Fortschritte der Physik 24, 259—323 (1976)

Amplitude Reconstruction for Usual Quasi Two Body Reactions with Unpolarized or Polarized Target

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Abstract

Tools for measuring joint polarization and polarization transfer are gathered. They allow the direct reconstruction of amplitudes in numerous quasi two body reactions with spinless beam and unpolarized or polarized target. Eight simple types of such reactions are worked out one by one; the practical results are summarized in Tables.

1. Introduction

Several experimental papers have been published [*1, 2, 3, 4*] in which quasi two body reaction amplitudes are reconstructed and tabulated. Old [*5, 6, 7*] and recent [*1, 8*] theoretical papers have described the method for reconstructing the amplitudes from the data in some simple cases. But each paper, limited to a particular reaction, introduces its own arbitrary conventions and this may obscure the future comparison of the experimental results for similar reactions. This paper aims to a systematics of amplitude reconstruction. It introduces a general terminology, it uses the most commonly accepted quantization conventions and it presents in details the practical method of amplitude reconstruction in the most usual reactions with unpolarized or polarized target.

We focus our attention on quasi two body reactions. We suppose therefore that resonances can be detected (we shall not enter into the problems of background separation) and that they have well known spins and parities. Furthermore, we restrict ourselves to a direct, model independent amplitude reconstruction, based only on angular momentum and parity conservation. Hence we shall not make use of other first principles as unitarity and analyticity, nor shall we relate amplitudes at different s and t values. Finally, this paper is limited to the simple initial state with a spinless meson beam and a spin 1/2 (polarized or unpolarized) baryon target.

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In Table 1 we list the most common reaction types and tabulate their number of amplitudes and observables for different target polarizations. The reaction type refers uniquely to the spins of the final particles and to the possibility of analyzing their polarization from their decay. We assume that for the spin l meson resonance only the even polarization can be measured (e.g. $\rho \to \pi\pi$, $K^* \to K\pi$), i.e. we do not consider the case of, e.g. $A_1 \to \rho\pi$. On the contrary, for the baryonic spin j resonance we consider both cases: i) measurement of even polarization only (j^e), this means no polarization measurement in the case $1/2^e$ (e.g. nucleons), and analysis of the simple, parity conserving decay in the case $3/2^e$, (e.g. $\Delta \to N\pi$). ii) Measurement of the whole polarization (j) by analysis of the parity violating decay of spin 1/2 baryons (e.g. $\Lambda \to p\pi$) or by analysis of the cascade decay of spin 3/2 resonances (e.g. $\Sigma^* \to \Lambda\pi$, $\Lambda \to p\pi$).
The tabulated number of amplitudes is the number of *real* independent amplitudes disregarding an overall phase.
For the observables the number tabulated in columns U and T is the number of real and imaginary components of a priori non vanishing multipole parameters and polarization transfer multipole parameters (cf. sect. 2 and 3). As we shall see, at fixed energy and momentum transfer they are not independent, they satisfy some linear and non linear (rank) constraints[1]). For this reason the number of observables is presented as the sum of two numbers: the number of independent observables + the number of constraints (when the second number is zero it has been omitted). Column U corresponds to an unpolarized target experiment, column T corresponds to a transversally polarized target experiment; the observables of column T includes of course the observables of column U. In column L is shown the number of independent observables of constraints that must be added to column U for an experiment with pure longitudinal polarization or to column T for an experiment with transverse and longitudinal polarization. Of course the number of independent observables can never the bigger than the number of amplitudes, but often it is one unit smaller; this is the case when the polarization of one spinning initial or final particle is not at all considered. This is an application of a general theorem derived by SIMONIUS [9], cf. Appendix 2. Practically the ghost amplitude appears as a relative phase between two sets of transversity amplitudes.
An analysis of the numbers in Table 1 suggests the following comments:

i) For higher spins the number of amplitudes increases linearly while the number of observables increases quadratically. When the number of observables becomes much bigger than the number of amplitudes it seems reasonable to communicate the amplitudes themselves, in so far as the statistics of the experiment allows their reconstruction.
ii) For some types of reactions all amplitudes but one can be reconstructed with unpolarized target, while for others amplitude reconstruction requires polarized target. For both reactions types, the most simplest cases are those for which the number of observables in columns U and T are set in boxes. It would be reasonable to give priority for the reconstruction of amplitudes to these types of reactions.
iii) As mentioned above, in all reaction types with unpolarized target and in those with a spin $1/2^e$ particle, it remains one ghost amplitude, the relative phase between two sets of transversity amplitudes. Since helicity amplitudes are linear combinations of amplitudes in both sets, they are ghosts too. Furthermore, parity conservation in the reaction has a simpler form in transversity quantization, and the observables are closer to the transversity amplitudes. All these arguments favor the reconstruction of transversity amplitudes. However to facilitate the comparison with models which use helicity amplitudes, for each reaction type we give the relations between the two kinds of amplitudes.

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[1]) Since we observe only quadratic expressions of the amplitudes, there are often discrete ambiquities in their reconstruction; the non linear constraints can be used to remove some of these ambiguites.

Table 1

Number of real amplitudes and of polarization observables for usual types of two-body reactions with meson beam and with unpolarized or polarized target. The reactions whose numbers are boxed will be studied in the text

Type of Reaction	Amplitudes	Observables: Unpolarized Target U	Polarized target: Transversal T	Polarized target: Longitudinal L
$\pi p \to \pi N \left(0, \frac{1^e}{2}\right)$	3	1	2	+0
$K\Lambda \left(0, \frac{1}{2}\right)$	3	2	3 + 3	+2
$\rho N \left(1^e, \frac{1^e}{2}\right)$	11	4	$\boxed{10}$	+2
$K^*\Lambda \left(1^e, \frac{1}{2}\right)$	11	$\boxed{10 + 2}$	11 + 25	+12
$A_2 N \left(2^e, \frac{1^e}{2}\right)$	19	9	$\boxed{18 + 6}$	+6
$K^{**}\Lambda \left(2^e, \frac{1}{2}\right)$	19	$\boxed{18 + 12}$	19 + 71	+30
$\left(l^e, \frac{1^e}{2}\right)$	$4(2l + 1) - 1$	$(l + 1)^2$†	$(l + 1)(3l + 2)$†	$+l(l + 1)$
$\left(l^e, \frac{1}{2}\right)$	$4(2l + 1) - 1$	$2(l + 1)(2l + 1)$†	$6(l + 1)(2l + 1)$	$+2(l + 1)(2l + 1)$
$\pi p \to \pi\Delta \left(0, \frac{3^e}{2}\right)$	7	4	$\boxed{7 + 3}$	+2
$K\Sigma^* \left(0, \frac{3}{2}\right)$	7	$\boxed{6 + 2}$	7 + 17	+8
$\rho\Delta \left(1^e, \frac{3^e}{2}\right)$	23	20	$\boxed{23 + 33}$	+16
$K^*\Sigma^* \left(1^e, \frac{3}{2}\right)$	23	$\boxed{22 + 26}$	23 + 121	+48
$\left(l^e, \frac{3^e}{2}\right)$	$8(2l + 1) - 1$	$2(l + 1)(3l + 2)$†	$2(l + 1)(9l + 5)$	$+2(l + 1)(3l + 1)$
$\left(l^e, \frac{3}{2}\right)$	$8(2l + 1) - 1$	$8(l + 1)(2l) + 1)$†	$24(l + 1)(2l + 1)$	$+8(l + 1)(2l + 1)$
(l^e, j^e)	$2(2l + 1) \times (2j + 1) - 1$	$(l + 1)\left(j + \frac{1}{2}\right) \times \left(2lj + j + \frac{1}{2}\right)$†	$(l + 1)\left(j + \frac{1}{2}\right) \times \left(6lj + 3j + \frac{1}{2}\right)$††	$(l + 1)\left(j + \frac{1}{2}\right) \times \left(2lj + j - \frac{1}{2}\right)$
(l^e, j)	$2(2l + 1) \times (2j + l) - 1$	$(l + 1)\left(j + \frac{1}{2}\right) \times (2l + 1)(2j + 1)$	$(l + 1)\left(j + \frac{1}{2}\right) \times 3(2l + 1)(2j + 1)$	$(l + 1)\left(j + \frac{1}{2}\right) \times (2l + 1)(2j + 1)$

† One amplitude is ghost.

†† One amplitude is ghost for $j = 1/2$.

Table I lists only general reaction types. In order to appreciate how many usual reactions correspond to each type, we present in Tables 2 and 3 several lists of such reactions. We have considered only reactions with practicable beam and targets and with well classified final particles (nonets or decuplets). We obtain in this way a list of 252 reactions belonging to the 8 types set in boxes in table 1 and whose amplitude reconstruction will be treated in detail below.

Furthermore in table 2 and 3 the isospin relations between these reactions are explicitly given. For reactions related via one isospin channel, the ratio of their amplitudes to the isospin amplitudes are fixed coefficients which are given as the coefficients of (f) in the brackets. Then the measurement of the amplitudes for only one reaction yields the amplitudes of all reactions in the same column. For reaction related via two isospin channels the amplitudes satisfy the relations indicated in part c) of each table. In these cases the reconstruction of amplitudes for two independent reactions, because of the overall phase and eventually of the ghost phase, does not fix the amplitudes of a third reaction. But this reconstruction for three (two by two linearly independent) reactions fixes the amplitudes of all the reactions in the same column[2]) up to an overall and eventually one ghost phases. The ghost phase of the whole set of reactions can be fixed by an experiment with polarized target for only one reaction of the set.

Sections 2 and 3 expose the general tools for the measurement of observables. In section 4 the concrete recipes for the amplitude reconstruction in each type of reaction are given. The hurried reader can directly skip to the reaction type in section 4 he is interested in.

Table 2

Listing of 140 reactions of types $\pi p \to K\Sigma^*$, $K^*\Lambda$, $K^{**}\Lambda$, $K^*\Sigma^*$ with their isospin relations. Their transversity amplitudes can be reconstructed with unpolarized target up to one phase (which can be measured with either longitudinally or transversally polarized target)

a) 30 reactions of type $K\Sigma^*$†. Similar reactions of type $K^*\Sigma^*$ are obtained by changing $(\pi\eta\eta' K) \to (\rho\omega\varphi K^*)$††.

$\pi N \to$	K	Σ^*	
$\pi^+ p$	+	+	(a_1)
$\pi^- p$	+	−	(a_2)
	0	0	(a_3)
$\pi^+ n$	+	0	(a_3)
	0	+	(a_2)
$\pi^- n$	0	−	(a_1)

$\bar{K}N \to$	$\eta(\eta')\ \Sigma^*$	π	Σ^*		K	Ξ^*	
$K^- p$	0 0 (f)	+	−	(b_4)	+	−	(c_1)
		0	0	(b_3)	0	0	(c_2)
		−	+	(b_2)			
$K^- n$	0 $-(\sqrt{2}\,f)$	0	−	$(-b_1)$	0	−	(c_3)
		−	0	(b_1)			
$\bar{K}^0 p$	0 $+(\sqrt{2}\,f)$	+	0	$(-b_1)$	+	0	(c_3)
		0	+	(b_1)			
$\bar{K}^0 n$	0 0 (f)	+	−	$(-b_2)$			
		0	0	$(-b_3)$	+	−	$(-c_1)$
		−	+	$(-b_4)$	0	0	(c_2)

† For their measurement and amplitude reconstruction, cf. tables 5—6.
†† For their measurement and amplitude reconstruction, cf. table 13.

[2]) The amplitude reconstruction for all other reactions in the column supplies checks of isospin invariance, and even for only three reactions, when several ampltudes with their relative phases are reconstructed.

If its experiment uses polarized target we advise him to read section 4.1 in order to see the connection of the terminology with the standard Wolfenstein parameters. For all other necessary tools he will find references to section 2 and 3.

Table 2 — cont'd

b) 40 reactions of type $K^*\Lambda$†††. Similar reactions of type $K^{**}\Lambda$ are obtained by changing $(\rho\omega\varphi K^*) \to (A_2 f f' K^{**})$††††.

$\pi N \to$	$K^*\Lambda$			$K^*\Sigma$		
$\pi^+ p$				+	+	(a_1)
$\pi^- p$	0	0	(f)	+	−	(a_2)
				0	0	(a_3)
$\pi^+ n$	+	0	$(-f)$	+	0	(a_3)
				0	+	(a_2)
$\pi^- n$				0	−	(a_1)

$\bar{K}N \to$	$\omega(\varphi)\Lambda$			$\rho\Lambda$			$\omega(\varphi)\Sigma$			$\rho\Sigma$			$K^*\Xi$		
$K^- p$	0	0	(f)	0	0	(f)	0	0	(f)	+	−	(b_4)	+	−	(c_1)
										0	0	(b_3)	0	0	(c_2)
										−	+	(b_2)			
$K^- n$				−	0	$(\sqrt{2}\, f)$	0	−	$(\sqrt{2}\, f)$	0	−	$(-b_1)$	0	−	(c_3)
										−	0	(b_1)			
$\bar{K}^0 p$				+	0	$(\sqrt{2}\, f)$	0	+	$(\sqrt{2}\, f)$	+	0	$(-b_1)$	+	0	(c_2)
										0	+	(b_1)			
										+	−	$(-b_2)$			
$\bar{K}^0 n$	0	0	$(-f)$	0	0	(f)	0	0	(f)	0	0	$(-b_3)$	+	−	(c_2)
										−	+	$(-b_4)$	0	0	(c_1)

c) Triangular isospin relations between the amplitudes of the reactions in a) and b)

$$a_1 = a_2 + \sqrt{2}\, a_3 \qquad \sqrt{2}\, b_1 = 2(b_2 + b_3) = b_2 - b_4 = -2(b_3 + b_4) \qquad c_1 + c_2 = c_3$$

††† For their measurement and amplitude reconstruction, cf. tables 8—9.
†††† For their measurement and amplitude reconstruction, c.f. table 11.

Table 3

Listing of 112 reactions of types $\pi p \to \pi\Delta$, ρN, A_2N, $\rho\Delta$ with their isospin relations. Their transversity amplitudes can be reconstructed completely or up to one phase) with transversally polarized target

a) 34 reactions of type $\pi\Delta$†. Similar reactions of type $\rho\Delta$ are obtained by changig $(\pi\eta\eta'K) \to \rho\omega\varphi K^*)$†† In both cases the amplitudes can be completely reconstructed.

$\pi N \uparrow \to$	π	Δ		$\eta(\eta')$	Δ	
π^+p	+	+	$(-\sqrt{2}\, d_1)$	0	++	$(\sqrt{3}\, f)$
	0	++	$(\sqrt{3}\, d_1)$			
π^-p	+	−	(d_2)			
	0	0	(d_3)	0	0	(f)
	−	+	(d_4)			
π^+n	+	0	$(-d_4)$			
	0	+	$(-d_3)$	0	+	(f)
	−	++	$(-d_2)$			
π^-n	0	−	$(-\sqrt{3}\, d_1)$			
	−	0	$(\sqrt{2}\, d_1)$	0	−	$(\sqrt{3} f)$

$KN \uparrow \to$	K	Δ	
K^+p	+	+	(f)
	0	++	$(-\sqrt{3} f)$
K^+n	+	0	(f)
	0	+	$(-f)$
K^0p	+	0	(f)
	0	+	$(-f)$
K^0n	+	−	$(\sqrt{3} f)$
	0	0	$(-f)$

$\bar{K}N \uparrow \to$	$\bar{K}$	Δ	
K^-p	0	0	(f)
	−	+	$(-f)$
K^-n	0	−	$(\sqrt{3}\, f)$
	−	0	$(-f)$
$\bar{K}^0p$	0	+	(f)
	−	++	$(-\sqrt{3}\, f)$
$\bar{K}^0n$	0	0	(f)
	−	+	$(-f)$

† For their measurement and amplitude reconstruction, cf. table 7.
†† For their measurement and amplitude reconstruction, c.f. table 14.

Table 3 — cont'd

b) 22 reactions of type ρN†††. Similar reactions of type A_2N are obtained by changing ($\rho\omega\varphi$K*) → (A_2 ff′K**)††††. In both cases the transversity amplitudes can be reconstructed up to one phase.

$\pi N \uparrow \to$	ρ N		$\omega(\varphi)$ N	
π^+p	+ +	(a_1)		
π^-p	0 0	(a_3)	0 0	(f)
	− +	(a_2)		
π^+n	+ 0	(a_2)	0 +	$(-f)$
	0 +	(a_3)		
π^-n	− 0	(a_1)		

KN $\uparrow \to$	K* N	
K$^+$p	+ +	(c_3)
Kn	+ 0	(c_2)
	0 +	(c_1)
K^{0}p	+ 0	(c_1)
	0 +	(c_2)
K^{0}n	0 0	(c_3)

$\overline{\text{K}}N \uparrow \to$	$\overline{\text{K}}$* N	
K$^-$p	0 0	(c_1)
	− +	(c_2)
K$^-$n	− 0	(c_3)
$\overline{\text{K}}^0$p	0 +	(c_3)
$\overline{\text{K}}^0$n	0 0	(c_2)
	− +	(c_1)

c) Triangular isospin relations between the amplitudes of the reactions in a) and b)

$$a_1 = a_2 + \sqrt{2a_3} \qquad c_1 + c_2 = c_3 \qquad d_1 = -\sqrt{\frac{2}{3}}\, d_2 - d_3 = -\sqrt{\frac{1}{6}}\, d_2 + \sqrt{\frac{1}{2}}\, d_4 = d_3 + \sqrt{2}\, d_4$$

††† For their measurement and amplitude reconstruction, cf. table 10.
†††† For their measurement and amplitude reconstruction, cf. table 12.

2. Some Basic Tools of Quasi Two Body Reaction Analysis

In this section we present some well known features of the formalism used in the study of quasi two body reactions. For more details one refers to [*10, 11, 12, 13*].
Let us first precise our notations. A quasi two body reaction will be denoted by

$$1 + 2 \to 3 + 4 .$$

1 denotes the beam, 2 the target, and we call 3 and 4 respectively the particles which share some physical properties with the beam and the target (for instance 1 and 3 are mesons, 2 and 4 are baryons) so that the t- and u-channels are well defined. The 4-momenta of the particles are denoted by $p_i (i = 1, 2, 3, 4)$ with $p_1 + p_2 = p_3 + p_4$, their spins by j_i and their masses by m_i $(m_i \neq 0)$.

2.1. Covariant quantization systems

To describe the polarization of the initial and final states one must fix a quantization frame for each spinning particle. Several different choices are possible, the most popular are the helicity and transversity frames in the s-, t- and u-channels[3]). Unfortunately there is not as yet a universal agreement on the definition of these quantization frames; the most usual conventions are the following.

i) For each particle and for each channel, the transversity quantization axis ${}^T n^{(3)}$ and the helicity second axis ${}^H n^{(2)}$ are along the "Basel normal" n to the reaction plane, defined by

$$n \cdot p_i = 0 \quad (i = 1, 2, 3), \qquad n^2 = -1, \quad \det(n, p_1, p_2, p_3) > 0 \tag{2.1}$$

where the last condition is equivalent to $\boldsymbol{n} \cdot \boldsymbol{p}_1 \times \boldsymbol{p}_3 > 0$ in the laboratory system, or in the center of mass system.

ii) For each particle and for each channel the helicity and the transversity frames have the same first axis, ${}^T n^{(1)} = {}^H n^{(1)}$.
Note that with these two conventions, the transversity second axis ${}^T n^{(2)}$ and the helicity quantization axis ${}^H n^{(3)}$ have oposite directions.[4]) Furthermore the transversity frame is transformed into the helicity frame by a rotation[5]) $\tilde{R} = (-\pi/2, \pi/2, \pi/2)$ of $+\pi/2$ around the common axis $n^{(1)}$. The unitary representations $D^j(\tilde{R})$ of this rotation have several useful properties (cf. Ref. [*13*], [*14*]).

iii) For each particle i ($i = 1, 2, 3, 4$) and for each channel a ($a = s, t, u$), the helicity quantization axis ${}^H_a n_i^{(3)}$ and the transversity second axis ${}^T_a n_i^{(2)}$ are defined by

$$ {}^H_a n_i^{(3)} = -{}^T_a n_i^{(2)} = \varepsilon q_i(a), \qquad \varepsilon = \pm 1, \tag{2.2}$$

with

$$q_i(a) = [\sinh \phi_i(a)]^{-1} (\hat{p}_i \cosh \phi_i(a) - \hat{p}_{ai}) \tag{2.3}$$

where $\hat{p}_i = p_i/m_i$, $\cosh \phi_i(a) = \hat{p}_i \cdot \hat{p}_{ai}$, $\sinh \phi_i(a) > 0$, and where ai is the particle associated to particle i in the channel a, i.e., for $i = (1, 2, 3, 4)$, $si = (2, 1, 4, 3)$, $ti = (3, 4, 1, 2)$, $ui = (4, 3, 2, 1)$[6]).

iv) The sign ε in eq. (2.2) is a last convention to be chosen. Jacob and Wick [*17*] and Cohen-Tannoudji, Morel and Navelet [*18*] define $\varepsilon = +1$ so that the s-helicity quantization axis of particle i, ${}^H_s \boldsymbol{n}_i^{(3)}$, be along the 3-momentum $\boldsymbol{p}_i$ of this particle, in the center of mass system ($\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0 = \boldsymbol{p}_2 + \boldsymbol{p}_4$). On the contrary, Gottfried and Jackson [*19*] define $\varepsilon = -1$ so that their t-helicity quantization axis ${}^H_t \boldsymbol{n}_i^{(3)}$, in the rest system of particle i, is along the 3-momentum $\boldsymbol{p}_{ti}$ of the particle ti associated to i in the t-channel.

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[3]) Cf. refs. [*10, 11*].

[4]) This means that in any channel the x, y, z axes are related by $({}^T x, {}^T y, {}^T z) = {}^H x, -{}^H z, {}^H y)$. Ref. [*14*] and the Cracow group use the same convention although a misprint in ref. [*14*] says the contrary. Many other conventions are occasionally used. Ref. [*2*] and [*3*] use $(-{}^T x, {}^T y, {}^T z) = ({}^H x, {}^H z, {}^H y)$. Ref. [*8*] uses $({}^T x, {}^T y, {}^T z) = ({}^H z, {}^H x, {}^H y)$. Ref. [*15*] compares the conventions of ref. [*8*] and that of the text and it recommends the latter.

[5]) We use the Euler angles and the rotation matrices of Rose [*16*].

[6]) We call these frames the s-, t-, or u-helicity frames and the s-, t-, or u-transversity frames. Some people use the following vocabulary for helicity frames: t-helicity frame = Jackson-frame, s-helicity frame = helicity frame; and they extend it to transversity frames: t-transversity frame = Jackson transversity frame, s-transversity frame = helicity transversity frame! We find that this last expression is an awful barbarism since helicity and transversity are two mutually exclusive notions.

For convenience in this paper we quantize the spin of the target in the s-channel transversity frame with $\varepsilon = +1$, i.e., in the laboratory system the second axis $\boldsymbol{n}^{(2)}$ is in the direction of the beam 3-momentum $\boldsymbol{p}_1$. Then the longitudinal polarization of the target (with respect to the beam) is in the y-direction, and the transverse polarization is in the (x, z) plane with the z-direction along the normal to the reaction plane (cf. Fig. 1).
For the final particles we quantize the spins in a transversity frame. We do not precise the channel since all subsequent equations are independent of this choice of channel.

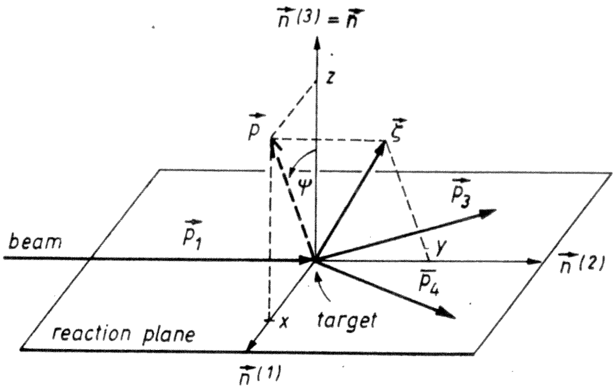

Fig. 1. The s-transversity frame for the target quantization (read $\vec{l}$ instead of $\vec{p}$)

2.2. Amplitudes

At fixed energy and momentum transfer, the Hilbert space $\mathscr{H}_e = \mathscr{H}_1 \otimes \mathscr{H}_2$ of initial particles has $(2j_1 + 1)(2j_2 + 1)$ dimensions. The transition operator for the reaction is a linear map T between the two spaces. The transversity and helicity amplitudes are the $\mathscr{N} = \prod_{i=1}^{4} (2j_i + 1)$ matrix elements of the transition operator in the transversity and helicity bases. For channel a they are respectively denoted by ${}_aT^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}$ and ${}_aH^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}$. They are related by

$$ {}_aT = [D^{j_3}(\tilde{R}) \otimes D^{j_4}(\tilde{R})]\, {}_aH [D^{j_1}(\tilde{R})^\dagger \otimes D^{j_2}(\tilde{R})^\dagger]. \tag{2.4} $$

The reflection through the reaction plane, called Bohr-symmetry or B-symmetry, leaves invariant the 4-momenta of the four particles. It acts on the polarization space of a spin-parity j^η particle by the operator $B(j) = \eta D^j(n, \pi)$, the product of the parity η of the particle times the unitary representation of the rotation by π around the normal to the reaction plane. If parity is conserved in the reaction, the transition operator is invariant by B-symmetry i.e.

$$ [B(j_3) \otimes B(j_4)]\, T [B(j_1)^\dagger \otimes B(j_2)^\dagger] = T, \tag{2.5} $$

and the matrix elements of T satisfy the relations

$$ \eta(-1)^{\lambda_3+\lambda_4-\lambda_1-\lambda_2}\, T^{\lambda_3\lambda_4}_{\lambda_1\lambda_2} = T^{\lambda_3\lambda_4}_{\lambda_1\lambda_2} \tag{2.6a} $$

$$ \eta(-1)^{j_3-\lambda_3+j_4-\lambda_4+j_1-\lambda_1+j_2-\lambda_2}\, H^{-\lambda_3-\lambda_4}_{-\lambda_1-\lambda_2} = H^{\lambda_3\lambda_4}_{\lambda_1\lambda_2} \tag{2.6b} $$

where η is the relative parity of the particles, i.e., $\eta = \eta_1\eta_2\eta_3\eta_4$ with η_i = parity of particle i.

In the collinear case of forward or backward scattering, the transition operator must be invariant under rotation around the reaction axis. This imposes "collinearity constraints" among its matrix elements, which are most simply written in the helicity basis

$$H_{\lambda_1\lambda_2}^{\lambda_3\lambda_4} = 0, \quad \text{for} \quad \lambda_1 + \lambda_2 \neq \lambda_3 + \lambda_4 \tag{2.6c}$$

2.3. Density matrices

For a single particle of spin j and fixed 4-momentum, the polarization state is described by a Hermitian, positive, trace one operator acting on the Hilbert space $\mathscr{H}(j)$ and represented by a matrix $\varrho(j)$. For a system of two particles (spins j and j') of fixed 4-momenta, the polarization operator acts on the space $\mathscr{H}(j) \otimes \mathscr{H}(j')$ and is represented by the joint density matrix $\varrho(j, j')$. When the particles are uncorrelated, this matrix can be written in the form of a tensor product $\varrho(j) \otimes \varrho(j')$.

For later use, especially for the study of decay angular distributions, it is useful to introduce the polarization multipole parameters of these density matrices.

i) The single particle density matrix $\varrho(j)$ is expanded on a set of basis matrices $T(j)_M^L$. ($L = 0, \ldots, 2j$; $M = -L, \ldots, +L$), the matrix elements of which are Clebsch-Gordan coefficients

$$\left(T(j)_M^L\right)_{\lambda'}^{\lambda} = \langle jL\lambda' M \mid j\lambda\rangle. \tag{2.7}$$

The multipole expansion of $\varrho(j)$ reads

$$\varrho(j) = \frac{1}{2j+1}\left[\mathbf{1} + \sum_{L=1}^{2j}(2L+1)\sum_{M=-L}^{+L}\overline{t_M^L}\,T(j)_M^L\right], \tag{2.8}$$

the expansion coefficients $t_M{}^L$ are the multipole parameters. We have exhibited the trace of the matrix. We could have written an expansion from $L = 0$ to $2j$, with $T(j)_0{}^0 = 1$ and $t_0{}^0 = 1$.

ii) A similar multipole expansion can be written for the joint density matrix $\varrho(j, j')$. The set of basis matrices is the tensor products $T(j)_M^L \otimes T(j')_{M'}^{L'}$ and the corresponding multipole parameters are denoted by $t_{MM'}^{LL'}$. The expansion is

$$\varrho(j, j') = \sum_{L=0}^{2j}\sum_{L'=0}^{2j'}\frac{(2L+1)(2L'+1)}{(2j+1)(2j'+1)}\sum_{M=-L}^{+L}\sum_{M'=-L'}^{+L'}\overline{t_{MM'}^{LL'}}\,T(j)_M^L \otimes T(j)_{M'}^{L'}. \tag{2.9}$$

Note that the multipole parameters $t(j)_M{}^L$ and $t(j')_{M'}^{L'}$ of the single particle density matrices[7] $\varrho(j) = \text{tr}_{j'}\,\varrho(j, j')$ and $\varrho(j') = \text{tr}_j\,\varrho(j, j')$ are $t(j)_M^L = t_{M0}^{L0}$ and $t(j')_{M'}^{L'} = t_{0M'}^{0L'}$.

A density matrix ϱ can be split into a B-symmetric part ϱ^B and a B-antisymmetric part ϱ^A satisfying the conditions

$$B\varrho^B B^\dagger = \varrho^B, \qquad B\varrho^A B^\dagger = -\varrho^A.$$

i) For a single particle, in transversity quantization, the matrix elements of ϱ^B and ϱ^A satisfy the relations

$$(\varrho^B)_{\lambda'}^{\lambda} = (-1)^{\lambda-\lambda'}(\varrho^B)_{\lambda'}^{\lambda}, \qquad (\varrho^A)_{\lambda'}^{\lambda} = -(-1)^{\lambda-\lambda'}(\varrho^A)_{\lambda'}^{\lambda} \tag{2.10}$$

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[7]) tr_j (and $\text{tr}_{j'}$) represent the partial trace in the space $\mathscr{H}(j)$ (and $\mathscr{H}(j')$), e.g., $\varrho(j)_\nu{}^\mu = (\text{tr}_{j'}\,\varrho(j, j'))_\nu{}^\mu = \sum_\lambda \varrho(j, j')_{\nu\lambda}{}^{\mu\lambda}$.

and the multipole parameters satisfy the conditions

$$t_M^{LB} = (1)^M\, t_{\bar{M}}^{LB}, \qquad t_M^{LA} = -(-1)^M\, t_{\bar{M}}^{LA}. \tag{2.11}$$

ii) For a joint density matrix, in transversity quantization, the matrix elements satisfy

$$(\varrho^{B,A})^{\lambda\mu}_{\lambda'\mu'} = \pm(-1)^{\lambda+\mu-\lambda'-\mu'}\,(\varrho^{B,A})^{\lambda\mu}_{\lambda'\mu'}, \begin{cases} + \text{ for } B \\ - \text{ for } A \end{cases} \tag{2.12}$$

and the joint multipole parameters satisfy

$$(t^{LL'}_{MM'})^{B,A} = \pm(-1)^{M+M'}\,(t^{LL'}_{\bar{M}\bar{M}'})^{B,A} \begin{cases} + \text{ for } B \\ - \text{ for } A. \end{cases} \tag{2.13}$$

In a quasi two body reaction with spin zero beam and spin 1/2 target, the transition matrix has 2 columns ($T^{\lambda_3\lambda_4}_{0\pm 1/2}$). If the target is unpolarized, the density matrix of the final particles, at fixed energy and momentum transfer, is obtained by[8])

$$\sigma\varrho_f = \frac{1}{2}\,TT^\dagger \tag{2.14}$$

(where σ is the differential cross section) and ϱ_f has rank 2. If furthermore T is B-symmetric, ϱ_f is B-symmetric too and in transversity quantization it can be written in the form of a direct sum $\varrho_f = \varrho_1 \oplus \varrho_2$, each of which has rank one. This property enforces very strong relations between the matrix elements or the multipole parameters of ϱ_f.

2.4. Decay angular distributions

If the final particles of the quasi two body reaction are unstable, the angular distribution of their decay products (and occasionally the cascade angular distribution) provides some information on their polarization state. In this paper we limit ourselves to two body decays, however we give some results for 3 body decays in sect. 2.4.4.

2.4.1. Single decay angular distribution

Assume first that only one of the final particles is unstable and let j be the spin of this particle ($j = j_3$ or $j = j_4$). In the rest system of this particle the kinematics of the decay is determined by the polar angle θ and azimuthal angle ϕ of one of the decay products with respect to the quantization frame of the decaying particle. If M is the decay operator, the normalized angular distribution is defined by

$$I(\theta, \phi) = \operatorname{tr} M\varrho(j)\, M^\dagger \,/ \int \operatorname{tr} M\varrho(j)\, M^\dagger d(\cos\theta)\, d\phi. \tag{2.15}$$

This angular distribution is linear in the multipole parameters, it can be written

$$I(\theta, \phi) = \frac{1}{4\pi} + \sum_{L=1}^{2j} C(L) \sum_{M=-L}^{+L} \overline{t^L_M}\, Y^L_M(\theta, \phi) \tag{2.16}$$

where the coefficients $C(L)$ depend on the spins of the decay products and on the dynamics of the decay. If parity is conserved in the decay (e.g. $\rho \to \pi\pi \quad \Delta \to N\pi$) the coeffi-

[8]) This relation fixes the normalization of T.

cients $C(L)$ vanish for $L = $ odd; the angular distribution is an analyser of the even polarization. If parity is violated in the decay (e.g. $\Lambda \to p\pi$, $\Omega \to \Xi\pi$) no $C(L)$ coefficient vanishes; the angular distribution is an analyser of the complete polarization. In many cases angular momentum conservation implies that only one amplitude contributes to the decay. Then the coefficients $C(L)$ are pure numbers. If two amplitudes contribute they depend on one dynamical parameter. Here are the values of the coefficients $C(L)$ for some usual decays

$$\begin{aligned} &1^- \to 0^-0^- : \sqrt{4\pi}\, C(2) = -\sqrt{2}\,, \\ &2^+ \to 0^-0^- : \sqrt{4\pi}\, C(2) = -\sqrt{10/7}\,, \qquad \sqrt{4\pi}\, C(4) = \sqrt{18/7}\,, \\ &2^+ \to 1^-0^- : \sqrt{4\pi}\, C(2) = -\sqrt{5/14}\,, \qquad \sqrt{4\pi}\, C(4) = -\sqrt{8/7}\,, \\ &\frac{1}{2} \to \frac{1}{2}\, 0 : \sqrt{4\pi}\, C(1) = \alpha\,, \\ &\frac{3^+}{2} \to \frac{1^+}{2}\, 0^- : \sqrt{4\pi}\, C(2) = -1\,, \end{aligned} \tag{2.17}$$

where α is the asymmetry parameter of the parity violating decay $1/2 \to 1/2\ 0$. With these known values of the $C(L)$ coefficients, the t^L_M are obtained by a maximum likelihood analysis of the angular distribution, or by a moment analysis which yields $\big(\Omega = (\theta, \phi),\ d\Omega = d(\cos\theta)\, d\phi\big)$

$$C(L)\, t^L_M = \langle Y^L_M(\Omega)\rangle \equiv \int I(\Omega)\, Y^L_M\, d\Omega\,. \tag{2.18}$$

Experimentally the moments $\langle Y^L_M(\Omega)\rangle$ of the angular distribution are the values of the spherical harmonics $Y^L_M(\theta, \phi)$ for all events in an ensemble of fixed energy and momentum transfer

$$\langle Y^L_M(\Omega)\rangle = \frac{1}{N} \sum_{i=1}^{N} Y^L_M(\theta_i, \phi_i)$$

where the index i specifies the event which is considered, and N is the total number of events in the ensemble.

2.4.2. Joint decay angular distribution

If both final particles of the quasi two body reaction are unstable, one may study the correlations between the directions of their decay products. Let j and j' be the spins of the final particles. The joint decay angular distribution reads

$$I(\theta, \phi; \theta', \phi') = \sum_{L=0}^{2j} \sum_{L'=0}^{2j'} C(L)\, C'(L') \sum_{M=-L}^{+L} \sum_{M'=-L'}^{+L'} \overline{t^{LL'}_{MM'}}\, Y^L_M(\theta, \phi)\, Y^{L'}_{M'}(\theta', \phi') \tag{2.19}$$

where $C(L)$ and $C'(L')$ are the coefficients of the single decays, with $C(0) = C'(0) = 1/\sqrt{4\pi}$. These coefficients being known, the parameters $t^{LL'}_{MM'}$ are obtained by a best fit analysis of the angular distribution or by a moment analysis $\big(\Omega = (\theta, \phi),\ \Omega' = (\theta, \phi'),\ d\Omega = d(\cos\theta)\, d\phi,\ d\Omega' = d(\cos\theta')\, d\phi'\big)$

$$C(L)\, C'(L')\, t^{LL'}_{MM'} = \langle Y^L_M(\Omega)\, Y^{L'}_{M'}(\Omega')\rangle \equiv \int I(\Omega, \Omega')\, Y^L_M(\Omega)\, Y^{L'}_{M'}(\Omega')\, d\Omega\, d\Omega'\,. \tag{2.20}$$

2.4.3. Cascade decay angular distribution

Consider the cascade decay $C \to A + B$, $A \to A_1 + B_1$ (with spins $j(C) = j$, $j(A) = j(A_1) = 1/2$, $j(B) = j(B_1) = 0$); the first decay is parity conserving and the second decay is parity violating. (e.g. $\Sigma^* \to \Lambda\pi$, $\Lambda \to p\pi$). We denote by θ and ϕ the angles of A with respect to the quantization frame of C, and by θ_1 and ϕ_1 the angles of A_1 with respect to the canonical quantization frame for A, deduced from the quantization frame of C by a pure Lorentz transformation (boost). Then the cascade angular distribution is

$$I(\theta, \phi; \theta_1, \phi_1) = \sum_{L=0}^{2j} \sum_{L_1=0}^{1} \sum_{J\,\text{even}} C(L, L_1, J) \sum_{M,N,M_1} \langle J L_1 N M_1 \mid L M\rangle \times \overline{t_M^L}\, Y_N^J(\theta, \phi)\, Y_{M_1}^{L_1}(\theta_1, \phi_1). \tag{2.21a}$$

Instead of the canonical quantization frame for particle A one may use the helicity frame, deduced from the previous one by the rotation $R(\phi, \theta, 0)$. We denote by $\theta_1{}^h$, $\phi_1{}^h$ the angles of A_1 with respect to this frame, then the cascade angular distribution is

$$I(\theta, \phi, \theta_1{}^h, \phi_1{}^h) = \sum_{L=0}^{2j} \sum_{L_1=0}^{1} C^h(L\; L_1, M_1) \sum_{M,M_1} \overline{t_M^L} \sqrt{\frac{2L+1}{4\pi}}\, D^L(\phi, \theta, 0)_M^{M_1}\, Y_{M_1}^{L_1}(\theta_1{}^h, \phi_1{}^h). \tag{2.21b}$$

The coefficients $C(L, L_1, J)$ in Eq. (2.21a) and $C^h(L, L_1, M_1)$ in Eq. (2.21b) depend on the spins and parities of the particle and on the dynamics of the decays, if these involve more than one amplitude.
The most usual decay of this type is $3^+/2 \to 1^+/2\; 0^-$, $1^+/2 \to 1^+/2\; 0^-$. For this cascade decay the non vanishing coefficients $C(L, L_1, J)$ and $C^h(L, L_1, M_1)$ are

$$\begin{aligned} &4\pi C(0, 0, 0) = 1, && 4\pi C(2, 0, 2) = -1 \\ &4\pi C(1, 1, 0) = \alpha \sqrt{5/9}, && 4\pi C(1, 1, 2) = \alpha \sqrt{2/45} \\ &4\pi C(3, 1, 2) = -\alpha \sqrt{7/5}, && \end{aligned} \tag{2.22a}$$

$$\begin{aligned} &(4\pi)\, C^h(0, 0, 0) = 1, && (4\pi)\, C^h(2, 0, 0) = -1 \\ &(4\pi)\, C^h(1, 1, 0) = \alpha \sqrt{1/15}, && (4\pi)\, C^h(1, 1, \pm 1) = \alpha \sqrt{4/15} \\ &(4\pi)\, C^h(3, 1, 0) = -\alpha \sqrt{3/5}, && (4\pi)\, C^h(3, 1, \pm 1) = -\alpha \sqrt{2/5} \end{aligned} \tag{2.22b}$$

where α is the asymmetry parameter of the second decay. With these known values of the coefficients, the t_M^L parameters are deduced by a best fit adjustment of the decay angular distribution or by a moment analysis $\left(\Omega = (\theta, \phi), \Omega_1 = (\theta_1, \phi_1), \Omega_1{}^h = (\theta_1{}^h, \phi_1{}^h)\right)$

$$C(L, L_1, J)\, t_M^L = \sum_{N,M_1} \langle J L_1 N M_1 \mid L M\rangle \langle Y_N^J(\Omega)\, Y_{M_1}^{L_1}(\Omega_1)\rangle, \tag{2.23a}$$

$$C^h(L, L_1, M_1)\, t_M^L = \sqrt{\frac{2L+1}{4\pi}} \sum_{M_1} \langle D^L(\phi, \theta, 0)_{M_1}^M\, Y_{M_1}^{L_1}(\Omega_1{}^h)\rangle. \tag{2.23b}$$

Note that in the above example, since for $L = 1$ one has two non vanishing coefficients, $C(1, 1, 0)$ and $C(1, 1, 2)$ or ($C^h(1, 1, 0)$ and $C^h(1, 1, \pm 1)$) the parameters $t_M{}^1$ can be measured by two different experimental expressions.

2.4.4. Three body decays

Some well known unstable resonances undergo 3 body decays, (e.g. $\eta \to \pi^0\pi^0\pi^0$ or $\pi^+\pi^-\pi^0$ $\omega \to \pi^+\pi^-\pi^0$ $\phi \to \pi^+\pi^-\pi^0$). The final state is determined by 5 quantities, often split into the 2 Dalitz plot variables and the 3 angles which fix the orientation of the decay plane. In the rest system of the decaying particle, let us denote by θ and ϕ the angles of the normal to the decay plane with respect to the quantization frame of the decaying particle. Then the angular distribution $I(\theta, \phi)$ can be written in the same form as eq. (2.16) with coefficients $C(L)$ depending on the spins and parities of the particles and on the dynamics of the decay. For the most usual decay, i.e., $1^- \to 0^-0^-0^-$ (e.g. ω and ϕ decays), the non vanishing $C(L)$ coefficient is a pure number

$$1^- \to 0^-0^-0^- : \sqrt{4\pi}\, C(2) = -\sqrt{2}, \tag{2.24}$$

which happens to be equal to the coefficient of the two body decay $1^- \to 0^-0^-$, (cf. eq. (2.17)).

3. Quasi Two Body Reactions with Polarized Target

In this section we study the observables of a quasi two body reaction with polarized target. In the particular case of a spin zero beam and a spin 1/2 polarized target we give explicitly the structure of the final state density matrix and of the differential cross section in terms of the initial polarization. We also show how to measure the multipole parameters which describe the *polarization transfer* between the initial and final states.

3.1. Observables

Consider a quasi two body reaction $1 + 2 \to 3 + 4$. The beam and the target are prepared independently hence the initial state is described by a (Hermitian, positive, trace one) density matrix which is a tensor product $\varrho_e = \varrho(j_1) \otimes \varrho(j_2)$. On the contrary, the polarizations of the final particles are generally correlated and the final state is described by a joint density matrix $\varrho_f = \varrho(j_3, j_4)$ which cannot generally be written in a tensor product form. These matrices are related through the transition matrix T by

$$\sigma\varrho_f = T\varrho_e T^\dagger . \tag{3.1}$$

We have denoted by σ the double differential cross section

$$\sigma \equiv \frac{d\sigma}{dt\, d\psi} = \operatorname{tr} T\varrho_e T^\dagger \tag{3.2}$$

where t is the momentum transfer and ψ is, in the laboratory system, the angle between the Basel normal $\boldsymbol{n}$ and a direction $\boldsymbol{l}$, perpendicular to the beam direction, fixed by the initial polarization. If the initial state is unpolarized, $\varrho_e = \mathbf{1}/n_1 \otimes \mathbf{1}/n_2$ with $n_i = (2j + 1)$, the double differential cross section is denoted by σ_0

$$\sigma_0 \equiv \left.\frac{d\sigma}{dt\, d\psi}\right|_{\varrho_e = \mathbf{1}/n_1 n_2} = \frac{1}{n_1 n_2} \operatorname{tr} TT^\dagger . \tag{3.3}$$

In this case, the initial state has no preferential direction $\boldsymbol{l}$ in the laboratory and the double differential cross section σ_0 is isotropic in ψ. Then one may consider the simple differential cross section

$$\frac{d\sigma}{dt} = \int_0^{2\pi} \sigma_0 \, d\psi = 2\pi\sigma_0 . \tag{3.4}$$

At fixed energy and momentum transfer, a complete measurement of the reaction includes the measurement of the double differential cross section σ and of the joint final density matrix ϱ_f, as functions of the initial polarization ϱ_e and of the angle ψ. We call *observables* of the reaction the set of quantities which parametrize these functions and can be effectively measured. The measure is obtained by an analysis of the differential cross section and of the combined angular distribution of the normal $\boldsymbol{n}$ and of the decay products of the final particles (cf. section 3.3. below) for different initial polarizations.

3.2. Description of the final state when the target is polarized

From now on we assume that the initial state consists simply of beam of spin zero particles and a target of spin 1/2 particles.

3.2.1. The initial state

The initial density matrix ϱ_e is a 2×2 matrix, which in the laboratory system is described by the polarization pseudo vector $\boldsymbol{\zeta}(\zeta^2 \leqq 1)$, also called the Stokes vector. The projection of this vector on a plane perpendicular to the beam fixes the direction $\boldsymbol{l}$ alluded to previously. As we defined it in section 2.1., for s-transversity quantization, in the laboratory system the $\boldsymbol{n}^{(2)}$ axis of the target is in the direction of the beam momentum $\boldsymbol{p}_1$, while $\boldsymbol{n}^{(3)}$ is along the Basel normal $\boldsymbol{n}$ to the reaction plane and $\boldsymbol{n}^{(1)}$ is perpendicular to the beam and to this normal. Then the density matrix ϱ_e, in s-transversity quantization, is

$$\varrho_e = \frac{1}{2}\,(\mathbf{1} + x\tau_x + y\tau_y + z\tau_z) \tag{3.5}$$

where τ_x, τ_y, τ_z are the Pauli matrices, and x, y, z are the projections of the vector $\boldsymbol{\zeta}$ on the s-transversity axes $\boldsymbol{n}^{(1)}$, $\boldsymbol{n}^{(2)}$, $\boldsymbol{n}^{(3)}$ respectively. These components can be written

$$\boldsymbol{\zeta} : x = P_T \sin\psi , \qquad y = P_L , \qquad z = P_T \cos\psi \tag{3.6}$$

where ψ is the angle between $\boldsymbol{n}$ and $\boldsymbol{l}$ with the sign of $\boldsymbol{n} \times \boldsymbol{l} \cdot \boldsymbol{p}_1$, see fig. 1. By definition P_T is the length of the projection of $\boldsymbol{\zeta}$ on the (x, z) plane; it is the degree of transverse polarization $0 \leqq P_T \leqq 1$. P_L is the projection of $\boldsymbol{\zeta}$ on the beam; it may be positive or negative and its modulus $|P_L|$ is the degree of longitudinal polarization $0 \leqq |P_L| \leqq 1$. Note that $P_T{}^2 + P_L{}^2 = \zeta^2$ is the degree of polarization of the target. It is important to remark that in general the initial state is not B-symmetric. Indeed the matrices $\mathbf{1}$ and τ_z are B-symmetric in transversity quantization but τ_x and τ_y are not. Then, except in the case of normal polarization, $\boldsymbol{\zeta} \simeq \boldsymbol{n}$ the initial state is not invariant by reflection through the reaction plane.

3.2.2. The density matrix of the final state

The density matrix ϱ_f computed from eq. (3.1) with the initial state (3.5) is linear in the components of $\boldsymbol{\zeta}$. It can be written in the form

$$\sigma\varrho_f = \sigma_0(\varrho_0 + z\varrho_z + x\varrho_x + y\varrho_y), \tag{3.7}$$

where ϱ_0 is the density matrix of the final state when the target is unpolarized ($\boldsymbol{\zeta} = 0$). If parity is conserved in the reaction the transition matrix T is B-symmetric and since the matrices $\mathbf{1}$ and τ_z are B-symmetric, the matrices ϱ_0 and ϱ_z are B-symmetric too and have non vanishing trace while the matrices ϱ_x and ϱ_y are B-antisymmetric and hence traceless. The density matrix ϱ_0 has trace 1 whereas the trace of ϱ_z depends on the dynamics of the reaction; this trace, P_R, is sometimes called the "reaction polarization"

$$\operatorname{tr}\varrho_0 = 1, \qquad \operatorname{tr}\varrho_z = P_R, \qquad \operatorname{tr}\varrho_x = \operatorname{tr}\varrho_y = 0. \tag{3.8}$$

By definition the 4 matrices $\sigma_0\varrho_\alpha = 1/2T\tau_\alpha T^\dagger(\alpha = 0, x, y, z, \tau_0 = \mathbf{1})$ are not independent of each other. It is easy to show that the so-called "polarization transfer matrix" (cf. Appendix 1 and ref. [20])

$$W = \sum_\alpha \tilde{\tau}_\alpha \otimes \sigma_0\varrho_\alpha = \sigma_0 \left|\begin{array}{c|c} \varrho_0 + \varrho_z & \varrho_x + i\varrho_y \\ \hline \varrho_x - i\varrho_y & \varrho_0 - \varrho_z \end{array}\right| \tag{3.9}$$

($\sim$ = transposition in the initial space) must be positive and have rank 1, since σW can be written

$$W = \frac{1}{2}\tilde{T}\tilde{T}^\dagger \tag{3.9'}$$

where $\tilde{T}$ is the column matrix obtained from the transition matrix T by transposition in the initial space, i.e., its elements are

$$\tilde{T}^{\lambda_1\lambda_2\lambda_3\lambda_4} = \tilde{T}^{\lambda_3\lambda_4}_{\lambda_1\lambda_2}.$$

The matrix W is B-symmetric; in transversity it can be written in the form of a direct sum $W = W_1 \oplus W_2$. The line and column indices of W_1 satisfy $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 =$ even, those of W_2 satisfy $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 =$ odd. The rank 1 condition on W implies that either $W_2 = 0$ and rank $W_1 = 1$ or $W_1 = 0$ and rank $W_2 = 1$. Which submatrix is nul depends on the relative parity of the particles. From eq. (2.6a), if $\eta = +1$, $W_2 = 0$, if $\eta = -1$, $W_1 = 0$. The nullity of W_1 (or W_2) yields linear constraints between the elements of ϱ_0 and ϱ_z and of ϱ_x and ϱ_y, while the rank 1 condition for W_2 (or W_1) gives quadratic constraints between the elements of all matrices. Often the decay of the final particles does not allow a complete measurement of the matrices W_1 and W_2. Then one may obtain constraints on the observable parameters by elimination of the unobserved quantities from the previous equations. This elimination keeps the degree of the linear constraints, but it generally raises the degree of the quadratic constraints.

3.2.3. Multipole expansions

i) If only the final particle 4 has spin ($j_3 = 0$, $j_4 = j$), the density matrix ϱ_f is a single particle density matrix. Its multipole expansion is (cf. eq. (2.8))

$$\frac{\sigma}{\sigma_0}\varrho_f = \frac{1}{2j+1}\left[(1 + P_R z)\,\mathbf{1} + \sum_{L=1}^{2j}\sum_{M=-L}^{+L}(2L+1)\left\{\overline{t^L_M} + z\,\overline{{}^zt^L_M} + x\,\overline{{}^xt^L_M} + y\,\overline{{}^yt^L_M}\right\}T(j)^L_M\right], \tag{3.10}$$

where t_M^L and ${}^zt_M^L$ are the multipole parameters of the B-symmetric matrices ϱ_0 and ϱ_z, and ${}^xt_M^L$ and ${}^yt_M^L$ are the parameters of the B-antisymmetric matrices ϱ_x and ϱ_y. We have exhibited the trace $(1 + P_R z)$ of the matrix $\sigma/\sigma_0\, \varrho_f$; we could keep this term inside the summation $(L = 0, 2j)$, with the conventions $t_0{}^0 = 1$, ${}^zt_0{}^0 = P_R$.

ii) If both final particles 3 and 4 have spin $(j_3 = j,\ j_4 = j')$, the density matrix ϱ_f is a joint density matrix $\varrho_f(j, j')$. Its multipole expansion is (cf. eq. (2.9)).

$$\frac{\sigma}{\sigma_0}\,\varrho_f = \sum_{L=0}^{2j} \sum_{L'=0}^{2j'} \frac{(2L+1)(2L'+1)}{(2j+1)(2j'+1)} \sum_{M=-L}^{+L} \sum_{M'=-L'}^{+L'} \{\overline{t_{MM'}^{LL'}} + z\overline{{}^zt_{MM'}^{LL'}}$$

$$+ x\overline{{}^xt_{MM'}^{LL'}} + Y\overline{{}^yt_{MM'}^{LL'}}\}\, T(j)_M^L \otimes T(j')_{M'}^{L'} \tag{3.11}$$

where $t_{MM'}^{LL'}$, ${}^zt_{MM'}^{LL'}$, ${}^xt_{MM'}^{LL'}$, ${}^yt_{MM'}^{LL'}$ are the multipole parameters of the B-symmetric matrices ϱ_0 and ϱ_z and B-antisymmetric matrices ϱ_x and ϱ_y respectively, with the conventions $t_{00}^{00} = 1$ and ${}^zt_{00}^{00} = P_R$.
The multipole parameters ${}^\alpha t_M^L$ and ${}^\alpha t_{MM'}^{LL'}$ $(\alpha = x, y, z)$ are called "polarization transfer" multipole parameters.

3.3. Measurement of the observables of a reaction

If the final particles are unstable and undergo two body decays, the final state is characterized by the production angle ψ and by the decay angles θ, ϕ and θ', ϕ'.

3.3.1. The double differential cross sections

From the general form (3.7) of $\sigma\varrho_f$ and from the trace conditions (3.8), one gets

$$\sigma = \operatorname{tr} \sigma\varrho = \sigma_0(1 + P_R z). \tag{3.12}$$

Then from the value (3.6) of the z-component of the polarization vector ζ, the ψ dependence of the double differential cross section is

$$\sigma(\psi) = \sigma_0(1 + P_R P_T \cos\psi). \tag{3.13}$$

i) The unpolarized double differential cross section σ_0 may be obtained by several different ways

$$\sigma_0 = \sigma(\psi)|_{P_T=0} \tag{3.14a}$$

$$\sigma_0 = \sigma\left(\frac{\pi}{2}\right) \tag{3.14b}$$

$$\sigma_0 = \frac{1}{2}\left(\sigma(0) + \sigma(\pi)\right)$$

$$\sigma_0 = \frac{1}{2\pi}\langle\sigma(\psi)\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} \sigma(\psi)\, d\psi. \tag{3.14d}$$

One may verify that all these ways lead to the same result.

ii) Similarly the asymmetry $P_R P_T$ of the differential cross section can be obtained by several ways. We denote by $\sigma\uparrow$ (resp. $\sigma\downarrow$) the cross sections of the events with the polarization vector ζ above (resp. under) the reaction plane

$$\sigma\uparrow = \int_{-\pi/2}^{\pi/2} \sigma(\psi)\, d\psi\,, \qquad \sigma\downarrow = \int_{\pi/2}^{3\pi/2} \sigma(\psi)\, d\psi\,, \tag{3.15a}$$

and we use the notation

$$\langle f(\psi)\rangle = \int_0^{2\pi} f(\psi)\, d\psi\,. \tag{3.15b}$$

Then the asymmetry can be obtained by

$$P_R P_T = \frac{\sigma(0) - \sigma(\pi)}{\sigma(0) + \sigma(\pi)} \tag{3.16a}$$

$$P_R P_T = \frac{\pi}{2}\,\frac{\sigma\uparrow - \sigma\downarrow}{\sigma\uparrow + \sigma\downarrow} \tag{3.16b}$$

$$P_R P_T = \frac{\langle \sigma(\psi)\, 2\cos\psi\rangle}{\langle\sigma(\psi)\rangle}\,. \tag{3.16c}$$

3.3.2. Production and single decay angular distribution

Assume first that only one final particle undergo a two body decay (e.g. $\pi p \to \pi\Delta$, $\Delta \to \pi N$; or $\pi p \to \rho N$, $\rho \to \pi\pi$). Then the final state is characterized by 3 angles ψ, θ, ϕ. Les us call T the reaction transition matrix and $\boldsymbol{M}$ the decay transition matrix. The normalized combined angular distribution is defined by

$$I(\psi; \theta, \phi) = \frac{\operatorname{tr} MT\varrho_e T^\dagger M^\dagger}{\int \operatorname{tr} MT\varrho_e T^\dagger M^\dagger\, d(\cos\theta)\, d\phi\, d\psi}\,. \tag{3.17}$$

From eq. (3.1) this can be written

$$I(\psi; \theta, \phi) = \frac{\operatorname{tr} M \dfrac{\sigma}{\sigma_0}\, \varrho_f M^\dagger}{\displaystyle\int \operatorname{tr} M \frac{\sigma}{\sigma_0}\, \varrho_f M^\dagger\, d(\cos\theta)\, d\phi\, d\psi}\,. \tag{3.18}$$

Then, by comparison of the multipole expansion of $\sigma/\sigma_0\, \varrho_f$ (cf. eq. (3.10)) with the expansion (2.8) and from the usual decay angular distribution (2.16) one gets the combined normalized angular distribution

$$I(\psi; \theta, \phi) = \frac{1}{2\pi}\left[\frac{1 + P_R P_T \cos\psi}{4\pi} + \sum_{L=1}^{2j} C(L) \sum_{M=-L}^{+L} \left\{\overline{t^L_M} + P_T \cos\psi\, \overline{{}^z t^L_M}\right.\right.$$
$$\left.\left. + P_T \sin\psi\, \overline{{}^x t^L_M} + P_L \overline{{}^y t^L_M}\right\} Y^L_M(\theta, \phi)\right] \tag{3.19}$$

where the coefficients $C(L)$ are defined in section 2.4 and are, for the most usual decays, well known numerical coefficients (cf. eq. (2.17)). By inspection of this expression one sees that it allows the measure of the quantities $P_R P_T$, $t^L_M + P_L {}^y t^L_M$, $P_T {}^z t^L_M$, $P_T {}^x t^L_M$, either by a best fit adjustment or by a moment analysis which yields

$$P_R P_T = \langle 2\cos\psi\rangle \equiv \int I(\psi;\Omega)\, 2\cos\psi\, d\Omega\, d\psi \tag{3.20a}$$

$$C(L)\,(t^L_M + P_L{}^y t^L_M) = \langle Y^L_M(\Omega)\rangle \equiv \int I(\psi;\Omega)\, Y^L_M(\Omega)\, d\Omega\, d\psi \tag{3.20b}$$

$$C(L)\, P_T{}^z t^L_M = \langle 2\cos\psi\, Y^L_M(\Omega)\rangle \equiv \int I(\psi;\Omega)\, Y^L_M(\Omega)\, 2\cos\psi\, d\Omega\, d\psi \tag{3.20c}$$

$$C(L)\, P_T{}^x t^L_M = \langle 2\sin\psi\, Y^L_M(\Omega)\rangle \equiv \int I(\psi;\Omega)\, Y^L_M(\Omega)\, 2\sin\psi\, d\Omega\, d\psi \tag{3.20d}$$

with $\Omega = (\theta, \phi)$ and $d\Omega = d(\cos\theta)\, d\phi$. Note that eq. (3.20a) is a particular case of eq. (3.20c) for $L = 0$ (with the conventions ${}^z t_0{}^0 = P_R$, $C(0) = 1/\sqrt{4\pi}$) and is equivalent to eq. (3.16c). By different choices of the initial polarization ζ, i.e. of P_T and P_L, one easily deduces from these equations the value of the observables:

i) The target is unpolarized, i.e., $P_L = P_T = 0$. One must first verify that the angular distribution is isotropic around the direction of the beam. Then eq. (3.20b) gives the B-symmetric parameters t^L_M, and one must verify that the B-antisymmetric moments vanish.
(We recall that in transversity quantization B-symmetric parameters have M = even and B-antisymmetric parameters have M = odd).

ii) The target is transversally polarized, i.e., $P_T \neq 0$, $P_L = 0$. Eq. (3.20c) gives the B-symmetric polarization transfer parameters ${}^z t^L_M$, and one must verify that the B-antisymmetric moments vanish. Similarly, eq. (3.20d) gives the B-antisymmetric polarization transfer parameters ${}^x t^L_M$ and one must verify that the corresponding B-symmetric moments vanish.
Furthermore one may verify that eq. (3.20b) yields the same results as in case i).

iii) The target is longitudinally polarized, i.e., $P_T = 0$, $P_L \neq 0$. One must verify that the angular distribution is isotropic around the common direction of the beam and of the polarization vector ζ. Then eq. (3.20b) gives the B-symmetric parameters t^L_M (which should be equal to the parameters obtained in i)) and the B-antisymmetric polarization transfer parameters ${}^y t^L_M$.

iv) The target is arbitrarily polarized, i.e., $P_T \neq 0$, $P_L \neq 0$. One obtains all the parameters. Eq. (3.20b) gives the B-symmetric parameters t^L_M and the B-antisymmetric parameters ${}^y t_M{}^L$. Eq. (3.20c, d) give the B-antisymmetric parameters ${}^z t^L_M$ and ${}^x t^L_M$, and one may verify that their B-symmetric moments vanish.
Of course, if the decay is parity conserving, this analysis yields only the L = even parameters (cf. section 2.4.1). The moments with L = odd must be found compatible with zero.

3.3.3. Production and joint decay angular distribution

If both final particles (3 and 4) are untable and undergo two body decay (e.g. $\pi p \to \rho\Delta$, $\rho \to \pi\pi$, $\Delta \to N\pi$) the combined production and joint decay angular distribution is (cf. eq. (2.9, 2.19, 3.11))

$$I(\psi:\theta,\phi,\theta',\phi') = \frac{1}{2\pi}\sum_{L=0}^{2j}\sum_{L'=0}^{2j} C(L)\,C'(L') \sum_{M=-L}^{+L}\sum_{M'=-L'}^{+L'} \{\overline{t^{LL'}_{MM'}} + P_T\cos\psi\,\overline{{}^z t^{LL'}_{MM'}}$$

$$+ P_T\sin\psi\,\overline{{}^x t^{LL'}_{MM'}} + P_L\overline{{}^y t^{LL'}_{MM'}}\}\; Y^L_M(\theta,\phi)\; Y^{L'}_{M'}(\theta',\phi') \tag{3.21}$$

where $t^{00}_{00} = 1$, ${}^z t^{00}_{00} = P_R\, C(0) = C'(0) = 1/\sqrt{4\pi}$, $Y_0{}^0 = 1/\sqrt{4\pi}$. The moment analysis of this distribution gives $\big(\Omega = (\theta, \phi),\ \Omega' = (\theta', \phi')\big)$.

$$\begin{aligned}
C(L)\, C'(L')\, (t^{LL'}_{MM'} + P_L{}^y t^{LL'}_{MM'}) &= \langle Y^L_M(\Omega)\, Y^{L'}_{M'}(\Omega')\rangle \\
C(L)\, C'(L')\, P_T{}^z t^{LL'}_{MM'} &= \langle 2\cos\psi\, Y^L_M(\Omega)\, Y^{L'}_{M'}(\Omega')\rangle \qquad (3.22)\\
C(L)\, C'(L')\, P_T{}^x t^{LL'}_{MM'} &= \langle 2\sin\psi\, Y^L_M(\Omega)\, Y^{L'}_{M'}(\Omega')\rangle .
\end{aligned}$$

A similar discussion to that of the preceding section 3.3.2. can be made. We shall not repeat it. We only recall that in the present case, the B-symmetric parameters (in transversity) have $M + M' =$ even and the B-antisymmetric ones have $M + M' =$ odd. Furthermore, if both decays are parity conserving, one gets only the $L =$ even and L' = even parameters; all the moments with L or L' odd must vanish. If the decay in (θ', ϕ') is parity violating one gets the multipole parameters with $L =$ even and $L' =$ even or odd. All other moments vanish.

3.3.4. Production and cascade decay angular distribution

Assume again that only one final particle decays, but that it undergoes a cascade decay of the type discussed in section 2.4.3. (e.g. $Kp \to \pi\Sigma^*$, $\Sigma^* \to \Lambda\pi$, $\Lambda \to p\pi$). Then one may study the angular distribution of the production and of the cascade decay. It reads (cf. eq. (2.21a) for the canonical quantization frame and (2.21b) for the helicity quantization frame).

$$\begin{aligned}
I(\psi; \theta, \phi; \theta_1, \phi_1) = \frac{1}{2\pi} \sum_{L=0}^{2j} \sum_{L_1=0}^{1} \sum_{J\,\text{even}} C(L, L_1, J) \sum_{M, M_1, N} \langle JL, NM_1 \mid LM\rangle \big\{\overline{t^L_M} + P_T \cos\psi\, \overline{{}^z t^L_M} \\
+ P_T \sin\psi\, \overline{{}^x t^L_M} + P_L \overline{{}^y t^L_M}\big\}\, Y^J_N(\theta, \phi)\, Y^{L_1}_{M_1}(\theta, \phi_1)
\end{aligned} \qquad (3.23\text{a})$$

$$\begin{aligned}
I(\psi; \theta, \phi; \theta_1{}^h, \phi_1{}^h) = \frac{1}{2\pi} \sum_{L=0}^{2j} \sum_{L_1=0}^{1} C^h(L, L_1, M_1) \sum_{M, M_1} \big\{\overline{t^L_M} + P_T \cos\psi\, \overline{{}^z t^L_M} \\
+ P_T \sin\psi\, \overline{{}^x t^L_M} + P_L \overline{{}^y t^L_M}\big\} \sqrt{\frac{2L+1}{4\pi}}\, \overline{D^L(\phi, \theta, 0)^M_{M_1}}\, Y^{L_1}_{M_1}(\theta_1{}^h, \phi_1{}^h).
\end{aligned} \qquad (3.23\text{b})$$

where the coefficients $C(L, L_1, J)$ of Eq. (3.23a) and $C^h(L, L_1, M_1)$ of (3.23b) are given in Eq. (2.22a) and (2.22b). With these known values of the coefficients, the parameters $t_M{}^L$ and ${}^\varkappa t_M{}^L (\varkappa = x, y, z)$ are deduced by a best fit adjustment of the combined angular distributions or by a moment analysis $\big(\Omega = (\theta, \phi),\ \Omega_1 = (\theta_1, \phi_1),\ \Omega_1{}^h = (\theta_1{}^h, \phi_1{}^h)\big)$

$$\begin{aligned}
C(L, L_1, J)\, (t^L_M + P_L{}^y t^L_M) &= \sum_{N, M_1} \langle JL_1 NM_1 \mid LM\rangle \langle Y^J_N(\Omega)\, Y^{L_1}_{M_1}(\Omega_1)\rangle, \\
C(L, L_1, J)\, P_T{}^z t^L_M &= \sum_{N, M_1} \langle JL_1 NM_1 \mid LM\rangle \langle 2\cos\psi\, Y^J_N(\Omega)\, Y^{L_1}_{M_1}(\Omega_1)\rangle, \qquad (3.24\text{a})\\
C(L, L_1, J)\, P_T{}^x t^L_M &= \sum_{N, M_1} \langle JL_1 NM_1 \mid LM\rangle \langle 2\sin\psi\, Y^J_N(\Omega)\, Y^{L_1}_{M_1}(\Omega_1)\rangle,
\end{aligned}$$

$$C^h(L, L_1, M_1)\,(t^L_M + P_L{}^y t^L_M) = \sqrt{\frac{2L+1}{4\pi}}\,\langle \overline{D^L(\phi,\theta,0)^M_{M_1}\; Y^{L_1}_{M_1}(\Omega_1{}^h)} \rangle,$$

$$C^h(L, L_1, M_1)\,P_T{}^z t^L_M = \sqrt{\frac{2L+1}{4\pi}}\,\langle 2\cos\psi \overline{D^L(\phi,\theta,0)^M_{M_1}\; Y^{L_1}_{M_1}(\Omega_1{}^h)} \rangle, \tag{3.24 b}$$

$$C^h(L, L_1, M_1)\,P_T{}^x t^L_M = \sqrt{\frac{2L+1}{4\pi}}\,\langle 2\sin\psi \overline{D^L(\phi,\theta,0)^M_{M_1}\; Y^{L_1}_{M_1}(\Omega_1{}^h)} \rangle.$$

A discussion identical to that of Section 3.3.2. can be made. Note however that in this case all parameters (L = even and L = odd) can be measured.

3.3.5. More complex combined production and decay angular distributions

One may consider more intricate situations. For example, if the two final particles are unstable and one of them undergo a cascade decay (e.g. $\mathrm{Kp} \to \rho\Sigma^*$, $\rho \to \pi\pi$, $\Sigma^* \to \Lambda\pi$, $\Lambda \to \mathrm{p}\pi$) the complete angular distribution involves 7 angles. Still more complex is the case where both final particles undergo cascade decays (e.g. $\mathrm{Kp} \to A_1\Sigma^*$, $A_1 \to \rho\pi$, $\rho \to \pi\pi$, $\Sigma^* \to \Lambda\pi$, $\Lambda \to \mathrm{p}\pi$). Then the complete angular distribution involves 9 angles. The expressions of such angular distributions are easily written down, however the present day experimentalists are not yet interested in such complex reactions with polarized target.

4. Amplitude Reconstruction in Usual Reactions

In the previous sections we have shown the way of measuring the observables of a reaction with unpolarized target (section 2) or with polarized target (section 3). They are embodied in the final polarization $\sigma\varrho_f$ or in the transfer polarization matrix W, which are quadratic expressions of the transition matrix T or $\tilde{T}$, namely (c.f. eq. (2.14) and (3.9′)).

$$\sigma\varrho_f = \frac{1}{2}\,TT^\dagger, \qquad W = \frac{1}{2}\,\tilde{T}\tilde{T}^\dagger.$$

We call amplitudes the elements of these transition matrices. Their reconstruction consists essentially in obtaining an explicit expression for T or $\tilde{T}$ by inverting the quadratic expressions $TT^\dagger$ or $\tilde{T}\tilde{T}^\dagger$. Theoretically this can easily be done, and one obtains T or $\tilde{T}$ up to some unknown phases (by the procedure of "conventional amplitude reconstruction" of Appendix 1,2). Practically each concrete case needs a separate study since generally the observable matrices $\sigma\varrho_f$ or W are not completely measured.
In this section we present the practical method of reconstructing the amplitudes in usual reactions with unpolarized and/or polarized target. For pedagogical purposes we first recall the method of reconstructing the amplitudes in the simplest reaction type $\pi\mathrm{p} \to \mathrm{K}\Lambda$, by measurement of the classical Wolfenstein parameters P, R, A. In view of further generalizations to higher final spins, we introduce, already in this simple case, the multipole formalism and some complex spin rotation parameters. This is discussed in section 4.1., and summarized in table 4. In the following sections we present the details of the generalization to reactions of the types $\pi\mathrm{p} \to \mathrm{K}\Sigma^*$, $\pi\Delta$, $\mathrm{K}^*\Lambda$, ρN as simple comments to the tables 5 to 10, in which all the recipes for measurements and amplitude reconstructions have been gathered.

Table 4

Amplitude reconstruction for reactions of type $\pi p \to K\Lambda$ with polarized target
(Comparison with the PRA parameters and the spin flip and non flip helicity amplitudes)

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a) Combined production and decay angular distribution and measurement of the polarization transfer by the method of moments.

$$I(\psi, \theta\phi) = \frac{1}{2\pi} \sum_L C(L) \sum_M \left[\overline{t^L_M} + P_T\left(\cos\psi \overline{{}^z t^L_M} + \sin\psi \overline{{}^x t^L_M}\right) + P_L \overline{{}^y t^L_M}\right] Y^L_M(\theta\phi)$$

$$C(L)\,(t^L_M + P_L {}^y t^L_M) = \langle Y^L_M(\theta\varphi)\rangle$$

$$C(L)\,P_T {}^z t^L_M = \langle 2\cos\psi\, Y^L_M(\theta\varphi)\rangle$$

$$C(L)\,P_T {}^x t^L_M = \langle 2\sin\psi\, Y^L_M(\theta\varphi)\rangle$$

with: $C(0) = 1/\sqrt{4\pi}$, $C(1) = \alpha_\Lambda/\sqrt{4\pi}$ (all other C coefficients vanish)

---

b) The 8 real observables in transversity quantization

$$\sigma_0,\ t_0{}^1$$

$$P = {}^z t_0{}^0,\ {}^z t_0{}^1,\ \mathrm{Re}\,{}^x t_1{}^1,\ \mathrm{Im}\,{}^x t_1{}^1$$

$$\mathrm{Re}\,{}^y t_1{}^1,\ \mathrm{Im}\,{}^y t_1{}^1$$

(M = even for t^L_M, ${}^z t^L_M$, and odd for ${}^x t^L_M$, ${}^y t^L_M$ by B-symmetry; $t^L_{-M} = (-)^M \overline{t^L_M}$)

---

c) The 4 linear constraints and the 3 spin rotation parameters

$$P_0 = \frac{1}{2}\left[1 + \sqrt{3}\, t_0{}^1\right] = \frac{1}{2}\left[P + \sqrt{3}\, {}^z t_0{}^1\right] = P_0$$

$$P_0' = \frac{1}{2}\left[1 - \sqrt{3}\, t_0{}^1\right] = \frac{-1}{2}\left[P - \sqrt{3}\, {}^z t_0{}^1\right] = P_0'$$

$$P_0 + P_0' = 1^\dagger$$

$$R_0 = \frac{1}{2}\left[-\sqrt{6}\, \overline{{}^y t_1{}^1}\right] = \frac{1}{2}\left[-\sqrt{6}\, \overline{{}^x t_1{}^1}\right] = R_0$$

$$P_0 = \frac{1}{2}(1 + P)$$

$$P_0' = \frac{1}{2}(1 - P)$$

$$R_0 = \frac{1}{2}(R + iA)$$

---

d) The non linear constraint

$$|R_0|^2 = P_0 P_0' \qquad P_0 \geqq 0$$

$$P^2 + R^2 + A^2 = 1$$

---

† The two first constraints imply $\sqrt{3}\, t_0{}^1 = P$, $\sqrt{3}\, {}^z t_0{}^1 = 1$

Table 4 — cont'd

e) Transfer of polarization

	$\sigma = \sigma_0(1 + Pz)$
$\sigma\varrho_{++} = \sigma_0 P_0(1 + z)$	$\sigma Z = \sigma_0(P + z)$
$\sigma\varrho_{--} = \sigma_0 P_0'(1 - z)$	$\sigma X = \sigma_0(Rx + Ay)$
$\sigma\varrho_{+-} = \sigma_0 R_0(x - iy)$	$\sigma Y = \sigma_0(-Ax + Ry)$

f) Transversity amplitudes and the usual spin non flip and spin flip helicity amplitudes

$$\begin{array}{c|cc} & \multicolumn{2}{c}{\lambda_p} \\ \lambda_\Lambda & +\frac{1}{2} & -\frac{1}{2} \\ \hline +\frac{1}{2} & a & \\ -\frac{1}{2} & & a' \end{array} = T_{2\lambda_\Lambda, 2\lambda_p} \qquad \begin{pmatrix} F & G \\ -G & F \end{pmatrix} = H_{2\lambda_\Lambda, 2\lambda_p}$$

$$a = F + iG$$

$$a' = F - iG$$

f') Collinearity constraints

$$G = 0 \Leftrightarrow a = a'$$

g) Expression of the observables of c), e) as functions of the transversity amplitudes

$$\sigma_0 = \frac{1}{2}(|a|^2 + |a'|^2) = |F|^2 + |G|^2$$

$2\sigma_0 P_0 = \|a\|^2$	$\sigma_0 P = \frac{1}{2}(\|a\|^2 - \|a'\|^2)$	$= 2\ \mathrm{Im}\ F\bar{G}$
$2\sigma_0 P_0' = \|a'\|^2$	$\sigma_0 R = \mathrm{Re}\ a\overline{a'}$	$= \|F\|^2 - \|G\|^2$
$2\sigma_0 R_0 = a\bar{a}'$	$\sigma_0 A = \mathrm{Im}\ a\overline{a'}$	$= 2\ \mathrm{Re}\ F\bar{G}$

Table 5

Amplitude reconstruction for reactions of type $\pi p \to K\Sigma^*(0^-1/2^+ \to 0^-3/2^+)$ with unpolarized target

a) Angular distribution of the Σ^* decay and measurement of the even multipole parameters by the method of moments.

$$I(\theta, \Phi) = \sum_L C(L) \sum_M \overline{t^L_M}\, Y^L_M(\theta, \Phi)$$

$$C(L)\, t^L_M = \langle Y^L_M(\theta, \Phi)\rangle$$

with: $C(0) = 1/\sqrt{4\pi}$, $C(2) = -1/\sqrt{4\pi}$ (all other C coefficients vanish)

Table 5 — cont'd

b) Cascade angular distribution of Σ^* decay ($\Sigma^* \to \Lambda\pi$, $\Lambda \to p\pi$) and measurement of all multipole parameters by the method of moments.

b 1) Canonical frame for the Λ

$$I(\theta, \phi, \theta_1, \phi_1) = \sum_{LL_1J} C(LL_1J) \sum_{MNM_1} \langle JL_1NM_1 \mid LM\rangle \overline{t^L_M}\, Y^J_N(\theta\phi)\; Y^{L_1}_{M_1}(\theta_1\phi_1)$$

$$C(LL_1J)\, t^L_M = \sum_N \langle JL_1NM_1 \mid LM\rangle \langle Y^J_N(\theta, \phi)\; Y^{L_1}_{M_1}(\theta_1, \phi_1)\rangle$$

b 2) Helicity frame for the Λ

$$I(\theta, \phi, \theta_1{}^h, \phi_1{}^h) = \sum_{LL_1M_1} C^h(LL_1M_1) \sum_M \overline{t^L_M} \sqrt{\frac{2L+1}{4\pi}}\; \overline{D^L(\phi\theta 0)^{M_1}_M}\; Y^{L_1}_{M_1}(\theta_1{}^h, \phi_1{}^h)$$

$$C^h(LL_1M_1)\, t^L_M = \sqrt{\frac{2L+1}{4\pi}}\; \langle \overline{D^L(\phi\theta 0)^M_{M_1}}\; Y^{L_1}_{M_1}(\theta_1{}^h, \phi_1{}^h)\rangle$$

b 3) Values of the non vanishing coefficients in b 1) and b 2)

$C(000) = 1/4\pi$,	$C^h(000) = 1/4\pi$,
$C(202) = -1/4\pi$,	$C^h(200) = -1/4\pi$,
$C(110) = (\alpha/4\pi)\sqrt{5/9}$,	$C^h(110) = (\alpha/4\pi)\sqrt{1/15}$,
$C(112) = (\alpha/4\pi)\sqrt{2/45}$,	$C^h(11 \pm 1) = (\alpha/4\pi)\sqrt{4/15}$
$C(312) = -(\alpha/4\pi)\sqrt{7/5}$,	$C^h(310) = -(\alpha/4\pi)\sqrt{3/5}$
	$C^h(31 \pm 1) = -(\alpha/4\pi)\sqrt{2/5}$

c) The 8 real observables in transversity quantization

σ_0, $t_0{}^2$, Re $t_2{}^2$, Im $t_0{}^1$, $t_0{}^3$, Re $t_2{}^3$, Im $t_2{}^3$.

(M = even by B-symmetry, $t^L_{-M} = (-1)^M\, t_M{}^L$)

d) Observable density matrix elements in transversity quantization

$$P_1 \equiv \varrho_{11} = 1/4\left[1 - \sqrt{5}\, t_0{}^2 + \sqrt{3/5}\, t_0{}^1 - \sqrt{63/5}\, t_0{}^3\right]$$

$$P_1' \equiv \varrho_{-1-1} = 1/4\left[1 - \sqrt{5}\, t_0{}^2 - \sqrt{3/5}\, t_0{}^1 + \sqrt{63/5}\, t_0{}^3\right]$$

$$P_2 \equiv \varrho_{-3-3} = 1/4\left[1 + \sqrt{5}\, t_0{}^2 - \sqrt{27/5}\, t_0{}^1 - \sqrt{7/5}\, t_0{}^3\right]$$

$$P_2' \equiv \varrho_{33} = 1/4\left[1 + \sqrt{5}\, t_0{}^2 + \sqrt{27/5}\, t_0{}^1 + \sqrt{7/5}\, t_0{}^3\right]$$

$$Q \equiv \varrho_{1-3} = 1/4\left[\sqrt{10}\, \overline{t_2{}^2} - \sqrt{14}\, \overline{t_2{}^3}\right], \qquad Q' \equiv \varrho_{3-1} = 1/4\left[\sqrt{10}\, \overline{t_2{}^2} + \sqrt{14}\, \overline{t_2{}^3}\right]$$

e) Positivity and rank constraints

$P_1 \geq 0$, $P_1' \geq 0$, $P_2 \geq 0$, $P_2' \geq 0$, $|Q|^2 = P_1P_2$, $|Q'^2| = P_1'P_2'$

Table 5 — cont'd

f) Transversity and helicity amplitudes

$$\begin{array}{c|cc|}
 & \multicolumn{2}{c}{\lambda_{\mathrm{p}}} \\
\lambda_{\Sigma^*} & \tfrac{1}{2} & -\tfrac{1}{2} \\
\hline
3/2 & \cdot & b' \\
1/2 & a & \cdot \\
-1/2 & \cdot & a' \\
-3/2 & b & \cdot \\
\hline
\end{array} = T_{2\lambda_{\Sigma^*}, 2\lambda_{\mathrm{p}}} \qquad \begin{array}{|cc|}
\hline
B' & -B \\
A & A' \\
A' & -A \\
B & B' \\
\hline
\end{array} = H_{2\lambda_{\Sigma^*}, 2\lambda_{\mathrm{p}}}$$

$$A + iA' = -1/2\left(a + \sqrt{3}\, b\right) \qquad B + iB' \equiv -1/2\left(\sqrt{3}\, a - b\right)$$

$$A - iA' = 1/2\left(a' + \sqrt{3}\, b'\right) \qquad B - iB' = 1/2\left(\sqrt{3}\, a' - b'\right)$$

f′) Collinearity constraints

$$A' = B' = B = 0 \Leftrightarrow a = -a' = \sqrt{\frac{1}{3}}\, b = -\sqrt{\frac{1}{3}}\, b'$$

g) Expression of the observables in d) and e) as functions of the transversity amplitudes

$$2\sigma_0 P_1 = |a|^2 \qquad 2\sigma_0 P_1{}' = |a'|^2$$

$$2\sigma_0 P_2 = |b|^2 \qquad 2\sigma_0 P_2{}' = |b'|^2$$

$$2\sigma_0 Q = a\bar{b} \qquad 2\sigma_0 Q' = b'\bar{a}'$$

Table 6

Amplitude reconstruction for reactions of type p$\pi \to$KΣ^* ($0^-\ 1/2^+ \to 0^-\ 3/2^+$) with polarized target.

a) Combined production and cascade angular distribution for general target polarization, and measurement of the polarization transfer by the method of moments

a 1) Canonical frame for the Λ

$$I(\psi, \theta, \Phi, \theta_1, \Phi_1) = \frac{1}{2\pi} \sum_{LJL_1} C(LL_1J) \sum_{MNM_1} \langle JL_2NM_1 | LM\rangle \Big[\overline{t^L_M} + P_T\left(\cos\psi\, \overline{{}^z t^L_M} + \sin\psi\, \overline{{}^x t^L_M}\right) + P_L \overline{{}^y t^L_M}\Big] Y^J_N(\theta, \Phi)\, Y^L_M(\theta_1, \Phi_1)$$

$$C(LL_1J)\, (t^L_M + P_L {}^y t^L_M = \sum_{NM_1} \langle JL_1NM_1 \mid LM\rangle \langle Y^J_N(\theta, \Phi)\, Y^{L_1}_{M_1}(\theta_1, \Phi_1)\rangle$$

$$C(LL_1J)\, P_T {}^z t^L_M = \sum_{NM_1} \langle JL_1NM_1 \mid LM\rangle \langle 2\cos\psi\, Y^J_N(\theta, \Phi)\, Y^{L_1}_{M_1}(\theta_1, \Phi_1)\rangle$$

$$C(LL_1J)\, P_T {}^x t^L_M = \sum_{NM_1} \langle JL_1NM_1 \mid LM\rangle \langle 2\sin\psi\, Y^J_N(\theta, \Phi)\, Y^{L_1}_{M_1}(\theta_1, \Phi_1)\rangle$$

Table 6 — cont'd

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a 2) Helicity frame for the Λ

$$I(\psi, \theta, \phi, \theta_1^h, \phi_1^h) = \frac{1}{2\pi} \sum_{LL_1M_1} C^h(LL_1 M_1) \sum_M \Big[\overline{t^L_M} + P_T\big(\cos\psi\, \overline{{}^z t^L_M} + \sin\psi\, \overline{{}^x t^L_M}\big) + P_L\, \overline{{}^y t^L_M}\Big] \sqrt{\frac{2L+1}{4\pi}}\; \overline{D^L(\phi\theta 0)^M_{M_1}}\; Y^{L_1}_{M_1}(\theta_1^h, \phi_1^h)$$

$$C^h(LL_1M_1)\,(t^L_M + P_L\, {}^y t^L_M) = \sqrt{\frac{2L+1}{4\pi}}\, \langle \overline{D^L(\phi\theta 0)^M_{M_1}}\; Y^{L_1}_{M_1}(\theta_1^h, \phi_1^h)\rangle$$

$$C^h(LL_1M_1)\, P_T\, {}^z t^L_M = \sqrt{\frac{2L+1}{4\pi}}\, \langle 2\cos\psi\, \overline{D^L(\phi\theta 0)^M_{M_1}}\; Y^{L_1}_{M_1}(\theta_1^h, \phi_1^h)\rangle$$

$$C^h(LL_1M_1)\, P_T\, {}^x t^L_M = \sqrt{\frac{2L+1}{4\pi}}\, \langle 2\sin\psi\, \overline{D^L(\phi\theta 0)^M_{M_1}}\; Y^{L_1}_{M_1}(\theta_1^h, \phi_1^h)\rangle$$

with the coefficients $C(LL_1J)$ and $C^h(LL_1M_1)$ of Table 5 b 3)

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b) The 32 real observables in transversity quantization

σ_0, t_0^2, Re t_2^2, Im t_2^2, t_0^1, t_0^3, Re t_2^3, Im t_2^3,

$P_R = {}^z t_0^0$, ${}^z t_0^2$, Re ${}^z t_2^2$, Im ${}^z t_2^2$, ${}^z t_0^1$, ${}^z t_0^3$, Re ${}^z t_2^3$, Im ${}^z t_2^3$,

Re ${}^x t_1^2$, Im ${}^x t_1^2$, Re ${}^x t_1^1$, Im ${}^x t_1^1$, Re ${}^x t_1^3$, Im ${}^x t_1^3$, Re ${}^x t_3^3$, Im ${}^x t_3^3$,

Re ${}^y t_1^2$, Im ${}^y t_1^2$, Re ${}^y t_1^1$, Im ${}^y t_1^1$, Re ${}^y t_1^3$, Im ${}^y t_1^3$, Re ${}^y t_3^3$, Im ${}^y t_3^3$.

$\big(M$ = even for t^L_M and ${}^z t^L_M$, M = odd for ${}^x t^L_M$ and ${}^y t^L_M$ by B-symmetry, $t^L_{-M} = (-1)^M\, \overline{t^L_M}\big)$

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c) The 16 linear constraints and the 15 generalized spin rotation parameters

$$P_1 = \frac{1}{4}\left[1 - \sqrt{5}\, t_0^2 + \sqrt{\frac{3}{5}}\, t_0^1 - \sqrt{\frac{63}{5}}\, t_0^3\right] = \frac{1}{4}\left[P_R - \sqrt{5}\, {}^z t_0^2 + \sqrt{\frac{3}{5}}\, {}^z t_0^1 - \sqrt{\frac{63}{5}}\, {}^z t_0^3\right] = P_1$$

$$P_1' = \frac{1}{4}\left[1 - \sqrt{5}\, t_0^2 - \sqrt{\frac{3}{5}}\, t_0^1 + \sqrt{\frac{63}{5}}\, t_0^3\right] = \frac{-1}{4}\left[P_R - \sqrt{5}\, {}^z t_0^2 - \sqrt{\frac{3}{5}}\, {}^z t_0^1 + \sqrt{\frac{63}{5}}\, {}^z t_0^3\right] = P_1'$$

$$P_2 = \frac{1}{4}\left[1 + \sqrt{5}\, t_0^2 - \sqrt{\frac{27}{5}}\, t_0^1 - \sqrt{\frac{7}{5}}\, t_0^3\right] = \frac{1}{4}\left[P_R + \sqrt{5}\, {}^z t_0^2 - \sqrt{\frac{27}{5}}\, {}^z t_0^1 - \sqrt{\frac{7}{5}}\, {}^z t_0^3\right] = P_2$$

$$P_2' = \frac{1}{4}\left[1 + \sqrt{5}\, t_0^2 + \sqrt{\frac{27}{5}}\, t_0^1 + \sqrt{\frac{7}{5}}\, t_0^3\right] = \frac{-1}{4}\left[P_R + \sqrt{5}\, {}^z t_0^2 + \sqrt{\frac{27}{5}}\, {}^z t_0^1 + \sqrt{\frac{7}{5}}\, {}^z t_0^3\right] = P_2'$$

$$P_1 + P_1' + P_2 + P_2' = 1, \qquad P_1 - P_1' + P_2 - P_2' = P_R$$

Table 6 — cont'd

c) (continued)

$$Q = \frac{1}{4}\left[\sqrt{10}\,\overline{t_2{}^2} - \sqrt{14}\,\overline{t_2{}^3}\right] = \frac{1}{4}\left[\sqrt{10}\,\overline{{}^zt_2{}^2} - \sqrt{14}\,\overline{{}^zt_2{}^3}\right] = Q$$

$$Q' = \frac{1}{4}\left[\sqrt{10}\,\overline{t_2{}^2} + \sqrt{14}\,\overline{t_2{}^3}\right] = \frac{1}{4}\left[\sqrt{10}\,\overline{{}^zt_2{}^2} + \sqrt{14}\,\overline{{}^zt_2{}^3}\right] = Q'$$

$$R = \frac{i}{4}\left[-\sqrt{10}\,\overline{{}^yt_1{}^2} - \sqrt{\frac{18}{5}}\,\overline{{}^yt_1{}^1} - \sqrt{\frac{28}{5}}\,\overline{{}^yt_1{}^3}\right] = \frac{1}{4}\left[-\sqrt{10}\,\overline{{}^xt_1{}^2} - \sqrt{\frac{18}{5}}\,\overline{{}^xt_1{}^1} - \sqrt{\frac{28}{5}}\,\overline{{}^xt_1{}^3}\right] = R$$

$$R' = \frac{i}{4}\left[\sqrt{10}\,\overline{{}^yt_1{}^2} - \sqrt{\frac{18}{5}}\,\overline{{}^yt_1{}^1} - \sqrt{\frac{28}{5}}\,\overline{{}^yt_1{}^3}\right] = \frac{1}{4}\left[\sqrt{10}\,\overline{{}^xt_1{}^2} - \sqrt{\frac{18}{5}}\,\overline{{}^xt_1{}^1} - \sqrt{\frac{28}{5}}\,\overline{{}^xt_1{}^3}\right] = R'$$

$$R_1 = \frac{i}{4}\left[-\sqrt{\frac{24}{5}}\,\overline{{}^yt_1{}^1} - \sqrt{\frac{84}{5}}\,\overline{{}^yt_1{}^3}\right] = \frac{1}{4}\left[-\sqrt{\frac{24}{5}}\,\overline{{}^xt_1{}^1} - \sqrt{\frac{84}{5}}\,\overline{{}^xt_1{}^3}\right] = R_1$$

$$R_2 = \frac{-i}{4}\left[-\sqrt{28}\,\overline{{}^yt_3{}^3}\right] = \frac{1}{4}\left[-\sqrt{28}\,\overline{{}^xt_3{}^3}\right] = R_2$$

---

d) The 9 non linear constraints and the positivity conditions

$$|Q|^2 = P_1P_2, \quad |R_1|^2 = P_1P_1', \quad P_1R' = Q\bar{R}_1, \quad P_1 \geqq 0, \quad P_2 \geqq 0$$

$$|Q'|^2 = P_1'P_2', \quad RR' = \bar{R}_1R_2, \quad P_1'R = Q'\bar{R}_1, \quad P_1' \geqq 0, \quad P_2' \geqq 0$$

---

e) Transfer of polarization

$\sigma\varrho_{11} = \sigma_0 P_1(1+z)$	$\sigma\varrho_{-1-1} = \sigma_0 P_1'(1-z)$
$\sigma\varrho_{-3-3} = \sigma_0 P_2(1+z)$	$\sigma\varrho_{33} = \sigma_0 P_2'(1-z)$
$\sigma\varrho_{1-3} = \sigma_0 Q(1+z)$	$\sigma\varrho_{3-1} = \sigma_0 Q'(1-z)$
$\sigma\varrho_{31} = \sigma_0 R(x+iy)$	$\sigma\varrho_{-1-3} = \sigma_0 R'(x+iy)$
$\sigma\varrho_{1-1} = \sigma R_1(x-iy)$	$\sigma\varrho_{3-3} = \sigma_0 R_2(x+iy)$

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f) Transversity and helicity amplitudes as in Table 5f)

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g) Expression of the observables in c) and e) as functions of the transversity amplitudes)

$2\sigma_0 P_1 = \|a\|^2$	$2\sigma_0 P_1' = \|a'\|^2$
$2\sigma_0 P_2 = \|b\|^2$	$2\sigma_0 P_2' = \|b'\|^2$
$2\sigma_0 Q = a\bar{b}$	$2\sigma_0 Q' = b'\bar{a}'$
$2\sigma_0 R = b'\bar{a}$	$2\sigma_0 R' = a'\bar{b}$
$2\sigma_0 R_1 = a\bar{a}'$	$2\sigma_0 R_2 = b'\bar{b}$

Table 7

Amplitude reconstruction for reactions of type $\pi p \to \pi\Delta(0^- 1/2^+ \to 0^- 3/2^+)$ with polarized target.

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a) Combined production and decay angular distribution and measurement of the polarization transfer by the method of moments

$$I(\psi, \theta, \phi) = \frac{1}{2\pi} \sum_L C(L) \sum_M \left[\overline{t^L_M} + P_T(\cos\psi\,\overline{{}^z t^L_M} + \sin\psi\,\overline{{}^x t^L_M}) + P_L \overline{{}^y t^L_M}\right] Y^L_M(\theta, \phi)$$

$$C(L)\,(t^L_M + P_L\,{}^y t^L_M) = \langle Y^L_M(\theta, \phi)\rangle$$

$$C(L)\,P_T\,{}^z t^L_M = \langle 2\cos\psi\, Y^L_M(\theta, \phi)\rangle$$

$$C(L)\,P_T\,{}^x t^L_M = \langle 2\sin\psi\, Y^L_M(\theta, \phi)\rangle$$

with $C(0) = 1/\sqrt{4\pi}$, $C(2) = 1/\sqrt{4\pi}$ (all other C coefficients vanish)

---

b) The 12 real observables in transversity quantization

σ_0, t_0^2, Re t_2^2, Im t_2^2

$P_R = {}^z t_0^0$, ${}^z t_0^2$, Re ${}^z t_2^2$, Im ${}^x t_2^2$, Re ${}^x t_1^2$, Im ${}^x t_1^2$, Re ${}^y t_1^2$, Im ${}^y t_1^2$.

(M = even for t^L_M and ${}^z t^L_M$, M = odd for ${}^x t^L_M$ and ${}^y t^L_M$, by B-symmetry; $t^L_{-M} = (-1)^M\, t^L_M$)

---

c) The 9 generalized spin rotation parameters and the 2 linear constraints

$$P_1 = \frac{1}{4}\left[1 + P_R - \sqrt{5}\,(t_0^2 + {}^z t_0^2)\right]$$

$$P_1' = \frac{1}{4}\left[1 - P_R - \sqrt{5}\,(t_0^2 - {}^z t_0^2)\right] \qquad P_1 + P_1' + P_2 + P_2' = 1$$

$$P_2 = \frac{1}{4}\left[1 + P_R + \sqrt{5}\,(t_0^2 + {}^z t_0^2)\right] \qquad P_1 - P_1' + P_2 - P_2' = P_R$$

$$P_2' = \frac{1}{4}\left[1 - P_R + \sqrt{5}\,(t_0^2 - {}^z t_0^2)\right]$$

$$Q = \frac{1}{4}\sqrt{10}\left(\overline{t_0^2} + \overline{{}^z t_2^2}\right), \qquad Q' = \frac{1}{4}\sqrt{10}\left(\overline{t_2^2} - \overline{{}^z t_2^2}\right)$$

$$R_- = -\frac{1}{2}\sqrt{10}\,\overline{{}^x t_1^2} = \frac{i}{2}\sqrt{10}\,\overline{{}^y t_1^2}$$

---

d) The 3 non linear constraints and the positivity conditions

$$|Q|^2 = P_1 P_2, \quad |Q'|^2 = P_1' P_2', \quad |R_-|^2 + |Q + Q'|^2 = (P_1 + P_1')(P_2 + P_2')$$

$$P_1 \geqq 0, \quad P_2 \geqq 0, \quad P_1' \geqq 0, \quad P_2' \geqq 0.$$

Table 7 — cont'd

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e) Transfer of polarization

$$\sigma\varrho^e_{11} = \sigma_0 \left[\frac{1}{2}(P_1 + P_1') + \frac{1}{2}(P_1 - P_1')\, z\right]$$

$$\sigma\varrho^e_{33} = \sigma_0 \left[\frac{1}{2}(P_2 + P_2') + \frac{1}{2}(P_2 - P_2')\, z\right]$$

$$\sigma\varrho^e_{3-1} = \sigma_0 \left[\frac{1}{2}(Q + Q') + \frac{1}{2}(Q - Q')\, z\right]$$

$$\sigma\varrho^e_{31} = \sigma_0 R_-(x + iy)$$

---

f) Transversity and helicity amplitudes

$$\begin{array}{c|cc|} & \multicolumn{2}{c}{\lambda_p} \\ \lambda_\Delta & \tfrac{1}{2} & -\tfrac{1}{2} \\ \hline 3/2 & . & b' \\ 1/2 & a & . \\ -1/2 & . & a' \\ 3/2 & b & . \\ \hline \end{array} = T_{2\lambda_\Delta, 2\lambda_\mathrm{p}} \qquad \begin{array}{|cc|} \hline B' & -B \\ A & A' \\ A' & -A \\ B & B' \\ \hline \end{array} = H_{2\lambda_\Delta, 2\lambda_\mathrm{p}}$$

$$A + iA' = -\frac{1}{2}\left(a + \sqrt{3}\, b\right) \qquad B + iB' = -\frac{1}{2}\left(\sqrt{3}\, a - b\right)$$

$$A - iA' = \frac{1}{2}\left(a' + \sqrt{3}\, b'\right) \qquad B - iB' = \frac{1}{2}\left(\sqrt{3}\, a' - b'\right)$$

f') Collinearity constraints

$$A' = B' = B = 0 \Leftrightarrow a = -a' = \sqrt{\frac{1}{3}}\, b = -\sqrt{\frac{1}{3}}\, b'$$

---

g) Expression of the observables in c)—e) in terms of the transversity amplitudes in f)

$$2\sigma_0 P_1 = |a|^2 \qquad 2\sigma_0 P_1' = |a'|^2$$

$$2\sigma_0 P_2 = |b|^2 \qquad 2\sigma_0 P_2' = |b'|^2$$

$$2\sigma_0 Q = a\bar{b} \qquad 2\sigma_0 Q' = b'\bar{a}'$$

$$2\sigma_0 R_- = b'\bar{a} - a'\bar{b} \rightarrow [a\bar{a}' = 2\sigma_0(P_1 Q' - P_1' Q)/R_-]$$

Table 8

Amplitude reconstruction for reactions of type $\pi p \to K^*\Lambda(0^-1/2^+ \to 1^{-e}\ 1/2^+$ with unpolraized target

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a) Joint angular distribution of K* and Λ decays and measurement of the double multipole parameters by the method of moments

$$I(\theta,\phi,\theta',\phi') = \sum_{L,L'} C_{K^*}(L)\, C_\Lambda(L') \sum_{M,M'} \overline{t^{LL'}_{MM'}}\, Y^L_M(\theta,\phi)\ Y^{L'}_{M'}(\theta,\phi)$$

$$C_{K^*}(L)\ C_\Lambda(L')\ t^{LL'}_{MM'} = \langle Y^L_M(\theta,\ \phi) Y^{L'}_{M'}(\theta'\ \phi')\rangle$$

with

$$C_{K^*}(0) = C_\Lambda(0) = 1/\sqrt{4\pi}, \qquad C_{K^*}(2) = -1/\sqrt{2\pi}, \quad C_\Lambda(1) = \alpha_\Lambda/\sqrt{4\pi}$$

(all other C coefficients vanish)

---

b) The 12 real observables in transversity quantization

$$\sigma_0,\ t^{20}_{00},\ t^{01}_{00},\ t^{21}_{00},\quad \left.\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\right\}\ t^{20}_{20},\ t^{21}_{20},\ t^{21}_{11},\ t^{21}_{1-1}$$

$$\left(M + M' = \text{even, by B-symmetry, } t^{LL'}_{-M-M'} = (-1)^{M+M'}\ \overline{t^{LL'}_{MM'}}\right)$$

---

c) Observable density matrix elements in transversity quantization

$$\left.\begin{matrix} P_0 = \varrho^{00}_{++} \\ P_0' = \varrho^{00}_{--}\end{matrix}\right\} = \frac{1}{6}\left[1 - \sqrt{10}\, t^{20}_{00} \pm \left(\sqrt{3}\, t^{01}_{00} - \sqrt{30}\, t^{21}_{00}\right)\right]$$

$$\left.\begin{matrix} P = \varrho^{11e}_{--} \\ P' = \varrho^{11e}_{++}\end{matrix}\right\} = \frac{1}{6}\left[1 + \sqrt{\frac{5}{2}}\ t^{20}_{00} \mp \left(\sqrt{3}\, t^{01}_{00} + \sqrt{\frac{15}{2}}\ t^{21}_{00}\right)\right]$$

$$\left.\begin{matrix} Q = \varrho^{1-1}_{--} \\ Q' = \varrho^{1-1}_{++}\end{matrix}\right\} = \frac{1}{6}\left[\sqrt{15}\, \overline{t^{20}_{20}} \mp \sqrt{45}\ \overline{t^{21}_{20}}\right]$$

$$S_1 = \varrho^{10e}_{-+} = \frac{-1}{6}\sqrt{45}\ \overline{t^{21}_{1-1}}$$

$$S_2 = \varrho^{10e}_{+-} = \frac{1}{6}\sqrt{45}\ \overline{t^{21}_{11}}$$

---

d) Positivity and diacritical constraints

$$P_0 \geqq 0, \qquad P \geqq |Q|, \qquad P_0' \geqq 0, \qquad P' \geqq |Q'|$$

$$32 P_0 P = |\Gamma - \sqrt{\Delta}|^2/|S_2|^2 + |\Gamma + \sqrt{\Delta}|^2/|S_1|^2$$

$$32 P_0' P' = |\Gamma' + \sqrt{\Delta}|^2/|S_2|^2 + |\Gamma' - \sqrt{\Delta}|^2/|S_1|^2$$

Table 8 — cont'd

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with

$\Gamma = 4S_1S_2 - P_0Q + P_0'Q'$ $\qquad$ $\Gamma' = 4S_1S_2 + P_0Q - P_0'Q'$

$\Delta = \Delta(4S_1S_2, -P_0Q, -P_0'Q']$ $\qquad$ $\Delta(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$

$\sqrt{\Delta}$ = one of the two complex roots, to be fixed by the diacritical constraints.

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e) Transversity and helicity amplitudes

$$\begin{array}{ccc} & & \lambda_p \\ \lambda_\Lambda & \lambda_{K^*} & +\tfrac{1}{2} \quad -\tfrac{1}{2} \end{array}$$

$$\begin{array}{cc|cc|} & 1 & . & c' \\ +\dfrac{1}{2} & 0 & a & . \\ & -1 & . & b' \\ \hline & 1 & b & . \\ -\dfrac{1}{2} & 0 & . & a' \\ & -1 & c & . \end{array} = T^{\lambda_{K^*}}_{2\lambda_\Lambda, 2\lambda_p}; \qquad \begin{array}{|cc|} C' & -C \\ A & A' \\ B' & -B \\ \hline B & B' \\ A' & -A' \\ C & C \end{array} = H^{\lambda_{K^*}}_{2\lambda_\Lambda, 2\lambda_p}$$

$2(A + iA') = -\sqrt{2}\,(b + c),$ $\qquad$ $2(A - iA') = \sqrt{2}\,(b' + c')$

$2(B + iB') = -\sqrt{2}\,a + (b - c),$ $\qquad$ $2(B - iB') = \sqrt{2a'} - (b' - c'),$

$2(C + iC') = -\sqrt{2}\,a - (b - c),$ $\qquad$ $2(C - iC') = \sqrt{2}\,a' + (b' - c').$

The Byers-Yang [5] amplitudes are given by

$a_+ = -a', \quad b_+ = -(b' + c')/\sqrt{2},$ $\qquad$ $c_+ = -i(-b' + c')/\sqrt{2}$

$a_- = -a, \quad b_- = (b + c)/\sqrt{2},$ $\qquad$ $c_- = i(-b + c)/\sqrt{2}$

if one uses standard s, t, and u transversity frames for the quantization of p, K*, and Λ respectively.

e') Collinearity constraints

$$A' = B' = C' = C = 0 \;\Leftrightarrow\; b = -b', \qquad a = -a' = \sqrt{\frac{1}{2}}\,(c - b), \quad c = -c'.$$

f) Expression of the observables in c) and d) as functions of the transversity amplitudes

$2\sigma P_0 = |a|^2$ $\qquad$ $2\sigma P_0' = |a'|^2$

$2\sigma P = \dfrac{1}{2}\,(|b|^2 + |c|^2)$ $\qquad$ $2\sigma P' = \dfrac{1}{2}\,(|b'|^2 + |c'|^2)$

$2\sigma Q = b\bar{c}$ $\qquad$ $2\sigma Q' = c'\bar{b}'$

$2\sigma S_1 = \dfrac{1}{2}\,(b\bar{a} - a'\bar{b}')$ $\qquad$ $2\sigma S_2 = \dfrac{1}{2}\,(c'\bar{a}' - a\bar{c})$

Table 8 — cont'd

g) Algebraic reconstruction of the amplitudes

$$|a|^2 = 2\sigma P_0 \qquad |a'|^2 = 2\sigma P_0$$

$$|b|^2 = 2\sigma\left(P + \varepsilon\sqrt{P^2 - |Q|^2}\right) \qquad |c'|^2 = 2\sigma\left(P' + \varepsilon\sqrt{P'^2 - |Q'|^2}\right)$$

$$|c|^2 = 2\sigma\left(P - \varepsilon\sqrt{P^2 - |Q|^2}\right) \qquad |c'|^2 = 2\sigma\left(P' - \varepsilon\sqrt{P'^2 - |Q'|^2}\right)$$

$$\phi_b - \phi_c = \operatorname{Arg} Q \qquad \phi_{c'} - \phi_{b'} = \operatorname{Arg} Q'$$

$$\begin{cases}\phi_b - \phi_a = \operatorname{Arg}\left(\Gamma - \sqrt{\Delta}\right)/S_2 \\ \phi_a - \phi_c = \operatorname{Arg}\left(-\Gamma - \sqrt{\Delta}\right)/S_1\end{cases} \qquad \begin{cases}\phi_{a'} - \phi_{b'} = \operatorname{Arg}\left(-\Gamma' - \sqrt{\Delta}\right)/S_2 \\ \phi_{c'} - \phi_{b'} = \operatorname{Arg}\left(\Gamma' - \sqrt{\Delta}\right)/S_1\end{cases}$$

with the expression for Γ, Γ', $\sqrt{\Delta}$ given in d), and the signs ε, ε' fixed by:

$$\varepsilon[|S_1|^2\,|\Gamma - \sqrt{\Delta}|^2 - |S_2|^2\,|\Gamma + \sqrt{\Delta}|^2] \geqq 0, \qquad \varepsilon'[|S_1|^2\,|\Gamma' + \sqrt{\Delta}|^2 - |S_2|^2\,|\Gamma' - \sqrt{\Delta}|^2] \geqq 0$$

Table 9

Amplitude reconstruction for reactions of type $\pi p \to K^*\Lambda(0^-1/2^+ \to 1^{-\varepsilon}\,1/2^+)$ with polarized target

a) Combined production and joint decay angular distribution, and measurement of the polarization transfer by the method of moments

$$I(\psi, \theta\phi, \theta'\phi') = \frac{1}{2\pi}\sum_{L,L'} C_{K^*}(L)\, C_\Lambda(L')$$

$$X \sum_{M,M'}\left[\overline{t^{LL'}_{MM'}} + P_T\cos\psi\,\overline{{}^z t^{LL'}_{MM'}} + P_T\sin\psi\,\overline{{}^x t^{LL'}_{MM'}} + P_T\,\overline{{}^y t^{LL'}_{MM'}}\right] Y^L_M(\theta,\phi)\, Y^{L'}_{M'}(\theta',\phi')$$

$$C_{K^*}(L)\, C_\Lambda(L')\left(t^{LL'}_{MM'} + P_T\,{}^y t^{LL'}_{MM'}\right) = \langle Y^L_M(\theta,\phi)\, Y^{L'}_{M'}(\theta',\phi')\rangle$$

$$C_{K^*}(L)\, C_\Lambda(L')\, P_T\,{}^z t^{LL'}_{MM'} = \langle 2\cos\psi\, Y^L_M(\theta,\phi)\, Y^{L'}_{M'}(\theta'\phi')\rangle$$

$$C_{K^*}(L)\, C_\Lambda(L')\, P_T\,{}^x t^{LL'}_{MM'} = \langle 2\sin\psi\, Y^L_M(\theta,\phi)\, Y^{L'}_{M'}(\theta'\phi')\rangle$$

with

$$C_{K^*}(0) = C_\Lambda(0) = 1/\sqrt{4\pi}, \qquad C_{K^*}(2) = -1/\sqrt{2\pi}, \qquad C_\Lambda(1) = \alpha_1/\sqrt{4\pi}$$

(all other C coefficients vanish)