Breaking of the $SU_3 \times SU_3$ Symmetry in Hadronic Physics.

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1. – Introduction.

In this paper we analyze the properties of the three fundamental interactions (strong, electromagnetic, and weak) from the point of view of the $SU_3 \times SU_3$ group. For this analysis we will use an extension of the geometrical approach which we have introduced before [1, 2] for SU_3. In that case the three charges conserved by each interaction namely the hypercharge Y, the hadronic electric charge Q_H, and the weak hypercharge Z, are generators of the unitary linear representation of SU_3 on the Hilbert space of hadronic states. That is, in the representation $a \rightsquigarrow Q(a) \in \mathscr{L}(\mathscr{H})$ of the SU_3 Lie algebra on $\mathscr{H}$, Q_H, Y, Z are the images of three vectors — q, y, z of $\boldsymbol{R}^8$, the octet space, *i.e.*, the eight-dimensional real vector space of the Lie algebra of SU_3. As we have shown in ref. [1] the isotropy groups of these vectors are maximal subgroups of SU_3 and the vectors themselves are solutions of a nonlinear equation.

It is however clear that for a full understanding of the properties of the interactions and of their relations we need to consider the group $SU_3 \times SU_3$. Indeed the different behavior under space reflections of the three interactions, cannot be described in terms of the diagonal SU_3 subgroup alone.

We will see that some of the interesting geometrical properties of the vectors y, q, z can be carried over to $SU_3 \times SU_3$. We will show that the directions along which the symmetry group is broken are, also in this case, solutions of nonlinear equations of the type postulated by the bootstrap approach to symmetry breaking.

Two subgroups of $SU_3 \times SU_3$ are of special significance for hadron physics: SU_3 and $SU_2 \times SU_2$. Both represent approximate invariances of the strong interactions which are valid when one neglects either the difference between the K- and π-meson mass (for SU_3) or the pion mass (for $SU_2\times SU_2$).

Recently Gell-Mann, Oakes and Renner [3] have suggested that the strong Hamiltonian which breaks the $SU_3\times SU_3$ symmetry transforms approximately like an element of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation which is left invariant by $SU_2\times SU_2$. We will show that in the space of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation we can define two directions which are solutions of nonlinear equations and whose isotropy groups are precisely SU_3 and $SU_2\times SU_2$.

In Sect. **2** after a brief resumé of the relevant results of refs. [1] and [2] we will discuss the unique symmetrical algebra, on the space of the $(1, 8) \oplus \oplus (8, 1)$ and of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representations, which have $SU_3\times SU_3$ as a group of automorphism. The existence on these spaces of symmetrical algebras insures the possibility of having nonlinear equations whose solutions define the directions along which $SU_3\times SU_3$ is broken.

2. – Mathematical preliminares.

2·1. *Geometry of the octet.* – We begin by briefly reviewing a co-ordinate-free formulation [1] of the SU_3 invariant algebras on the octet space $\boldsymbol{R}^8$.

We can realize $\boldsymbol{R}^8$ as the real vector space of all 3×3 Hermitian traceless matrices $a, b, c, \ldots$. Any element u of the group SU_3 is the form $u = \exp[-i\varphi a/2]$, $a\in \boldsymbol{R}^8$. The action of SU_3 on $\boldsymbol{R}^8$ (which is the space of its adjoint representation) is

(1) $$a \overset{u}{\rightarrow} uau^* = uau^{-1}.$$

We can define on $\boldsymbol{R}^8$ an SU_3-invariant scalar product and two algebras which have SU_3 as automorphism group:

Scalar product:

(2) $$(a, b) = \tfrac{1}{2}\,\mathrm{tr}\, ab$$

SU_3 Lie algebra:

(3) $$a\wedge b = -\frac{i}{2}[a, b]\,.$$

Symmetrical algebra:

(4) $$a\vee b = \tfrac{1}{2}(ab + ba) - \tfrac{2}{3}(a, b) = \tfrac{1}{2}\{a, b\} - \tfrac{2}{3}(a, b)\,.$$

If a and $a\vee a$ are linearly independent they generate a two-plane $\mathscr{C}_a$ (*i.e.* a two-dimensional subspace of $\boldsymbol{R}^8$) which is a Cartan subalgebra (*i.e.* a maximal Abelian subalgebra) of the SU_3 Lie algebra. Thus $\mathscr{C}_a$ which is isomorphic to $U_1 \times U_1$ is the Lie algebra of the isotropy group (or little group) of a. If on the contrary

$$q \vee q + \eta(q) q = 0 \, , \tag{5}$$

the isotropy group is a U_2 group which we denote by $U_2(q)$. Any vector whose isotropy group is a U_2 will be called a « q-vector ». From now on we will consider only normalized « positive » q-vectors, *i.e.* such that: $(q, q) = 1$, $\eta(q) > 0$. This implies, $\eta(q) = 1/\sqrt{3}$.

The Cartan subalgebras of the SU_3 Lie algebra are all conjugate (*i.e.* transformed into each other) by the SU_3 group. One of them is of course made with the diagonal matrices $u \in SU_3$. It can be proved that any $\mathscr{C}$ contains three positive normalized q-vectors at 120° from each other. Conversely if $x, y \in \boldsymbol{R}^8$ commute, $\alpha x + \beta y$ and $\alpha' x + \beta' y$ commute, and generate a $\mathscr{C}$ (denoted $\mathscr{C}_{x,y}$). For positive normalized q-vectors we thus have

$$(q_i, q_j) = -\tfrac{1}{2} \Leftrightarrow q_i \wedge q_j = 0 \quad \text{and} \quad q_i \neq q_j \, . \tag{6}$$

Given a q-vector y, the vectors t_y of $U_2(y)$ which are orthogonal to y form the $SU_2(y)$ subalgebra of $U_2(y)$. They satisfy the following relations:

$$y \wedge t_y = y \wedge t'_y = 0; \quad (y, t) = (y, t') = 0; \quad \sqrt{3} t_y \vee t'_y = (t_y, t'_y) y \, . \tag{7}$$

The normalized t_q of the three q-vectors of a Cartan subalgebra $\mathscr{C}$ form the hexagon of the « roots » different from zero.

2·2. *The $SU_3 \times SU_3$ algebra.* – To extend this formalism to $SU_3 \times SU_3$ we consider the space $\boldsymbol{R}^{16} = \boldsymbol{R}^8 \oplus \boldsymbol{R}^8$. We call a_+ and a_- the elements of the first and the second $\boldsymbol{R}^8$, respectively (the index $\pm$ corresponds in physics to the chirality) and denote by $\tilde{a} = a_+ \oplus a_-$ an element of $\boldsymbol{R}^{16}$. The Lie algebra of $SU_3 \times SU_3$ is then defined by

$$\tilde{a} \wedge \tilde{b} = (a_+ \oplus a_-) \wedge (b_+ \oplus b_-) = (a_+ \wedge b_+) \oplus (a_- \wedge b_-) \, , \tag{8}$$

where $\wedge$ in the right-hand side has been defined on $\boldsymbol{R}^8$ in eq. (3). The scalar product invariant under $SU_3 \times SU_3$ is the Cartan-Killing form which we write

$$(a_+ \oplus a_-, b_+ \oplus b_-) = \tfrac{1}{2}(a_+, b_+) + \tfrac{1}{2}(a_-, b_-) \, . \tag{9}$$

It is also convenient to use another decomposition of $\boldsymbol{R}^{16}$ into a direct sum $\boldsymbol{R}^8 \oplus \boldsymbol{R}^8$. In this decomposition, which is symbolically illustrated in Fig. 1, we denote the element $\tilde{a} = a_+ \oplus a_-$ by $(a|a')$ with

$$a_+ = a + a' \quad \text{and} \quad a_- = a - a' \, . \tag{10}$$

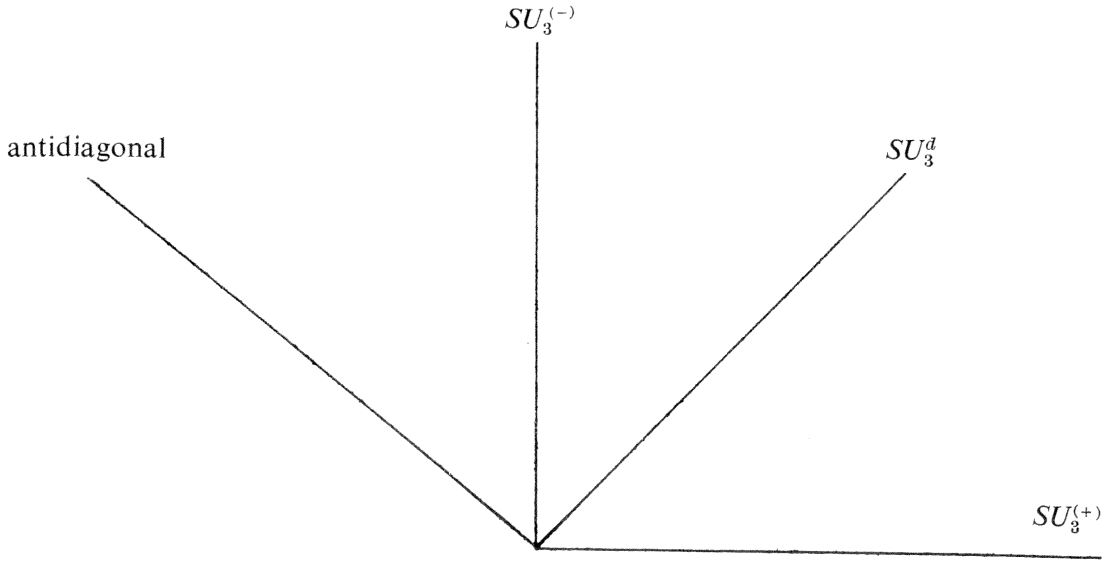

Fig. 1. – Decomposition of $\boldsymbol{R}^{16}$ into a direct sum of $\boldsymbol{R}^8 \oplus \boldsymbol{R}^8$.

In this notation the Lie algebra law (6) becomes

$$\tilde{a}\wedge\tilde{b} = (a|a')\wedge(b|b') = (a\wedge b + a'\wedge b'|a\wedge b' + a'\wedge b) \tag{11}$$

and the scalar product

$$(\tilde{a}, \tilde{b}) = (a, b) + (a', b') . \tag{12}$$

In a similar way we can extend to $\boldsymbol{R}^{16}$ the symmetrical algebra on $\boldsymbol{R}^8$:

$$\tilde{a}\vee\tilde{b} = (a|a')\vee(b|b') = (a\vee b + a'\vee b'|a\vee b' + a'\vee b) . \tag{13}$$

One verifies that the equation

$$\tilde{a}\vee\tilde{a} = \lambda\tilde{a} , \tag{14}$$

has only two types of solutions:

$$\tilde{a} = (q|0) \tag{15}$$

and

$$\tilde{a} = (q|\pm q) , \tag{16}$$

where q is a q-vector.

The subalgebra of $SU_3\times SU_3$ which leaves invariant (*i.e.* commutes with) a q-vector $(y|0)$ of the diagonal SU_3 subalgebra is the set of all $(a|a')$ such that $y\wedge a = 0$, $y\wedge a' = 0$; it will be denoted $(U_2|U_2)_y$. With the notation of (8) it is the direct sum $U_2^{(+)}(y)\oplus U_2^{(-)}(y)$.

2'3. *The* $(3, \bar{3}) \oplus (\bar{3}, 3)$ *representation.* – A special role in the physical applications is played by the $(3, \bar{3})$ representation of the group or of the Lie algebra $SU_3 \times SU_3$. We can realize the 9-dimensional space of this representation as the complex vector space of the 3×3 matrices $\boldsymbol{m}$. Under the transformation $u_+ \times u_- = \exp[-i\varphi a_+/2] \times \exp[-i\varphi a_-/2]$, $\boldsymbol{m}$ goes over into:

$$\boldsymbol{m} \rightsquigarrow u_+ \boldsymbol{m} u_-^* . \tag{17}$$

The representation of the Lie algebra is thus

$$D(\tilde{a})\boldsymbol{m} = D(a_+ \oplus a_-)\boldsymbol{m} = -\tfrac{1}{2}(a_+ \boldsymbol{m} - \boldsymbol{m} a_-) , \tag{18}$$

or

$$D(\tilde{a})\boldsymbol{m} = D(a|a')\boldsymbol{m} = -\frac{i}{2}([a, \boldsymbol{m}] + \{a', \boldsymbol{m}\}) . \tag{19}$$

(Equation (18) is obtained from (17) by differentation with respect to φ, at $\varphi = 0$.)

The representation is unitary for the group, *i.e.* it leaves invariant the Hermitian scalar product

$$\langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle = \tfrac{1}{2} \operatorname{tr}(\boldsymbol{m}_1^* \boldsymbol{m}_2) . \tag{20}$$

The 9-dimensional complex vector space $\mathbf{C}_{3,\bar{3}}$ can be considered as an 18-dimensional real vector space $\boldsymbol{R}^{18}$. The 18-dimensional representation of the group $SU_3 \times SU_3$ on this space is the direct sum of the $(3, \bar{3})$ and its complex conjugate $(\bar{3}, 3)$. It is real and unitary, hence orthogonal. It leaves invariant a Euclidean (*i.e.* real orthogonal) scalar product which is the real part of (20), while the imaginary part becomes an antisymmetrical real (*i.e.* sympletic) scalar product. Explicitely we have:

$$(\boldsymbol{m}_1, \boldsymbol{m}_2) = \operatorname{Re}\langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle = \tfrac{1}{4} \operatorname{tr}(\boldsymbol{m}_1^* \boldsymbol{m}_2 + \boldsymbol{m}_2^* \boldsymbol{m}_1) , \tag{21}$$

$$)\boldsymbol{m}_1, \boldsymbol{m}_2(= \operatorname{Im}\langle \boldsymbol{m}_1, \boldsymbol{m}_2 \rangle = \frac{1}{4i} \operatorname{tr}(\boldsymbol{m}_1^* \boldsymbol{m}_2 - \boldsymbol{m}_2^* \boldsymbol{m}_1) . \tag{22}$$

Any 3×3 complex matrix can be written in the form

$$\boldsymbol{m} = \sqrt{\tfrac{2}{3}}\mu \mathbf{1} + m + i\sqrt{\tfrac{2}{3}}\, \mathbf{1}\mu' + im' = (\mu|m||\mu'|m') , \tag{23}$$

where μ and μ' are real members and m and m' are vectors of the octet space. In this notation (21), (22), and (23) read

$$(\boldsymbol{m}_1, \boldsymbol{m}_2) = \mu_1\mu_2 + \mu_1'\mu_2' + (m_1, m_2) + (m_1', m_2') , \tag{24}$$

$$)\boldsymbol{m}_1, \boldsymbol{m}_2(= \mu_1\mu_2' - \mu_1'\mu_2 + (m_1, m_2') - (m_1', m_2) , \tag{25}$$

and eq. (19) reads

$$(26)\quad D(a|a')(\mu|m||\mu'|m') =$$
$$= \left(\sqrt{\tfrac{2}{3}}(a', m')|a\wedge m + a'\vee m' + \sqrt{\tfrac{2}{3}}\mu' a'||-\sqrt{\tfrac{2}{3}}(a', m)|a\wedge m' - a'\vee m - \sqrt{\tfrac{2}{3}}\mu a'\right).$$

Tensor operators which represent physical observable must be Hermitian on $\mathscr{H}$. It is therefore necessary that they belong to a real representation of the invariance group. This is the case of the $(3, \overline{3}) \oplus (\overline{3}, 3)$ representation which, we want to emphasize, is irreducible as a real representation.

The tensor product of $(3, \overline{3}) \oplus (\overline{3}, 3)$ by itself when decomposed into real irreducible representations contains the $(3, \overline{3}) \oplus (\overline{3}, 3)$ once.

Hence from two vectors $\boldsymbol{r}, \boldsymbol{s} \in (3, \overline{3}) \oplus (\overline{3}, 3)$ it is possible to form a new vector of the same representation which we denote $\boldsymbol{r}_T\boldsymbol{s}$. The symbol T is the law of a symmetrical algebra on $\boldsymbol{R}^{18}$ which has $SU_3 \times SU_3$ as automorphism group. By standard methods we find

$$(27)\quad \boldsymbol{r}_T\, \boldsymbol{s} = \tfrac{1}{2}\mathbf{1}\left(\operatorname{tr} \boldsymbol{r}^* \operatorname{tr} \boldsymbol{s}^* - \operatorname{tr}(\boldsymbol{r}^* \boldsymbol{s}^*)\right) - \tfrac{1}{2}\boldsymbol{r}^* \operatorname{tr} \boldsymbol{s}^* - \tfrac{1}{2}\boldsymbol{s}^* \operatorname{tr} \boldsymbol{r}^* + \tfrac{1}{2}\{\boldsymbol{r}^*, \boldsymbol{s}^*\}.$$

We leave to the reader to check that

$$(28)\qquad D(a|a')\boldsymbol{r}_T\boldsymbol{s} = \left(D(a|a')\boldsymbol{r}\right)_T \boldsymbol{s} + \boldsymbol{r}_T\left(D(a|a')\boldsymbol{s}\right),$$

which means that $SU_3 \times SU_3$ is a derivation algebra of the T-product.

With the notation (23) we can write eq. (27) in the form

$$(29)\qquad \boldsymbol{r}_T\, \boldsymbol{s} = (\tau|t||\tau'|t'),$$

where

$$\boldsymbol{r} = (\varrho|r||\varrho'|r') \quad \text{and} \quad \boldsymbol{s} = (\sigma|s||\sigma'|s')$$

and

$$(30)\qquad \left|\begin{aligned}
\tau &= \frac{1}{\sqrt{6}}\left(2\varrho\sigma - 2\varrho'\sigma' - (r, s) + (r', s')\right),\\
t &= \frac{1}{\sqrt{6}}\left(-\varrho s - \sigma r + \varrho' s' + \sigma' r'\right) + r\vee s - r'\vee s',\\
\tau' &= \frac{1}{\sqrt{6}}\left(-2\varrho\sigma' - 2\varrho'\sigma + (r, s') + (r', s)\right),\\
t' &= \frac{1}{\sqrt{6}}\left(\varrho s' + \varrho' s + \sigma' r + \sigma r'\right) - r\vee s' - r'\vee s.
\end{aligned}\right.$$

We add two more properties of this product

$$\langle x_T y, z\rangle = \langle x, y_T z\rangle, \tag{31}$$

$$\langle x, x_T x\rangle = \tfrac{3}{2}\det x = (x_T x, x) + i)x_T x, x(. \tag{32}$$

The $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation of $SU_3 \times SU_3$ has no invariant for the subgroup $(U_2|U_2)_y$, *i.e.* if for all $(a|a') \in (U_2 \times U_2)_y$, $D(a|a')m = 0$ eq. (26) shows that $m = 0$. However if a' is restricted to be in $SU_2(y)$, the same equation shows the existence of a two-dimensional invariant subspace spanned by the vectors

$$(\eta| -\sqrt{2}\eta y||\eta'| -\sqrt{2}\eta' y), \tag{33}$$

where y is a q-vector and η, η' are real numbers. We will denote the isotropy group (or its Lie algebra) of the vectors (33) by $(U_2(y)|SU_2(y))$. This Lie algebra is the following direct sum

$$(U_2(y)|SU_2(y)) = SU_2^{(+)}(y) \oplus SU_2^{(-)}(y) \oplus U_1^d(y), \tag{34}$$

where $U_1^d(y) = (U_1(y)|0)$ is the Lie algebra generated by $(y|0)$ (see Fig. 1).

The vectors (33) have an interesting property under the T-product. Let $y(\varphi)$ be the vector (33) with

$$\eta = \sqrt{\tfrac{1}{3}}\cos\varphi, \qquad \eta' = \sqrt{\tfrac{1}{3}}\sin\varphi . \tag{35}$$

These vectors are normalized

$$\langle y(\varphi), y(\varphi)\rangle = 1 . \tag{36}$$

They belong to the $SU_3 \times SU_3$ orbit of $y(0)$ and satisfy the quadratic equation

$$y(\varphi)_T y(\varphi') = 0 . \tag{37}$$

Moreover one shows that all unit vectors of $\mathbf{R}^{18}$ satisfying such an equation are on the $SU_3 \times SU_3$ orbit of $y(0)$.

Equation (37) is a particular case of the equation

$$m_T m = \lambda m . \tag{38}$$

If $\lambda \neq 0$ (and $m \neq 0$) one also shows that the only unit vectors which are solutions of (38) are on the two orbits of $\pm n = \pm\sqrt{2/3}\,\mathbf{1}$ which are SU_3^d invariants. The unit vectors which have SU_3^d as isotropy group form a circle

$$n(\varphi) = \sqrt{\tfrac{2}{3}}\exp[i\varphi]\mathbf{1} = (\cos\varphi|0||\sin\varphi|0) \tag{39}$$

$$n(\varphi \pm \pi) = -n(\varphi)$$

and they generate the T-subalgebra

$$n(\varphi)_T n(\varphi') = \sqrt{\tfrac{2}{3}}\, n(-\varphi-\varphi') \,. \tag{40}$$

3. – Geometrical properties of the three interactions.

3·1. *The SU_3 symmetry.* – We begin by recalling the basic properties of the interactions under the SU_3 group, *i.e.*, the diagonal subgroup SU_3^d of $SU_3^{(+)}\times SU_3^{(-)}$.

a) The hypercharge Y and the three isospin operators $T_1, T_2, T_3 = Q_H + \frac{1}{2}Y$ generate the invariance group $U_2(y)$ of the strong interactions. The extension of this invariance to SU_3 implies considering $U_2(y)$ as a subgroup of SU_3. This means that y is a q-vector of which Y is the image in the representation of the SU_3 algebra in the Hilbert space $\mathscr{H}$ of hadron physics.

The electric hadronic charge Q_H is the corresponding image of $-q$ and the relation $Q_H = T_3 + \frac{1}{2}Y$ implies that q is a q-vector. The $SU_2(q)$ group is called the « U-spin group ».

b) According to Cabibbo's hypothesis the two charged components of the vector current $v_\mu^\pm$ coupled to the leptons and the electromagnetic current $j_\mu^{\text{e.m.}}$ belong to the same SU_3^d octet. We denote by $c_1 \pm ic_2$ the directions. of $v_\mu^\pm$. Using the additional property that the electric charges of $v_\mu^\pm$ are ± 1 we can deduce

$$\sqrt{3}c_1 \vee c_1 = \sqrt{3}c_2 \vee c_2 = z \,, \tag{41}$$

where z is a q-vector. The operator Z, which is the image of z, is the weak hypercharge conserved in weak interactions.

The vector z commutes with q but not with y. We thus have in $\boldsymbol{R}^8$ two distinct algebras $\mathscr{C}_{qy}$ and $\mathscr{C}_{qz}$ which have q in common. The noncommunicativity of y and z reflects the existence of strangeness violating weak interactions. As one can see from (6) the difference from 0° or 120° of the angle between y and z gives a measure of the noncommunicativity of Y and Z and is therefore related to the Cabibbo angle θ. Explicitely we have

$$(y, z) = 1 - \tfrac{3}{2}\sin^2\theta \,. \tag{42}$$

It can be proved that two noncommuting q-vectors y and z uniquely define another q-vector which commutes with both of them. This vector is given by the relation

$$q = \big((y, z) - 1\big)^{-1}\big(\sqrt{3}y \vee z + \tfrac{1}{2}(y + z)\big) \,. \tag{43}$$

Thus the strong and weak interactions determine uniquely the direction of the electromagnetic interactions.

Cabibbo has also postulated that the axial currents $a_\mu^\pm$ belong to another SU_3^d octet in the same directions $c_1 \pm ic_2$ as $v_\mu^\pm$. The two assumptions about the vector and the axial vector currents are in good agreement with experiment.

3'2. $SU_3 \times SU_3$ *symmetry.* – Since the weak interactions have a definite (negative) chirality whereas the electromagnetic and strong interactions have a defined parity, their relations can only be fully understood by considering the enlarged group $SU_3 \times SU_3$. It has indeed been suggested [3, 4] that this group and its subgroups provide a reasonably approximate frame for the study of hadron physics. Cabibbo's hypothesis can be generalized to $SU_3 \times SU_3$ by assuming that $j_\mu^{\text{e.m.}}$, $v_\mu^\pm$, $a_\mu^\pm$ belong to the same representation of this group namely the adjoint representation $(8, 1) \oplus (1, 8)$. We can thus write for the currents:

$$j_\mu^{\text{e.m.}} = h_\mu(q|0)\,; \qquad v_\mu^\pm = h_\mu(c^\pm|0)\,; \qquad a_\mu^\pm = h_\mu(0|c^\pm)\,. \tag{44}$$

The weak currents are thus

$$h_\mu^\pm = h_\mu(c^\pm|-c^\pm)\,. \tag{45}$$

As Q_H is the integral over space of the time component of $j_\mu^{\text{e.m.}}$, the integrals

$$Q(a|a') = \int \mathrm{d}^3x\, h_0(a|a')\,, \tag{46}$$

are at a given time the generators of $SU_3 \times SU_3$. We shall now list the covariance properties of the three interactions under $SU_3 \times SU_3$:

a) The isotropy group of the electromagnetic current $h_\mu(q|0)$ and therefore of the electromagnetic interactions (see Sect. **2'2**) is $(U_2|U_2)_q = U_2^{(+)}(q) \oplus U_2^{(-)}(q)$.

b) The pair of weak currents $h_\mu(c^\pm|-c^\pm)$ and therefore the semi-leptonic weak interactions have for isotropy group $SU_3^{(+)} \oplus U_1^{(-)}(z)$.

c) The covariance of the *CP* conserving Hamiltonian $\mathscr{H}_{NL}$ for non-leptonic weak interactions is not yet established. If it involves only charged currents as many physicists would prefer [5] then it would have components outside the $(1, 8) \oplus (8, 1)$ representation. It is however compatible with the present evidence to assume that $\mathscr{H}_{NL}$ belongs entirely to the representations $(1, 8) \oplus (8, 1)$ in the direction $(z|-z)$ [6]. If this were the case the isotropy group of $\mathscr{H}_{NL}$ would be $SU_3^{(+)} \oplus U_2^{(-)}(z)$ which is a maximal subgroup of $SU_3 \times SU_3$. Nothing is known for the *CP* violating part.

d) We have said that $U_2(y)$ and SU_3^d are approximate invariances of the strong interactions. Another interesting approximate invariance has been recently proposed by Gell-Mann, Oakes, and Renner [3]. According to them, in the limit where the pion mass can be neglected the strong Hamiltonian is of the form

$$\mathscr{H}_S = \mathscr{H}_0 + \mathscr{H}(\boldsymbol{m}) , \tag{47}$$

where $\mathscr{H}_0$ is invariant under $SU_3 \times SU_3$ and $\mathscr{H}(\boldsymbol{m})$ transforms like the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representations. They also suggested that to a good approximation $\boldsymbol{m}$ coincides with the vector $\boldsymbol{y}(0)$ of eq. (37).

In this model the approximate isotropy group of the strong interactions would be $(U_2|SU_2)_y$ which is a maximal isotropy group for the nonzero vector of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation.

Even though the mass difference $m_K - m_\pi$ is larger than m_π, SU_3 remains an interesting approximation for the strong interactions. The $SU_3 \times SU_3$ breaking part in eq. (47) is in the SU_3 invariant direction denoted by $\boldsymbol{n}(0)$ in eq. (39). It is remarkable that its isotropy group (SU_3^d) is the other maximal isotropy group of the nonzero vectors of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation.

3·3. *$SU_3 \times SU_3$ and space reflections.* – In the limit where they are exact the $U_2(y)$ and SU_3^d symmetries of the strong interactions commute with the Poincaré group (without time reflection).

For the exact $SU_3 \times SU_3$ symmetry the invariance group is no longer a direct product of the internal symmetry group by the Poincaré group but the following semidirect product

$$(\mathscr{P}_0 \times SU_3^{(+)} \times SU_3^{(-)}) \dot{\times} Z_2 ,$$

where $\mathscr{P}_0$ is the connected Poincaré group and the nontrivial element r of Z_2 acts on $\mathscr{P}_0$ like the space inversion and interchanges $SU_3^{(-)}$ with $SU_3^{(+)}$. The action of r on the $(8, 1) \oplus (8, 1)$ representation is

$$(a|a') \overset{r}{\leftrightarrow} (a|-a') . \tag{48}$$

This allows to assign a parity to the elements of the $(8, 1) \oplus (1, 8)$ representation; the primed vectors have odd parity, the unprimed ones have even parity.

For the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation, eq. (26) shows that the primed and unprimed quantities which appear in (23) have opposite parity. For example $\boldsymbol{y}(0)$ and $\boldsymbol{y}(\pi/2)$ (see eq. (33)) are eigenvectors of r with opposite parity. As we have seen, $SU_3 \times SU_3$ implies the existence for this representation of the T-algebra and this fixes in the SU_3 limit the assignment of the parity. Indeed, as we have shown, the direction along which $SU_3 \times SU_3$ is broken in an SU_3-invariant way satisfies the nonlinear equation

$$(49) \qquad \boldsymbol{n}(0)_T\,\boldsymbol{n}(0) = \sqrt{\frac{2}{3}}\,\boldsymbol{n}(0) = \boldsymbol{n}\left(\frac{\pi}{2}\right)_T \boldsymbol{n}\left(\frac{\pi}{2}\right).$$

Thus under r

$$(50) \qquad \boldsymbol{m} = (\mu|m||\mu'|m') \overset{r}{\longleftrightarrow} (\mu|m||-\mu'|-m')\,.$$

4. – Remarks on symmetry breaking.

It has been suggested by several authors [7] that the SU_3 or the $SU_3 \times SU_3$ symmetries are spontaneously broken. Such a symmetry breaking occurs when the invariance group K of a stable state of a physical theory is only a subgroup of the invariance group G of the theory itself. In this case all states of the same orbit G/K of solutions are all stable states.

We have shown [1, 8] that in a theory based on a variational principle spontaneous symmetry breaking can occur and one expects the subgroup K to be a maximal isotropy group among those of all possible orbits. As we have seen the breaking of $SU_3 \times SU_3$ by the strong interactions has the above property both in the SU_3 or in the $SU_2 \times SU_2$ approximation [9]. The same is true of $\mathscr{H}_{NL}$ if its invariance group is $SU_3^{(+)} \oplus U_2^{(-)}(z)$ (see Sect. **3**·2). This may therefore suggest that the $SU_3 \times SU_3$ symmetry of the hadronic world is spontaneously broken by the strong and perhaps also the weak interactions.

The intersection between the two isotropy groups of the weak nonleptonic nteractions and of the strong interactions in the Gell-Mann, Oakes, and iRenner model is:

$$(51) \quad \big(SU_3^{(+)} \oplus U_2^{(-)}(z)\big) \cap \big(SU_2^{(+)}(y) \oplus SU_2^{(-)}(y) \oplus U_1^{d}(y)\big) = SU_2^{(+)}(y) \oplus U_1^{(-)}(q)\,,$$

where q is a q-vector commuting with y and z which, as we have seen, is uniquely defined once y and z are fixed. The intersection of the two groups in the left-hand side of (51) and SU_3^d is $U_1^d(q)$ which is thus the only invariance group for the interactions between hadrons (when the hadron-lepton interactions are disregarded) and corresponds to the conservation of the electromagnetic charge. We have thus the following scheme of decreasing invariance inside the hadronic world

$$SU_3 \times SU_3 \;\begin{matrix} \supset & (U_2|SU_2)_y & \supset \\ \supset & SU_3^d & \supset \end{matrix}\; U_2(y) \supset U_1(q)\,.$$

Let us remark that the isotropy group of the electromagnetic Hamiltonian is $(U_2|U_2)_q$. This group is not maximal in $SU_3 \times SU_3$ for the $(1, 8) \oplus (8, 1)$ representation.

However the direction $(q|0)$ of the electromagnetic interactions shares with the directions of the two other interactions, *i.e.* $(z|-z)$ and $y(0)$ or $\boldsymbol{n}(0)$ the following properties: they are the different types of solutions of $SU_3 \times SU_3$ *invariant nonlinear* equations:

$$(a|a') \vee (a|a') = \lambda(a|a'), \tag{52}$$

for $(q|0)$ and $(z|-z)$;

$$\boldsymbol{m}_T \boldsymbol{m} = \lambda \boldsymbol{m}, \tag{53}$$

for the two directions along which $SU_3 \times SU_3$ is broken with approximate SU_3 or SU_2 invariance. Bootstrap approaches to symmetry breaking lead to this quadratic type of nonlinear equations.

It is interesting to note that for an $SU_3 \times SU_3$ invariant theory, the space inversion operator r can only be defined modulo an inner $SU_3 \times SU_3$ automorphism. However, as we have discussed in Sect. **3**˙3, the existence of the T-algebra fixes naturally the parity of the vectors of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation of $SU_3 \times SU_3$ and the vector $\boldsymbol{n}(0)$ has even parity. Thus the requirement that the breaking due to strong interactions satisfies eq. (53) fixes the parity of the hadronic states.

It is also worthwhile to point out that one of the solutions of eq. (52), $(z|-z)$ has a pure chirality corresponding to maximal violation of the parity fixed by the strong interaction. The other solution $(q|0)$ has a definite parity and its direction q is fixed when $y(0)$ and $(z|-z)$ are known. From the point of view of $SU_3 \times SU_3$ the three directions according to which the symmetry is broken have thus fairly simple properties and correspond to all three types of solutions of the nonlinear equations (52) and (53).

There is nothing however to tell us why the directions y and z should make precisely the angle that is experimentally observed.

We do not want to discuss here the attempts [10, 11] to calculate θ. We only remark that in the $SU_3 \times SU_3$ scheme, $\boldsymbol{m}$ and $(z|-z)$ are not in the same representation space. Thus an $SU_3 \times SU_3$ invariant depending on these two vectors has to be at least quadratic in $\boldsymbol{m}$. For example if we define the vector (see eq. (18))

$$\boldsymbol{v} = D(z|-z)\boldsymbol{m} = \tfrac{1}{2}\boldsymbol{m}z,$$

we can form an $SU_3 \times SU_3$ invariant $\langle \boldsymbol{v}, \boldsymbol{v} \rangle$ which is a function of (y, z).

However the length of the vectors has not been given here a physical meaning as we did not take into consideration the strength of the coupling.

On the other hand a projective invariant such as $\mathscr{J}_3 = \langle \boldsymbol{v}_T, \boldsymbol{v} \rangle \langle \boldsymbol{v}, \boldsymbol{v} \rangle^{-\frac{3}{2}}$ depends upon both θ and the matrix elements of $\boldsymbol{m}$ which are functions of the physical masses. We note that in the limit where $\boldsymbol{m} = \boldsymbol{y}(0)$ (invariant under $(U_2|SU_2)_y$, $\mathscr{J}_3$ vanishes.

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