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An Extension of Voronoï's Theorem on Primitive Parallelotopes†

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Voronoï proved that any parallelotope that is the prototile of a face-to-face primitive tiling is the affine image of a Voronoï cell. We prove that the affinity is unique (up to an orthogonal transformation). Using a theorem of Zhitomirskii, this result can be extended to parallelotopes with maximal numbers of faces.

1. INTRODUCTION

G. Voronoï, the 125th anniversary of whose birth was celebrated in 1993, would surely be surprised by the rapid growth that the subject of Voronoï tilings, or Voronoï diagrams, has undergone in recent years. This is due not only to the increasing number of applications that mathematicians, scientists and scholars in many fields have found for them, but also because they are computationally tractable, and have many interesting combinatorial properties. Perhaps Voronoï would also be surprised to know that mathematicians continue to study and learn from his great work, 'Nouvelles applications des paramètres continus à la théorie des formes quadratiques' [5], published in two parts (the second posthumously) in 1908 and 1909. In this memoir, Voronoï studied in great detail the subtle bijection between n-ary quadratic forms and their associated tilings of n-dimensional euclidean space $\mathscr{E}^n$. Voronoï conjectured that the tiles of these tilings—which, following Delone, we call Voronoï cells—are representative of all tiles that fill space by translations; he proved this conjecture for the generic class of 'primitive' tilings (defined below). The present note is a postscript to this famous result.

Recall that, if $S \subset \mathscr{E}^n$ is any discrete set of points, the Voronoï cell $D(s)$ of $s \in S$ is the set of points of $\mathscr{E}^n$ that are at least as close to s as to any other point of S. (In this note, to say that S is discrete means that there is a minimum distance between its points.) The interiors of the Voronoï cells of distinct points have empty intersection, and the union of all of them is the whole of $\mathscr{E}^n$. A Voronoï tiling is, by definition, face-to-face.

A tiling is said to be convex if its tiles are convex; if the tiles are also compact and are translates of one another, they are called *parallelotopes.* A lattice tiling is one the translations of which form a lattice. Minkowski proved that every lattice parallelotope and its $(n-1)$-dimensional faces are centrosymmetric [3]. Voronoï tilings induced by lattices are convex lattice tilings with the special additional property that the line segments from o to the face centers of $D(o)$ are orthogonal to the faces. An n-dimensional face-to-face convex tiling, and its tiles, are said to be *primitive* if exactly $n+1$ tiles meet at every vertex. Voronoï's theorem ([5, p. 273]) states the following:

THEOREM 1 (*Voronoï*). *Any parallelotope that is the prototile of a face-to-face primitive tiling is the affine image of a Voronoï cell.*

† This paper is dedicated to the memory of G. Voronoï (1868–1908).

In fact, we need not explicitly assume that the tiling is face-to-face, since Venkov showed that every parallelotope admits a face-to-face tiling [4].

Voronoï's conjecture that every parallelotope in $\mathscr{E}^n$ is the affine image of a Voronoï cell was proved for $n \leq 4$ by Delone [1]; for $n > 4$ the question remains open.

Later, Zhitomirskii [6] generalized Theorem 1; we postpone a discussion of his work until Section 4. In this brief note we extend the theorems of Voronoï and Zhitomirskii, both of which assert existence, by proving that the affinity is unique.

2. Affine Symmetry of Voronoï Cells

We begin with point sets S that are more general than lattices: we will only require that S be discrete. As above, we define the Voronoï cell $D_S(o)$ (or $D(o)$, if S is understood) of a point $o \in S$ to be

$$D(o) = \{y \in \mathscr{E}^n \mid \forall u \in S, N(y-o) \leq N(y-u)\}, \tag{1}$$

where $N(x-o)$ is the squared length of the segment ox. From now on we choose o to be the origin of $\mathscr{E}^n$ and denote $x - o$ by $\vec{x}$.

$D(o)$ is smallest convex polytope bounded by the orthogonal bisectors of the vectors joining o to the other points of S. We say that $\vec{f}$ is a *face vector* of $D(o)$ if one of the faces of $D(o)$ lies in its orthogonal bisector; note that this bisector is the hyperplane

$$(\vec{f}, \vec{x}) = N(\vec{f})/2. \tag{2}$$

We denote the set of all face vectors of $D(o)$ by F_o. $D(o)$ is compact iff there are face vectors on both sides of every hyperplane through o.

Proposition 1. *If $D(o)$ is compact, then $|F_o| < \infty$.*

Proof. If $D(o)$ is compact, then $2D(o)$ is contained in a closed sphere $\sigma(r)$ of finite radius; by construction, the face vectors of $D(o)$ also lie in this sphere. Thus $F_o \subseteq (S \cap \sigma(r))$. Since S is discrete, $|\sigma(r) \cap S| < \infty$. □

Definition 1. If F_o is a disjoint union of subsets spanning mutually orthogonal subspaces, F_o is said to be reducible. If those subspaces are of dimension 1, F_o is completely reducible. If F_o is not reducible, it is irreducible.

Definition 2. We will say that a volume-preserving affine transformation ψ is an affine symmetry of $D(o)$ if it maps $D(o)$ to itself and F_o to F_o.

Such a ψ need not preserve lengths. Since o is fixed, we can represent ψ by a matrix $A \in SL_n(R)$ with respect to an orthonormal basis for $\mathscr{E}^n$ with origin o.

For each $\vec{f} \in F_o$, we have $A\vec{f} \in F_o$, and there is a constant $c_f > 0$ such that

$$N(A\vec{f}) = c_f N(\vec{f}). \tag{3}$$

By Definition 2, if $(\vec{x}, \vec{f}) = \alpha N(\vec{f})$ then $(A\vec{x}, A\vec{f}) = \alpha N(A\vec{f})$, since ψ preserves the orthogonality of a face and its corresponding face vector. Thus

$$(A\vec{f}, A\vec{x}) = \alpha N(A\vec{f}) = \alpha c_f N(\vec{f}) = c_f(\vec{f}, \vec{x}).$$

By definition of the transpose, $(A\vec{x}, \vec{y}) = (A^{\mathrm{T}}\vec{y}, \vec{x})$ for all $\vec{x}$ and $\vec{y}$ in $\mathscr{E}^n$, so

$$(A^{\mathrm{T}}A\vec{f}, \vec{x}) = c_f(\vec{f}, \vec{x}). \tag{4}$$

Since (4) holds for all $\vec{x} \in \mathscr{E}^n$, we have the following:

PROPOSITION 2. *For all $\vec{f} \in F_o$,*

$$A^{\mathrm{T}}A\vec{f} = c_f\vec{f}. \tag{5}$$

This completes the proof of:

LEMMA 1. *If ψ is an affine symmetry of $D(o)$, the elements of F_o are eigenvectors of $A^{\mathrm{T}}A$ and the corresponding eigenvalues are the constants $c_f > 0$.*

$A^{\mathrm{T}}A$ is symmetric and has n positive eigenvalues, $\lambda_1, \ldots, \lambda_n$. Since A is volume-preserving and $\det A^{\mathrm{T}} = \det A$,

$$\prod_{i=1}^{n} (\lambda_i) = 1.$$

We distinguish three cases, as follows:
(a) If all eigenvalues of $A^{\mathrm{T}}A$ are distinct, then all eigenspaces are one-dimensional and mutually orthogonal.
(b) If all eigenvalues have the same value λ, then $1 = \det A^{\mathrm{T}}A = \lambda^n$, so $A^{\mathrm{T}}A = I_n$. Thus A is an orthogonal matrix.
(c) When some of the eigenvalues occur with multiplicity greater than 1, the situation is a combination of cases (a) and (b). $A^{\mathrm{T}}A$ decomposes $\mathscr{E}^n$ into orthogonal eigenspaces, each containing the face vectors belonging to the corresponding eigenvalue. On each of these subspaces, A acts as an isometry possibly followed by a dilation or contraction, and the subspaces are permutated by A.
In other words, we have the following:

THEOREM 2. *If F_o is completely reducible, then A is a monomial matrix. If F_o is irreducible, then A is orthogonal. If F_o is reducible, but not completely reducible, then A permutes (perhaps together with dilations and contractions) the eigenspaces of $A^{\mathrm{T}}A$.*

EXAMPLE 1. Let $n = 3$, and let $F_o = \pm\vec{b}_1, \pm\vec{b}_2, \pm\vec{b}_3$, where $\vec{f}_1 = (1, 0, 0)$, $\vec{b}_2 = (0, 2, 0)$ and $\vec{b}_3 = (0, 0, 3)$. Up to an orthogonal transformation the only possible affine symmetries, other than $A = \pm I$, are those that interchange the face vectors. Suppose, for example, that A permutes them cyclically. Then since $N(\vec{b}_1) = 1$, $N(\vec{b}_2) = 4$ and $N(\vec{b}_3) = 9$, we have $c_1 = 4$, $c_2 = 9/4$ and $c_3 = 1/9$, and

$$A = \begin{pmatrix} 0 & 0 & 1/3 \\ 2 & 0 & 0 \\ 0 & 3/2 & 0 \end{pmatrix}, \qquad A^{\mathrm{T}}A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9/4 & 0 \\ 0 & 0 & 1/9 \end{pmatrix}.$$

EXAMPLE 2. Now let $n = 2$, and let

$$F_o = \pm\vec{b}_1, \pm\vec{b}_2, \pm(\vec{b}_1 + \vec{b}_2),$$

where $\vec{b}_1 = (1, 0)$ and $\vec{b}_2 = (-\varepsilon, 1)$. Since $A^{\mathrm{T}}A$ has these three eigenvectors, there can be only one eigenvalue. Thus $A^{\mathrm{T}}A = I_n$ and A is an isometry. For generic values of ε, there are no isometries other than $\pm I_n$.

3. AFFINE EQUIVALENCE FOR VORONOÏ CELLS OF LATTICES

We first recall a few basic facts about lattices and their Voronoï cells (see, e.g., [2] and [5]). We will denote the origin of a lattice by o.
(i) Every point, and the midpoint between any pair of points, of a lattice is a symmetry center for the lattice, and every symmetry center belongs to one of these two classes. $D(o)$ is centrosymmetric and so are its $(n-1)$-dimensional faces. This means that these faces lie in *belts*, closed loops of faces with parallel $(n-2)$-dimensional faces.

(ii) Every belt contains either four or six faces; there are four faces in the belt iff the four face vectors are $\pm\vec{f}_1$ and $\pm\vec{f}_2$, where $(\vec{f}_1, \vec{f}_2) = 0$.
(iii) $\vec{f} \in F_o$ iff $\pm\vec{f}$ are the shortest vectors in their coset mod $2L$: if $\vec{v} \in L$, $\vec{v} \neq \pm\vec{f}$ then $N(\vec{f}) < N(\vec{f} + 2\vec{v})$.

Since there are 2^n cosets of $2L$ in L, including $2L$ itself, which has coset representative 0, and since $\vec{f} \in F_o \Leftrightarrow -\vec{f} \in F_o$, we have:
(iv) $|F_o| \leqslant 2(2^n - 1)$.

Finally:
(v) All primitive parallelotopes have $2(2^n - 1)$ faces.

Proposition 3. *$|F_o|$ is maximal iff all belts of $D(o)$ have six faces.*

Proof. Suppose that a belt of $D(o)$ has four faces with face vectors $\pm\vec{f}_1$, $\pm\vec{f}_2$. Then $\vec{f}_1 \pm \vec{f}_2$ have equal length and are in the same $2L$ coset, so neither is in F_o. We will show that in fact this coset does not have a representative in F_o, from which it follows that $|F_o|$ is not maximal. It is enough to show that $(\vec{f}_1 + \vec{f}_2)/2$ lies on the boundary of $D(o)$, by the following remark: if $\frac{1}{2}\vec{v} \in \partial D(o)$, where $\vec{v} \in L$ then, by the definition of $D(o)$, $N(\vec{v}/2) \leqslant N(\vec{v}/2 - \vec{x})$ for all $\vec{x} \in L$. Thus $N(\vec{v}) \leqslant N(\vec{v} - 2\vec{x})$, so no vector in the $2L$ coset of $\vec{v}$, other than $\pm\vec{v}$, can satisfy (iii). To complete the proof, let α_1 and α_2 be the two parallel $(n-2)$-faces in the belt that lie in the face with face vector $\vec{f}_1$, where $\alpha_2 = \alpha_1 + \vec{f}_2$. Obviously, if $\vec{x} \in \alpha_1$, then $\vec{x} + \vec{f}_2 \in \alpha_2$, but also, since $\vec{f}/2$ is a center of symmetry for the face, we also have $\vec{f}_1 - \vec{x} \in \alpha_2$. One easily sees that the midpoint of the line segment joining them is $(\vec{f}_1 + \vec{f}_2)/2$, so $(\vec{f}_1 + \vec{f}_2)/2 \in \alpha_2 \in D(o)$. The converse is immediate.

□

Proposition 4. *If every belt of $D(o)$ has six faces, then F_o is irreducible.*

Proof. Consider any $\vec{f} \in F_o$, and let $F^\perp$ be the set of face vectors orthogonal to $\vec{f}$ (of course, $F^\perp$ may be empty). Since $\vec{f}$ lies in belts of six faces, $F^\perp$ does not contain any of the vectors of these belts, and except for $\pm\vec{f}$, none of the vectors in the belts is orthogonal to $F^\perp$. These vectors span ε^n, so $F^\perp = \phi$. (Note that the converse of this proposition is false.) □

Corollary 1. *The set of face vectors of a primitive parallelotope is irreducible.*

Next, we note the following:

Lemma 2. *Every affine transform of a lattice Voronoï cell onto itself is an affine symmetry of the cell.*

Proof. The centers of the faces of $D(o)$ are symmetry centers of L, and any affine transformation of $D(o)$ into itself maps symmetry centers to symmetry centers. It follows that face vectors are mapped to face vectors. □

So far we have only discussed linear transformations of $D(o)$ onto itself, but in fact our discussion carries over unchanged to the case in which L and L' are two lattices, brought to the same origin by translation, and ψ is a volume-preserving linear transformation carrying $D_L(0)$ to $D_{L'}(0)$. In particular, Theorem 2 still holds.

Now suppose that P is any primitive parallelotope. By Theorem 1, there exists an affine transformation η that maps P onto some primitive Voronoï lattice parallelotope $D_L(0)$. If there were a second map ζ carrying P to a different Voronoï lattice parallelotope $D_{L'}(0)$, there would be, since we may assume that the origins of the two lattices coincide, a linear transformation ψ carrying $D_L(0)$ to $D_{L'}(0)$, where

$$\psi = \eta^{-1} \circ \zeta.$$

Thus, by Corollary 1 and Theorem 2, we have proved uniqueness for Theorem 1, up to an orthogonal transformation.

Remarkably, when $n \geqslant 4$, there are also non-primitive parallelotopes for which $|F_o|$ is maximal! Zhitomirskii generalized Theorem 1 to include such parallelotopes by proving the following:

THEOREM 3 (*Zhitomirskii*). *Every parallelotope all the belts of which are hexagonal is the affine image of a Voronoï cell.*

Our discussion in fact establishes uniqueness in this case as well, since we only used the fact that all belts have six faces. Thus we have proved the following:

THEOREM 4. *Every paralleltotope with* $|F_o| = 2(2^n - 1)$ *is affine equivalent to a unique Voronoï cell, up to an orthogonal transformation.*

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