{(Z\alpha)^2}{2}\right) - E_N$, where E_N is the energy taken by the nucleus. Here is the main difference with electron capture. The proton which absorbs the μ^- by $p^+ + \mu^- \to n + \nu$ would receive a kinetic energy of about $\frac{p^2}{2M} = \frac{(100)^2}{2 \cdot 10^3} \sim 5$ MeV. However, from the motion of p^+ inside the nucleus this energy can go up to 20 MeV. So every proton in the nucleus can receive enough energy to go to an empty nuetron state (generally a free neutron state). So we have to put nearly an additional factor Z and replace p_ν by μ nearly (we multiply by 1/2 for taking account of all these nearly's).

We then have obtained the rate formula

$$\frac{1}{\tau_{capture}} = \frac{g^2}{4\pi^2} Z^4 \alpha^3 \mu^5 \qquad ..(47)$$

As we shall see, this is a good approximation of Z if it is not too large. When $Z > 20$, the ratio : μ-Bohr orbit/nuclear radius is not large enough, so it is essential to take in account the size of the nucleus. Equations (29) and (46) yield the ratio,

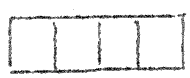
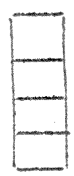
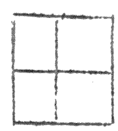
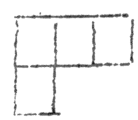
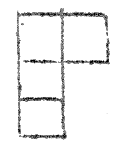
$$\frac{\tau_{\mu\,decay}}{\tau_{\mu\,capture}} = Z^4 \alpha^3 \pi \times 3 \times 2^7 Q_\mu^{-1} \sim \left(\frac{Z}{10}\right)^4 \text{ for } Q_\mu \sim 5.$$

This is well verified for light nuclei.

The equality of the Fermi coupling constants g for nuclear β-decay, μ-decay and μ-capture was emphasized first and independently by J. Tiomno and J.A. Wheeler, Pocono conference, Summer 1948, published in Rev. Mod. Phys. 21, 153 (1949) and Puppi N.Cim 5, 587 (1948) and 6, 194, (1949). See also T.D. Lee, N. Rosenbluth and Yang Phys. Rev. 79, 905 (1949).

Characters of the irreducible linear representations
of the four object permutation group

TABLE - 5

Class	No. of observables	Irreducible representations				
						
1111	1	1	1	2	3	3
211	6	1	- 1	0	1	- 1
22	3	1	1	2	- 1	- 1
31	8	1	1	- 1	0	0
4	6	1	- 1	0	-1	1