*Representations of the algebra of integrated contracted current commutators**

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Abstract

The use of the commutation relations of "field algebra" implies relinquishing the proposed identification of the *SU* (6) algebra (and *SU* $(6)_W$) with space integrals of current components. The latter now form an inhomogeneous algebra, whose representations we study in order to check whether they might still display a similarity to the *SU* (6) classification of the spectrum. The results are negative.

THREE ROLES FOR THE CHIRAL ALGEBRA

In non-relativistic atomic physics, the vector space of the energy-degenerate bound states of hydrogen is the carrier-space of an irreducible representation of an *SO* (4) Lie algebra generated by conserved observables (the components of angular momentum **L** and of the normalized Laplace–Lentz

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operators **A**). The Hamiltonian operator H is a scalar functional of these algebraic generators. We thus have three roles fulfilled by the $SO(4)$ physical algebra of the **L** and **A**:

a) They represent symmetries of the system since

$$[H, \mathbf{L}] = 0, \quad [H, \mathbf{A}] = 0. \tag{1}$$

can be used to generate by exponentiation a group of transformations leaving H invariant. In practice, this feature can sometimes be used to *observe* the algebraic generators indirectly, since they will impose selection rules on the energetically allowed transitions.

b) They appear as the dynamical variables of the theory, with H as a functional $H(\mathbf{L}, \mathbf{A})$:

$$-(2H)^{-1} = \mathbf{L}^2 + \mathbf{A}^2 + 1. \tag{2}$$

c) They can be observed directly in measurements using inertial or gravitational properties. This feature is due to the fact that energy–momentum and angular momentum components are related to the components of $\theta^{\mu\nu}$, the energy–momentum tensor, which is the source of the gravitational interaction.

The algebra of chiral $SU(3) \times SU(3)$ generates an approximate symmetry of the strong interactions, in analogy with (a). Note that the diagonal $SU(3)$ subalgebra ("plain" even parity $SU(3)$) can be extended to include in addition the law of baryon number conservation. Role (b) is as yet unclear, though an attempt has been made by Sugawara to embody it in a prescription for $\theta^{\mu\nu}$, written as a functional of current densities. To a limited extent, this is also achieved by writing a chiral-symmetric Lagrangian density as a functional of meson fields, which are assumed to be proportional to some current densities or to their divergences (though one is then faced in addition with the kinetic energy terms!).

Gell-Mann's approach emphasized role (c) for the chiral currents. Here it is the weak and electromagnetic interactions which allow us to observe directly the charge and current densities. In this approach the vector densities $v_i^\mu(x)$ and the axial vector densities $a_i^\mu(x)$ are defined through the observation of their matrix elements between various hadron states in these non strong transitions ($i = 1 \cdots 8$ are the unitary spin indices, with $i = 0$ by extension for the unitary scalar case: $\mu = 0, 1, 2, 3$ is the space–time index and x is a four vector). Their space integrals

$$V_i = \int d^3x\, v_i^0(x), \quad A_i = \int d^3x\, a_i^0(x)$$

obey $SU(3) \times SU(3)$ commutation relations

$$[V_i, V_j] = if_{ijk}V_k, \quad [V_i, A_j] = if_{ijk}A_k, \quad [A_i, A_j] = if_{ijk}V_k. \tag{3}$$

The free quark field model was introduced so as to provide an explicit prescription for the equal-time commutators between the densities, so that they should yield (3) upon integration (for $x_0 = y_0$):

$$\begin{aligned}
&[v_i^0(x), v_j^0(y)] = [a_i^0(x), a_j^0(y)] = if_{ijk}v_k^0(x)\,\delta^3(\mathbf{x}-\mathbf{y}),\\
&[v_i^0(x), a_j^0(y)] = if_{ijk}a_k^0(x)\,\delta^3(\mathbf{x}-\mathbf{y}),\\
&[v_i^0(x), v_j^\sigma(y)] = [a_i^0(x), a_j^\sigma(y)] = if_{ijk}v_k^\sigma(x)\,\delta^3(\mathbf{x}-\mathbf{y}) + \text{Schwinger terms},\\
&[v_i^0(x), a_j^\sigma(y)] = [a_i^0(x), v_j^\sigma(y)] = if_{ijk}\,a_k^\sigma(x)\,\delta^3(\mathbf{x}-\mathbf{y}) + \cdots,\\
&[v_i^\sigma(x), v_j^\tau(y)] = [a_i^\sigma(x), a_j^\tau(y)] = i\varepsilon^{\sigma\tau\varrho}d_{ijk}\,a_k^\varrho(x)\,\delta^3(\mathbf{x}-\mathbf{y})\\
&\qquad\qquad + i\delta^{\sigma\tau}f_{ijk}\,v_k^0(x)\,\delta^3(\mathbf{x}-\mathbf{y}) + \cdots,\\
&[v_i^\sigma(x), a_j^\tau(y)] = i\varepsilon^{\sigma\tau\varrho}\,d_{ijk}\,v_k^\varrho(x)\,\delta^3(\mathbf{x}-\mathbf{y}) + i\delta^{\sigma\tau}f_{ijk}\,a_k^0(x)\,\delta^3(\mathbf{x}-\mathbf{y}) + \cdots,
\end{aligned} \tag{4}$$

where the indices $\sigma, \tau = 1, 2, 3$ of space.

THE ANGULAR MOMENTUM SPECTRUM IN THE QUARK MODEL

The picture becomes muddled when we move on to $SU(6)$ and larger schemes, used to classify the correlated occurrence of unitary (eightfold) and rotational spins in the hadron spectrum. At first, Feynman *et al.*[1] suggested a mechanism which would have provided the generators of $SU(6)$ with a role (c). They identified the algebra of $SU(6)$ with the "diagonal" subalgebra (of even parity) in a system generated by space-integrals of all components of the vector and axial-vector current densities. These densities are assumed to follow the commutation relations corresponding to a fundamental quark field, generating at each point of space–time a chiral $SU(6) \times SU(6)$ with a variety of Schwinger terms in addition. Upon space-integration, we are left with global generators of $[SU(6) \times SU(6)]_{\gamma 5}$.

Feynman *et al.* were hoping to observe the entire $[SU(6) \times SU(6)_{\gamma 5}]$ as an appropriate symmetry of rest states. It then turned out that such a requirement is tantamount to imposing on the rest-symmetry group the condition that it should leave invariant a Lorentz scalar density (e.g., the mass-term in a Lagrangian). In the Dirac indices' space, the Lorentz scalar is represented by β, and this would imply that all odd parity generators be non-compact. The global chiral generators would close on $SL(6, C)$ rather than on $SU(6) \times SU(6)$. Considering that the Adler–Weisberger sum rule provides proof of the compact nature of the axial charges, Dashen *et al.*[2] suggested that role (c) be played by the $[U(6) \times U(6)]_\beta$ system of parity-even integrated currents. This rest-symmetry could then yield a further $SU(6)_W$ vertex-symmetry for collinear processes. With this picture, we thus even have a role (c) for both the Gursey–Radicati and the Lipkin–Meshkov (type (a)) symmetries. Moreover, Gell-Mann[3] could then also provide such an interpretation for the "intrinsic **L**", defined as the difference between angular momentum at rest (spin) and the "intrinsic spin" in $SU(6)$, i.e., the integrated axial vector unitary singlet current components.

Note that the integrated generators suffer from difficulties in the definition of their behavior as Lorentz tensors. We shall not deal with this aspect here, especially since we could simply go over to the more recent *local* treatment, working with the algebra generated in Eq. (4) by the current densities themselves. All the above Lie algebras would now appear for each point of space time (except for the additional complication due to Schwinger terms).

A further development in the interpretation of the hadron spectrum consisted[4] in embedding the above $[U(6) \times U(6)]_\beta$ in a non-compact $U(6,6)$. For the non-compact generators connecting levels of $[U(6) \times U(6)]_\beta$, no role of type (c) could be found. There were only some conjectures connecting them with strong-coupling theory—a role which may be considered to relate in a vague way to (b). Note that a scheme with a compact $SU(12)$ was also considered in the work of Dothan *et al.*[4]; this would provide a type (c) interpretation for a large part of the algebra, and by extension to its entirety.

THE CONTRACTED COMMUTATORS' ALGEBRA

Lee *et al.*[5] have suggested that the current densities behave like canonical boson fields, and thus obey contracted commutation relations (i.e., making the right-hand side of the quark field currents commutation relations in (4) vanish for commutators between space components) for $x^0 = y^0$:

$$\left.\begin{aligned}
&[v_i^0(x), v_j^0(y)] = [a_i^0(x), a_j^0(y)] \\
&[v_i^0(x), a_j^0(y)] \\
&[v_i^0(x), v_j^\sigma(y)] = [a_i^0(x), a_j^\sigma(y)] \\
&[v_i^0(x), a_j^\sigma(y)] = [a_i^0(x), v_j^\sigma(y)]
\end{aligned}\right\} \text{as in (4)}$$

$$[v_i^\sigma(x), v_j^\tau(y)] = [a_i^\sigma(x), a_j^\tau(y)] = [v_i^\sigma(x), a_j^\tau(y)] = 0. \tag{5}$$

Upon space-integration (or for each point), we now have non semi-simple algebras. While the $2n$ ($n = 3$ for $SU(2)$, 8 for $SU(3)$, 9 for $U(3)$) global charges (or time-component of the field $v_i^0(x)$, $a_j^0(x)$) still form a compact algebra $K = [SU(2) \times SU(2)]_{\gamma 5}$, $[SU(3) \times SU(3)]_{\gamma 5}$ or $[U(3) \times U(3)]_{\gamma 5}$, the other $2 \times 3n$ space integrals of the space components ($v_i^\sigma(x)$, $a_j^\tau(x)$) since they all commute, form a $6n$ dimensional abelian Lie algebra = vector space E. The commutation relations of one time component with a space component show that K acts on E by 3 times (one for each value of the space component index σ) its adjoint representation A (respectively 2 triplet, 2 octet, 2 nonet spaces). The total algebra (of dimension $8n$) will be denoted $I_A^3[W]$ (three times inhomogeneous W).

Hence for instance we have $\bar{G}' = I_{8+1+8+1}\,[U(3) \times U(3)]_{\gamma 5}$ for the "field algebra" instead of $[U(6) \times U(6)]_{\gamma 5}$ for the similar current algebra.

We will forget the center of these algebras (i.e., the two elements $v_o^0(x)$ and $a_o^0(x)$) and thus consider the 70 dimensional Lie algebra.

$$\bar{G} = I_{8+1+8+1}^3\,[SU(3) \times SU(3)]_{\gamma 5}. \tag{6}$$

The corresponding $[SU(6) \times SU(6)]_{\gamma 5}$ Lie algebra discussed earlier is a deformation[6] of $\bar{G}$. Although the Lie algebra $\bar{G}$ does not contain SU_6 (its maximal compact subalgebra is $[SU_3 \times SU_3]_{\gamma 5}$), it also can yield an $SU(6)$ multiplet spectrum by the following mechanism:

Let U be an irrep (irreducible unitary linear representation) of the group generated by G (U is infinite dimensional).

One can deform both $\bar{G}$ and U (keeping the maximal compact part $[SU(3) \times SU(3)]_{\gamma 5}$ fixed) such that $\bar{G} \to [SU(6) \times SU(6)]_{\gamma 5}$ and $U \to V$, an infinite dimensional unitary linear representation of the latter group. It is reducible, since $[SU(6) \times SU(6)]_{\gamma 5}$ is compact, and its reduction into a direct sum of irreps yields precisely the $[SU(6) \times SU(6)]_{\gamma 5}$ multiplet spectrum. The papers of reference[7] explain how to compute the number of linearly independent possible deformations and how to realize them.

We can even simplify a little this problem by noting that $\bar{G}$ is the direct product of $G \times G$ where G is the 35 dimensional

$$G = I^3_{8+1}\, SU(3) \tag{7}$$

which can be deformed into $SU(6)$.

Even so, such computation appears to us as a formidable task, and we have preferred another approach which is less direct but much simpler, since it is a straightforward application of some of Mackey's theorems on induced representations.

$SU(3)$ CONTENT OF THE IRREPS OF G: I^3_{8+1}

Since $\bar{G} = G \times G$, the irreps of $\bar{G}$ are tensor products of irreps of G. We study only those of G. This group, considered as the diagonal subgroup of $G \times G$ has a particular physical significance either as the group generated by the algebra of vector field or as the even-parity subgroup of $\bar{G}$, to be compared with $SU(6)$. Since G, defined in (7) is a semi-direct product

$$G = (E_8 + E_1 + E_8 + E_1 + E_8 + E_1) \times SU(3) = T \times SU(3) = T \times K \tag{7'}$$

to study its irreps, we need a list of all orbits of $SU(3)$ on T. Each orbit is determined by a set of 19 invariants.[8] For each point t of an orbit we can define the stabilizer (or little group) K_t. All points of T whose little groups are identical but for a conjugation in $K = SU(3)$ (i.e., $K_{t'} = gK_t g^{-1}, g \in K$) form a stratum. A stratum is the union of all orbits of a "given type" and it is characterized by the little group of any of its points. We call F such a little group. Then all equivalence classes of irreps of G are labelled by[9,10,11]

$$\left.\begin{array}{l} \text{an orbit of } SU(3) \text{ on } T \text{ characterized by its 19 invariants} \\ \text{whose set is denoted } \alpha; \\ \text{an irrep } D_F^{(t)} \text{ of the little group } F \text{ of this orbit.} \end{array}\right\} \tag{8}$$

Such an irrep of G will be denoted $U_G^{\alpha, D_F^{(t)}}$.

If one excepts the trivial case where the three octets are represented trivially and U is then finite dimensional (then $F = SU(3)$), the only possible little groups are[8,12]: 1, $U(1)$, $U(1) \times U(1)$, $U(2)$.

All irreps of the four possible F are well known, so (8) yields the complete list of (equivalent classes of) irreps of G.

The irreps of $SU(3)$ are also well known. They are characterized by two integers p, q and we shall denote them $\mathscr{D}^{(p,q)}$ (its dimension is $\frac{1}{2}(p+1)(q+1) \times \times (p+q+2)$). Consider an irrep $U_G^{\alpha, D_F^{(i)}}$ of G. Its restriction to $SU(3)$: $U_G^{\alpha, D_F^{(i)}}\big|_{SU(3)}$ is reducible:

$$U_G^{\alpha, D_F^{(i)}}\Big|_{SU(3)} = \bigoplus_{(p,q)} C_{(p,q)}^{(i)}\, \mathscr{D}(p, q), \tag{9}$$

where $C_{p,q}^{(i)}$ is the number of times that $\mathscr{D}^{(p,q)}$ appears in this direct sum decomposition of $U_G^{\alpha, D_F^{(i)}}|_{SU(3)}$. Mackey's induction reduction theorem tells us the value of the $C_{p,q}$ for the given α, $D_F^{(i)}$. We will use its easier version, Theorem E′ p. 128 of ref. 10. The closed subgroup inducing $U_G^{\alpha, D_F^{(i)}}$ is the semi-direct product $T \times F$. There is only one double coset $T \times F \cdot G/SU(3)$ since every element of G is a product of an element of $T \times F$ by an element of $SU(3)$.

Note that $T \times F \cap SU(3) = F$, so Mackey's theorem tells us that the $SU(3)$ representation (9) is equivalent to $U_{SU(3)}^{D_F^{(i)}}$, the $SU(3)$ representation induced by the representation $D_F^{(i)}$ of F, as a closed subgroup of $SU(3)$. If we denote

$$\mathscr{D}^{(p,q)}\Big|_F = \bigoplus_j \gamma_j^{(p,q)}\, D_F^{(j)} \tag{10}$$

the direct sum decomposition of $\mathscr{D}^{(p,q)}$ restricted to F, into irreps $D_F^{(i)}$ of F, then Frobenius' reciprocity theorem, quoted for compact groups (as $SU(3)$ and F) p. 35 of ref. 10, tells us that

$$C_{(p,q)}^{(i)} = \gamma_{(i)}^{(p,q)}. \tag{11}$$

The coefficients $\gamma_i^{(p,q)}$ are easy to know. They are given in physical terms in table 1. So we know the content, in irreps of $SU(3)$, of an irrep of G. We can then see if this content can be the same as the one arising from the reduction of some irreps of $SU(6)$ restricted to $SU(3)$. More generally we can see how much the "spectrum" of $SU(3)$ irreps obtained from an irrep of G fits with the physically known $SU(3)$ multiplet structure. This leads us to the following conclusion:

For mesons, we want at least an $SU(3)$ singlet ($\mathscr{D}^{(0,0)}$). This implies that $D_F^{(i)}$ is the trivial representation: the only one for $F = 1$, ${}_{U_1}D^{(0)}$, ${}_{U_1 \times U_1}D^{(0,0)}$, ${}_{U_2}D^{(0,0)}$.

For the baryons there exist at least two octets. The only irrep of F which

TABLE 1

F	Irreps of F	Value of $\gamma_i^{(p,q)}$
1	only one, trivial; $D_1^{(0)}$	${}_{(1)}\gamma_0^{(p,q)} = \dim$ of $\mathscr{D}^{(p,q)}$ $= \frac{1}{2}(p+1)(q+1)(p+q+2)$
$U(1)$	$D_{U_1}^{(m)}$, m integer	${}_{(U_1)}\gamma_m^{(p,q)} = \dim$ of $Y = m$ subspace of $\mathscr{D}^{(p,q)}$ space
$U(1)\times U(1)$	$D_{U_1\times U_1}^{(m,n)}$; m, n integers	${}_{(U_1\times U_1)}\gamma_{(m,n)}^{(p,q)} = \dim$ of $Y = m$, $2I_Z = n$ subspace of $\mathscr{D}^{(p,q)}$ space
$U(2)$	$D_{U(2)}^{(t,m)}$ $2t$, m integers	${}_{U(2)}\gamma_{(t,m)}^{(p,q)} = \dim$ of (I(isospin)) $= t$, $Y = m$ subspace of $\mathscr{D}^{(p,q)}$ space

Value of $\gamma_{(i)}^{(p,q)}$ for all irreps of the four possible F

appear at least twice in the octet $\mathscr{D}^{(1,1)}$ are $D_1^{(0)}$, $D_{U_1}^{(0)}$, $D_{U_1}^{(1)}$ or $D_{U_1}^{(-1)}$ (note that ${}_{U_1}\gamma_{(1)}^{(1,1)} = {}_{U_1}\gamma_{(-1)}^{(1,1)}$), $D_{(U_1\times U_1)}^{(0,0)}$. There are no irreps of $U(2)$ with these properties.

In table 2 we give the content in $SU(3)$ of $U_G^{\alpha, D_F^{(i)}}$ for the different $D_F^{(i)}$ selectioned by the above physical considerations.

We have also put in table 2 the contents, in irreps of $SU(3)$, of the "small" $SU(6)$ irreps in order to make easier the comparison with the $SU(3)$ content of the U_G which have passed the two tests.

TABLE 2

Nonexcluded $\gamma^{(p,q)}$ for mesons (at least one SU (3) singlet $\gamma_i^{(0,0)} \geq 1$)

D_F \ $\mathscr{D}^{(p,q)}$	(0, 0)	(1, 1)	(3, 0)	(0, 3)	(2, 2)		
$F = 1$	1	8	10	10	27	:	dim $\mathscr{D}^{(p,q)}$
$D_{U_1}^{(0)}$	1	4	3	3	9	:	dim $Y = 0$ subspace
$D_{U_1\times U_1}^{(0,0)}$	1	2	1	1	3	:	dim $I_Z = Y = 0$ subspace
$D_{U_2}^{(0,0)}$	1	1	0	0	1	:	dim $\mathbf{I} = Y = 0$ subspace
$SU(6)$ **35**	3	4	0	0	0		
405	6	12	3	3	9		

TABLE 2 (Continued)

Nonexcluded $\gamma^{(p,q)}$ for baryons (at least two SU (3) octets: $\gamma_{(i)}^{(1,1)} \geq 2$)

D_F \ $\mathscr{D}^{(p,q)}$	(0,0)	(1, 1)	(3, 0)	(0, 3)	(2, 2)		
$F = 1$	1	8	10	10	27	:	dim $\mathscr{D}^{(p,q)}$
$D_{U_1}^{(0)}$	1	4	3	3	9	:	dim $Y = 0$ subspace
$D_{U_1}^{(1)}$	0	2	4	2	6	:	dim $Y = 1$ subspace
$D_{U_1 \times U_1}^{(0,0)}$	1	2	1	1	3	:	dim $I_Z = Y = 0$ subspace
SU (6) **56**	0	2	4				
70	2	6	2				
20	4	2					
700	0	6	12	2	6	etc.	
1134	6	24	18	6	18	etc.	

CONCLUSIONS

The representation $D_{U_1}^{(1)}$ is the only one which ensures that all $\gamma_1^{(p,q)}$ be even, as required for baryons in an $SU(6)$ like classification.[13] It starts out with a $\frac{1}{2}^+$ **8** and $\frac{3}{2}^+$ **10**, then going on to $\frac{1}{2}^+$ $\overline{\mathbf{10}}$, $\frac{5}{2}^+$ **27** etc. It does accommodate the equivalent of the **56** of $SU(6)$ and is thus satisfactory. The appearance of higher irreps of $SU(3)$ such as the 27 etc. is a general feature, which occurs also for the candidate spectrum generating algebras in which $SU(6)$ has sometimes been embedded, such as the non-compact $U(6,6)$ or $SL(6,C)$. The $SU(6)$ classification has however the advantage of being alternatively utilizable in conjunction with intrinsic L assignments, possibly with the L embedded in $SL(2,C)$ or $SL(3,R)$. Such a scheme does not involve higher $SU(3)$ irreps; this may be an essential point, considering that the evidence for states such as the Z^+ baryon (with $Y = 2$) is rather flimsy to-date.

The meson 0^- and 1^- octets usually assigned to the $SU(6)$ **35** can be accommodated in $D_{U_1}^{(0)}$.

The key weakness of all these $I_{8+1}^3 SU(3)$ irreps is the impossibility of having more than one $SU(3)$ singlet per irrep, so that no single irrep can contain $Y^*(1405)$ or $Y^*(1520)$ for baryons ($\frac{1}{2}^-$ and $\frac{3}{2}^-$ respectively) or the ϕ-ω unitary singlet mixture (1^-), the similar $f^0 - f^{0\prime}$ (2^+) etc. The components of unitary singlet states are completely uncoupled.

Our result is thus in the nature of a no-go theorem, because of this last feature.

As a conclusion, the commutation relations should not be contracted and the "field algebra" ansatz given up if one is to conserve the connection with the spectrum. Alternatively, if the contracted commutators be kept, the generators of G will have very little to do with the observed clustering in the structure of the hadron spectrum.

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Footnotes and references

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8 See for instance L. Michel and L. Radicati "*Symmetry principles at high energy*", *Fifth Coral Gables Conference*, p.19 (Benjamin New York 1968) and unpublished notes on "The geometry of the $SU(3)$, octet space", § III.7. The three scalars yield 3 invariants and the 3 octet vector $(3 - 1) \times 8 = 16$ invariants.
9 The pioneer work of E.P. Wigner for the Poincaré group: *Ann. Math.* **40**, 149 (1939) can be applied to the semi-direct product we study here. This method is a generalization of a method invented by Frobenius for finite groups. Another (larger) generalization is the C.W. Mackey theory of induced group representation, *Ann. J. of Math.* **73**, 576 (1951); *Ann. Math.* **55**, 101 (1952) and **58**, 193 (1953). G. Mackey has exposed his theory for physicists in ref. 10 and we shall from now on quote from pages of this book.
10 G.W. Mackey *Induced Representations of Groups and Quantum Mechanics* (Benjamin New York 1968 and Boringhieri Torino 1968).
11 To be strict, we should have spoken of the dual group $\hat{T}$ of T and of the action of $SU(3)$ on $\hat{T}$. But here $\hat{T}$ and T are isomorphic and the $SU(3)$ action on both is identical. That this method yield all irreps of G is a consequence of Theorem B, p.43 of ref. 10.

12 The action of $SU(3)$ on the octet space is that of $SU(3)$ on its Lie algebra. Outside of the origin (a fixed point) there are two types of orbits; 1) the general type of orbits of regular elements of the Lie algebra; the corresponding little group is $U(1)\times U(1)$: it is a Cartan subgroup of $SU(3)$ = maximal abelian subgroup and they are all conjugated; 2) the orbit of exceptional elements (called "charges" or "q-vectors" in ref. 8) have for little group $U(2)$ (all conjugated). For the set of three vectors of three octets, the little group F is the intersection of the 3 little groups of the vectors. In the general case it is $F = 1$. Other cases are a) if the three vectors are collinear to a q vector, $F = U(2)$; b) the three vectors commute (for the Lie algebra law) but case a) is excluded, then $F = U(1)\times U(1)$. In the last case c) the three vectors are three q-vectors and either two are collinear and do not commute with the third one, or there are three linearly independent q vectors forming two commuting pairs and one non-commuting pair. Then $F = U_1$ and is the one parameter subgroup generated by a q-vector. All such U_1 are conjugated. (Note that in the traditional Gell-Mann basis only γ_8 is a q-vector.)

13 Indeed, for $Y = 1$, $T_3 = Q - Y/2$ is half-integer; so the number of states $Y = 1$ in an $SU(3)/Z_3$ multiplet is even.