Chapitre 5 .

LA GEOMETRIE DU DOMAINE DE POLARISATION ≠

1. Positivity.
2. Convexity.
3. Symmetry plane of a convex domain.
4. The one particle polarization state with B-symmetry and/or even polarization.

1. Positivity.

In this appendix we consider the Hermitian operators on the Hilbert space $\mathcal{H}_n$ of dimension n . They form a n^2 dimensional vector space $\mathcal{E}_{n^2}$ on which we can put the Euclidian scalar product (as we did in equation I(4)),

$$(\rho_1 , \rho_2) = \mathrm{tr}\, \rho_1 \rho_2 . \tag{1}$$

So $\mathcal{E}_{n^2}$ is an Euclidian space; the distance between the two points ρ_1 and ρ_2 is $|\rho_1 - \rho_2| = (\rho_1 - \rho_2 . \rho_1 - \rho_2)^{1/2}$.

A Hermitian matrix can be written

$$R = \Sigma_i \lambda_i P_i \tag{2}$$

where λ_i are its eigenvalues (real numbers) and P_i are Hermitian projectors, i.e.,

$$P_i^* = P_i = P_i^2 \tag{3}$$

the multiplicity of the λ_i eigenvalue is given by

$$\mathrm{tr}\, P_i = \mathrm{rank}\, P_i \tag{4}$$

≠ Ce chapitre est le texte, avec quelques corrections, de l'appendice I.A 6 de notre preprint "Polarization density matrices", First issue.

Definition. A Hermitian matrix is positive (or semi-positive) if all its eigenvalues λ_i are > 0 (or ≥ 0).

Outside this appendix we generally use the word positive for semi-positive In this more technical appendix we shall distinguish the two notions. We shall note $R > 0$, $R \geq 0$ for R positive, R semi-positive.

Theorem 1. $R > 0$ (or $R \geq 0$) $\Leftrightarrow$ any $|x> \in \mathcal{H}_n$, $<x|Rx> > 0$ (or ≥ 0).

Let $R = \Sigma_i \lambda_i P_i$

then $<x|Rx> = \Sigma_i \lambda_i \quad <x|P_i x> = \Sigma_i \lambda_i \quad <x|P_i P_i x> = \Sigma_i \lambda_i <P_i x|P_i x>$

since $<P_i x|P_i x> = \|P_i x\|^2 \geq 0 \quad , \quad \lambda_i \geq 0 \Rightarrow <x|Rx> \geq 0$

(Note $|x> \neq 0 \Rightarrow$ some $P_i|x> \neq 0$ since $\Sigma_i P_i = I$, so $\lambda_i > 0 \Rightarrow <x|Rx> > 0$).

Conversely if for all $|x> \in \mathcal{H}_n$, $<x|Rx> > 0$ (or ≥ 0) this is true for the eigenvectors of R ; e.g. if $P_i|x_i> = |i>$, then $<i|Ri> = \lambda_i > 0$ (or ≥ 0), hence all eigenvalues are > 0 (or ≥ 0).

Theorem 2. If $\alpha_i > 0$, $R_i > 0$, then $\Sigma \alpha_i R_i > 0$. Indeed for every $x> \in \mathcal{H}_n$, $<x|R_i x> > 0$ so $\Sigma_i \alpha_i <x|R_i x> = <x|\Sigma_i \alpha_i R_i x> > 0$.

Definition. A cone $\mathcal{C}$ in a vector space with vertex at the origin is a set of points such that $a \in \mathcal{C}$ and $\lambda \geq 0$ implies $\lambda a \in \mathcal{C}$.

As a corollary of theorem 2 we see that the positive matrices on $\mathcal{H}_n$ form a cone $\mathcal{C}_n$ in $\mathcal{E}_{n^2}$, which is an open set of $\mathcal{E}_{n^2}$. The boundary of the cone, $\partial \mathcal{C}_n$, is the set of semi-positive matrices which are not positive ; those matrices then have some eigenvalues zero. The rank of a matrix ρ is the dimension of its image space $\rho\mathcal{H}_n$. Positive matrices have rank n ; matrices of $\partial\mathcal{C}_n$ have rank $< n$. Let us call $\partial_k \mathcal{C}_n$ $(k < n)$ the set of semi-positive matrices of rank k

$$\partial \mathcal{C}_n = \bigcup_{0<k<n} \partial_k \mathcal{C}_n \tag{5}$$

As we saw in I, in some experiments an upper limit r of the rank of the polarization matrix is known. So it belongs to $\bigcup_{0<k\leq r} \partial_k \mathcal{C}_n$. We denote

$\overline{\mathcal{C}}_n = \mathcal{C}_n \cup \partial\mathcal{C}_n$ the closure of $\mathcal{C}_n$. It is the set of $\rho \geqslant 0$. Note that if $\rho_1 \geqslant 0$ and $\rho_2 \geqslant 0$, $\rho_1\rho_2$ is not necessary $\geqslant 0$. However :

<u>Theorem 3.</u> If $\rho_1 \geqslant 0$ and $\rho_2 \geqslant 0$, then $\mathrm{tr}\,\rho_1\rho_2 \geqslant 0$. Indeed ρ_1 can be written $\rho_1 = \Sigma_i \lambda_i P_i = \Sigma_i \lambda_i |i\rangle\langle i|$ where the $|i\rangle \in \mathcal{H}_n$ form an orthonormal basis of eigenvectors of ρ_1 and $\lambda_i \geqslant 0$. Then

$$\mathrm{tr}\,\rho_1\rho_2 = \Sigma_i \lambda_i \langle i|\rho_2|i\rangle$$

which is $\geqslant 0$ by theorem 1. More specifically, if $\rho_1 > 0$, $\rho_2 \geqslant 0$, theorem 1 shows that $(\rho_1, \rho_2) > 0$. Indeed

$$\mathrm{tr}\,\rho_1\rho_2 = \Sigma_i \lambda_i \,\mathrm{tr}\,\rho_2 P_i \quad \text{with} \quad \Sigma_i P_i = I \,, \quad \lambda_i > 0.$$

Since

$\Sigma_i \,\mathrm{tr}\,\rho_2 P_i = \mathrm{tr}\,\rho_2 = 1$, at least one of the terms $\mathrm{tr}\,\rho_2 P_i = \langle i|\rho_2|i\rangle$ is >0

so $\mathrm{tr}\,\rho_1\rho_2 > 0$.

Hence as a corollary, if two polarization states are orthogonal, they are both in $\partial\mathcal{C}_n$.

Consider now equations (6) to (8) of part I, where $\mathcal{E}_N$ is the n^2-1 dimensional subspace of $\mathcal{E}_{n^2}$ defined by the condition $\mathrm{tr}\,\rho = 1$. It is an Euclidian subspace of $\mathcal{E}_{n^2}$ and the polarization domain $\mathcal{D}_n$, i.e., the set of density matrices $\rho \geqslant 0$, $\mathrm{tr}\,\rho = 1$ is

$$\mathcal{D}_n = \overline{\mathcal{C}}_n \cap \mathcal{E}_N . \tag{6}$$

The density matrix of the unpolarized state is

$$\rho_o = \frac{1}{n} I \tag{7}$$

Since the vector ρ_o in $\mathcal{E}_{n^2}$ is orthogonal to the subspace $\mathcal{E}_N$ of the Euclidian space $\mathcal{E}_{n^2}$, with

$$\rho = \sqrt{\frac{n-1}{n}}\,\rho' + \rho_o \tag{8}$$

we deduce

$$(\rho, \rho) = \frac{n-1}{n}\,\mathrm{tr}\,\rho'^2 + \frac{1}{n}$$

so

$$1 \geqslant (\rho, \rho) \geqslant \frac{1}{n} = (\rho_o, \rho_o) \tag{9}$$

Relation (9) satisfied by spin density matrices has been used in the physics literature. It is much weaker than $\mathrm{tr}\,\rho = 1$, $\rho \geq 0$. We saw in Fig. 1 page I. 2-3, that $\mathcal{D}_n$ is inside the sphere $\mathcal{S}_{N-1}$, intersection of $\mathcal{E}_N$ $(\mathrm{tr}\,\rho = 1)$ and of the unit sphere $\mathcal{S}_N \,(\!(\rho, \rho)\!) = 1)$ of $\mathcal{E}_{n^2} = \mathcal{E}_{N+1}$. Furthermore, the domain

$$\mathcal{D}_n \cap \mathcal{S}_N = \partial_1 \mathcal{D}_n \tag{10}$$

is the set of density matrices of pure states, i. e., rank one projector $P_i = P_i^2$ $\mathrm{tr}\,P_i = \mathrm{tr}\,P_i^2 = 1$.

In part I and in all applications, we will multiply the length in the N dimensional Euclidian space $\mathcal{E}_N$ by the factor $\sqrt{\frac{n}{n-1}}$ so that the sphere $\mathcal{S}_{N-1}$ of center ρ_o has radius one in the new scale.

2. Convexity.

Definition of convexity. A domain D of a real vector space $\mathcal{E}$ is convex if all vectors $\alpha a + \beta b$ with $0 \leq \alpha$, $0 \leq \beta$, $\alpha + \beta = 1$ are elements of D when $a \in D$, $b \in D$. We can also say that all points between a and b on the straight line joining a and b belong to the domain when a and b do.

Example of a convex domain : the linear manifolds of $\mathcal{E}_n$ that we also call k-planes when their dimension is $k \leq n$.
It is easy to check that :

a) The intersection of convex domains is a convex domain.

b) The linear transformed of a convex domain is a convex domain.

c) A convex domain is connex.

Theorem 2 shows that the cone $\mathcal{C}$ of positive (also the cone $\overline{\mathcal{C}}$ of semi-positive) matrices in $\mathcal{H}_n$ is convex.

From property a) and equation (6) the polarization domain $\mathcal{D}_n$ is convex. Since $\mathcal{D}_n$ is in the Euclidian spaces $\mathcal{E}_N \subset \mathcal{E}_{n^2}$, we want to make some geometrical remarks on convex domains of Euclidian space $\mathcal{E}$ and their orthogonal projection on a subspace $\mathcal{P}$. Let P be this orthogonal projection; it is the identity on $\mathcal{P}$ and $P^2 = P$. Let $\mathcal{D}$ be a domain of $\mathcal{E}$, the domain $\Gamma = P\mathcal{D} \subset \mathcal{P}$ is called the projection of $\mathcal{D}$ on $\mathcal{P}$. From remark b), $\mathcal{D}$ convex $\Rightarrow$ $P\mathcal{D}$ convex,

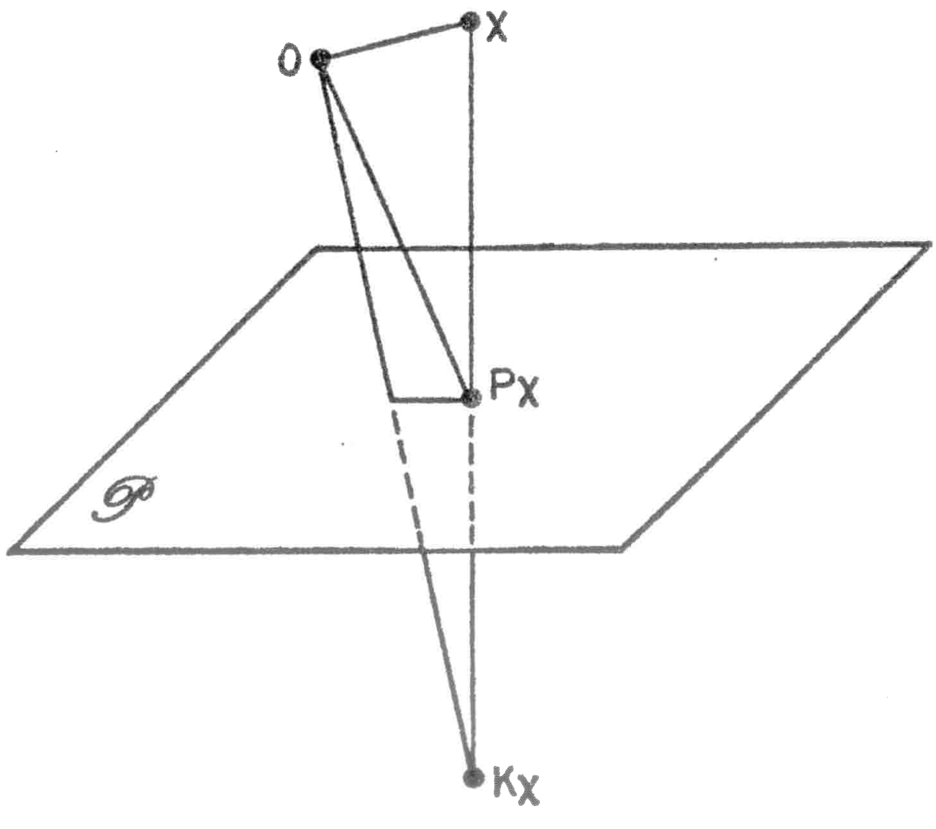

FIG.1

Fig. 1. - Point x of a Euclidian vector space, with its projection Px on the subspace $\mathcal{P}$, and its symmetric Kx through $\mathcal{P}$. Remark that $Kx = 2Px - x$.

of course $C = \mathcal{D} \cap \mathcal{P} \subset \Gamma$. We can always consider an Euclidian space $\mathcal{E}$ as a vector space, after we have chosen a point 0 of $\mathcal{E}$ to be the origin of the vector space, so if $a, b \in \mathcal{E}$, $a+b$ is defined. An involution K of $\mathcal{E}$ is an Euclidian transformation (i.e., it preserves the distance) whose square is the identity I in $\mathcal{E}$ i.e., $K^2 = I$. It can be shown that the fixed points of K form an Euclidian subspace $\mathcal{P} \subset \mathcal{E}$ and K is the symmetry through $\mathcal{P}$ i.e., if P is the projection on $\mathcal{P}$ and $x \in \mathcal{E}$ one sees (see Fig. 1) :

$$Kx = x - 2(x - Px) = -x + 2Px \tag{11}$$

$$\text{i.e.} \quad K = -I + 2P \tag{11'}$$

(Note that $\mathcal{P}$ may be reduced to a point !). From now on we denote by $K_{\mathcal{P}}$ the symmetry through $\mathcal{P}$.

Definition. If $K_{\mathcal{P}}\mathcal{D} = \mathcal{D}$, then $\mathcal{P}$ is called a symmetry p-plane for $\mathcal{D}$, where $p = \dim \mathcal{P}$.

Theorem 4. If $\mathcal{P}$ is a symmetry plane of the convex domain $\mathcal{D}$, then

$$C = \Gamma \quad \text{where} \quad C = \mathcal{D} \cap \mathcal{P} \quad \text{and} \quad \Gamma = P_{\mathcal{P}}\mathcal{D} .$$

Let $a \in \Gamma$, there exists $b \in \mathcal{D}$ such that $Pb = a$. Since $\mathcal{P}$ is a symmetry plane $K_{\mathcal{P}} b \in \mathcal{D}$ and from the convexity $\frac{1}{2}(b + K_{\mathcal{P}} b) \in \mathcal{D}$ and from (11) $\frac{1}{2}(b + Kb) = Pb = a$ so $a \in C$, hence $\Gamma \subset C$ and since $C \subset \Gamma$, therefore $C = \Gamma$. Of course, this was geometrically obvious.

The diagonal matrices of $\mathcal{D}_n$.

Let us remark that there are p-planes $\mathcal{P}$ such that $\mathcal{D} \cap \mathcal{P} = P_{\mathcal{P}}\mathcal{D}$ which are not symmetry planes of $\mathcal{D}$. Such an example is obtained by the domain Δ_n of the semi-positive diagonal matrix of trace 1. So $\Delta_n \subset \mathcal{D}_n$ the domain of density matrices. Let $\mathcal{P}$ be the n-plane of $\mathcal{E}_{n^2}$ containing all diagonal Hermitian matrices on $\mathcal{H}_n$. By definition $\Delta_n = \mathcal{P} \cap \mathcal{D}_n$. We now remark that if $\rho \geq 0$, each diagonal matrix element ρ_{ii} is positive. Indeed let $|i\rangle$ the vector of coordinates $\xi^{\alpha} = \delta_i^{\alpha}$ ($\alpha = 1$ to n) then $\langle i|\rho|i\rangle = \rho_{ii} \geq 0$ by theorem 2. Hence $P_{\mathcal{P}} \cap \mathcal{D}_n = \Delta$.

If $n = 1$ or $n = 2$, the n-plane of diagonal matrices is a symmetry plane of $\mathcal{D}_n$. This is not true for $n \geq 0$ as it is shown by the two Hermitian matrices

λ, λ' with $tr\lambda = tr\lambda' = 1$, $\lambda' = K_\rho \lambda$ (i.e., the non diagonal elements are changed of sign).

$$\lambda = \frac{1}{5}\left[\begin{array}{ccc|c} 1 & 1 & 1 & \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & \\ \hline & 0 & & 0 \end{array}\right] \qquad \lambda' = K_\rho \lambda = \left[\begin{array}{ccc|c} 1 & -1 & -1 & \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & \\ \hline & 0 & & 0 \end{array}\right]$$

the eigenvalues of λ are $\frac{1}{5}$, $\frac{1}{5}(2+\sqrt{2})$, $\frac{1}{5}(2-\sqrt{2})$, and n-3 times zeros, those of λ' are $\frac{3}{5}$, $\frac{1}{5}(1+\sqrt{2})$, $\frac{1}{5}(1-\sqrt{2})$, and n-3 times zeros and $1-\sqrt{2} < 0 \Rightarrow \lambda'$ is not semi-positive.

Note that the convex domain Δ_n is the regular polyhedra in $R^n \subset \mathcal{E}_{n^2}$ $0 \leqslant \lambda_i$, $\Sigma_{i=1}^n \lambda_i = 1$. It is in $\mathcal{E}_N$ ($tr\rho = 1$). It is the regular n-hedron (for n=3 : equilateral triangle, n=4 : tetrahedron, etc...) whose center is $\rho_o = \frac{1}{n}\mathbb{1}$. Its n vertices are the n diagonal rank one projectors P_i (one $\lambda_i = 1$, all other are zero). The straight line $P_i\rho_o$ cut the n-hedron in Q_i at the center of the face opposite to the vertex P_i and it is perpendicular to it.

(Note that $(Q_i)_{\alpha\beta} = \frac{1}{n-1}(\delta_{\alpha\beta} - \delta_{i\alpha}\delta_{i\beta})$).

The square of the distance $\rho_o P_i$ in $\mathcal{E}_N$ is $tr(\frac{1}{n} - P_i)^2 = \frac{n-1}{n}$, in $\mathcal{E}'_N$ is 1.

The square of the distance $\rho_o Q_i$ in $\mathcal{E}_N$ is $tr(\rho_o - Q_i)^2 = \frac{1}{n}$, in $\mathcal{E}'_N$ is $\frac{1}{n-1}$.

Finally, the scalar product in $\mathcal{E}'_N$ of two distinct pure states is ($i \neq j$), $\frac{n}{n-1} tr(P_i - \rho_o)(P_j - \rho_o) = -\frac{1}{n-1}$ as equation I (8) shows since

$$P_i P_j = 0 \Rightarrow (P_i, P_j) = 0 .$$

3. Symmetry planes of a convex domain.

Involutions in $\mathcal{E}$. We will now study some physically interesting examples of involution in $\mathcal{E}_{n^2}$.

Let U(n) be the unitary group of $n \times n$ matrices on $\mathcal{H}_n$. It acts on $\mathcal{E}_{n^2}$ as

$$\rho \rightsquigarrow u\rho u^* = u\rho u^{-1} \qquad (12)$$

Since such a unitary transformation respects the scalar product on $\mathcal{E}_{n^2}$

$$(\rho_1, \rho_2) = \operatorname{tr} \rho_1 \rho_2 = \operatorname{tr} u \rho_1 u^{-1} u \rho_2 u^{-1} = (u \rho_1 u^{-1}, u \rho_2 u^{-1}) \qquad (13)$$

it is an isometry of $\mathcal{E}_{n^2}$. Furthermore it preserves the eigenvalues of ρ so U(n) transforms $\mathcal{D}_n$ in itself. If $u^2 = 1$, the transformation (12) is an involution on $\mathcal{E}_{n^2}$. Let us give examples of such involution :

The B-symmetry. For the B-symmetry, that we shall denote K_B

$$u = D^{(j,n)}(S_n) \quad \text{or} \quad u = \bigotimes_k D^{(j_k, n_k)}(S_n) \qquad (14)$$

where S_n is the symmetry in space-time through the reaction plane (three-plane of $\underline{p}_A$, $\underline{p}_B$, $\underline{p}_C$, the energy momentum of the beam particle A, of the target particle B and of the observed particle C), $\underline{n}$ is the unit (space-like) vector normal to this three-plane. Since $S_n^2 = 1$, $u^2 = (-1)^{2j}$ (see equation A 1 (25) or A 1 (26)) then $\rho \rightsquigarrow u^2 \rho u^{-2} = \rho$, the B-symmetry does induce an involution on $\mathcal{E}_{n^2}$. The fixed points of this involution form the p-plane $\mathcal{B}$ of B-symmetric matrices, with

$$\left.\begin{aligned} p &= \frac{1}{2} n^2 && \text{if } n \text{ is even} \\ p &= \frac{1}{2}(n^2 + 1) && \text{if } n \text{ is odd} \end{aligned}\right\} \qquad (15)$$

So

$$C_{\mathcal{B}} = \mathcal{B} \cap \mathcal{D}_n = \Gamma_{\mathcal{B}} \qquad (16)$$

is the set of B-symmetric density matrices.

Indeed, (as we shall see in part IV) in an experiment with polarized target if we observe only the B-symmetric part of the polarization matrix, it is a positive matrix which is that which would have been observed if the target had not been polarized.

The set of alignment matrices. As we have seen in I. 4, if $\mathcal{H}_n(\underline{p})$ is the 2j+1 dimensional Hilbert space of polarization states of a particle of spin j and energy momentum $\underline{p}$, the little group $\mathcal{L}_p$ of the "rotations" and space-like symmetries of the Lorentz group $\mathcal{L}$ which leave $\underline{p}$ invariant, acts on $\mathcal{H}_n(\underline{p})$ through the 2j+1 dimensional irrep of the orthogonal O(3) group (three-dimensional rotation and symmetry group) $D^{(j,\eta)}$. Therefore this group acts on $\mathcal{E}_{n^2}$ by the representatio $D^{(j,\eta)} \otimes \overline{D^{(j,\eta)}} \sim D^{(j,\eta)} \otimes D^{(j,\eta)} \sim \oplus_{L=0}^{2j} D^{(L,+1)}$. This induces the decomposition of $\mathcal{E}_{n^2}$ into the direct sum of space

$$\mathcal{E}_{n^2} = \oplus_{L=0}^{2j} \mathcal{E}^{(L)} \tag{17}$$

$$\mathcal{E}_N = \oplus_{L=1}^{2j} \mathcal{E}^{(L)} \tag{17'}$$

and the corresponding decomposition of $\rho \in \mathcal{E}_{n^2}$

$$\rho = \rho_o + \Sigma_{L=1}^{2j} \rho^{(L)} \tag{18}.$$

A density matrix ρ is an alignment matrix if all its components $\rho^{(L)} = 0$ for L odd; we also say ρ has only even-multipoles. The generalization to $\mathcal{H}_n = \otimes_{k=1}^{r} \mathcal{H}_{2j_k+1}(p_k)$, the Hilbert space of polarization states of r particles with spin j_k, energy momentum p_k, is straightforward. (See I(11') for two particles). The irrep of O(3) on $\mathcal{H}_n$ is $\otimes_k D^{(j_k,\eta_k)}$ and ρ can be expanded in r- uple multipoles

$$\rho = \Sigma_i \Sigma_{L_{k_i}=0}^{2j_k} \rho^{(L_{k_1}, L_{k_2}, \dots, L_{k_r})} \tag{19}$$

ρ is an alignment matrix if $\rho^{(L_{k_1}, \dots, L_{k_r})} = 0$ when $\Sigma_i L_{k_i}$ is odd. Note that the polarization matrix of the i-particle is

$$\rho_i = \Sigma_{L_k=0}^{2j_i} \rho^{(0,0,\dots,L_k,\dots,0)} \tag{20}$$

so that the alignment matrix of ρ_i is just that obtained from the single-particle polarization matrix.

We can define the involution K_a on $\mathcal{E}_{n^2}$

$$K_a \rho = \Sigma_{L=0}^{2j} (-1)^L \rho^{(L)} \tag{21}$$

when ρ is given by (18) and

$$K_{\mathcal{A}}\rho = \Sigma_i \Sigma_{L_{k_i}=0}^{2j_i} (-1)^{\Sigma_i L_{k_i}} \rho^{(L_{k_1}, \dots, L_{k_n})} \qquad (22)$$

when ρ is given by (19).

Note that the fixed points of $K_{\mathcal{A}}$ form the p-plane $\mathcal{A}$ of aligned matrices, and $K_{\mathcal{A}}$ does not change the length of the matrices $(K_{\mathcal{A}}\rho\,,\, K_{\mathcal{A}}\rho) = (\rho, \rho)$.

We want to show that $K_{\mathcal{A}}$ leaves $\mathcal{D}_n$ invariant, so $\mathcal{A}$ is a symmetry plane of $\mathcal{D}_n$.

Consider the involution $K_{\mathcal{T}}$ which transposes the matrices of $\mathcal{E}_{n^2}$:

$$K_{\mathcal{T}}\rho = \rho^T \qquad (23)$$

Its fixed points form the subspace of the symmetrical matrices of $\mathcal{E}$; $K_{\mathcal{T}}$ does not change the eigenvalues of the matrices so it transforms $\mathcal{D}_n$ in itself. Let us call $K_{\mathcal{S}}$ the involution induced (as in 12) by

$$u = D^{(j,\eta)}(S_{n(2)}) = \eta\Gamma^j \,, \qquad \text{A 1.(53)}$$

where

$$(\Gamma^j)^{\lambda}{}_{\lambda'} = (-1)^{j-\lambda}\, \delta^{-\lambda}{}_{\lambda'} \,, \qquad \text{A 1.(50)}$$

indeed $\qquad u^2 = (\Gamma^j)^2 = (-1)^{2j}\mathbb{1}$.

We remark that

$$K_{\mathcal{T}}K_{\mathcal{S}} = K_{\mathcal{S}}K_{\mathcal{T}} \,, \qquad (24)$$

indeed

$$(\Gamma^j \rho (\Gamma^j)^{-1})^T = \Gamma^j \rho^T (\Gamma^j)^{-1}$$

because

$$(\Gamma^j)^T = (-1)^{2j}\,\Gamma^j \qquad . \qquad \text{A 1.(51)}$$

Furthermore the product of two commuting involutions is an involution. This is the case of $K_{\mathcal{T}}K_{\mathcal{S}} = K_{\mathcal{S}}K_{\mathcal{T}}$. We have shown in A 2. that

$$K_{\mathcal{A}} = K_{\mathcal{T}}K_{\mathcal{S}} \qquad (25)$$

Indeed equation A 2.(57) is

$$K_{\mathcal{S}}K_{\mathcal{T}}T^L_M = \Gamma^j (T^L_M)^T (\Gamma^j)^{-1} = (-1)^L T^L_M \,.$$

Since both $K_{\mathcal{C}}$ and $K_{\mathcal{J}}$ transform $\mathcal{D}_n$ into itself, this is also the case of $K_{\mathcal{A}}$.

To summarize, the subspace of alignment matrices is a symmetry plane of $\mathcal{D}_n$ and $C_{\mathcal{A}} = \Gamma_{\mathcal{A}}$.

Theorem 5. Let $\mathcal{V}$ and $\mathcal{W}$ be two symmetry planes (which might be of different dimension) of the convex domain $\mathcal{D}_n$. If

$$P_{\mathcal{V}} P_{\mathcal{W}} = P_{\mathcal{W}} P_{\mathcal{V}} \tag{26}$$

then $\mathcal{W}$ is a symmetry plane of $C_{\mathcal{V}} = \mathcal{V} \cap \mathcal{D} = P_{\mathcal{V}} \mathcal{D} = \Gamma_{\mathcal{V}}$. (By syntactic symmetry $\mathcal{V}$ is a symmetry plane of $\mathcal{W} \cap \mathcal{D}$).

From (11'), $K_{\mathcal{C}} = -I + 2P_{\mathcal{C}}$ we obtain that $P_{\mathcal{V}}, P_{\mathcal{W}}, K_{\mathcal{V}}, K_{\mathcal{W}}$ form a set of commuting operators. Let $a \in \mathcal{V} = P_{\mathcal{V}} \mathcal{E}$. So there exists x such that $a = P_{\mathcal{V}} x$

$$K_{\mathcal{W}} a = K_{\mathcal{W}} P_{\mathcal{V}} x = P_{\mathcal{V}} K_{\mathcal{W}} x \in P_{\mathcal{V}} \mathcal{E} = \mathcal{V}.$$

Hence $\mathcal{W}$ is a symmetry plane of $\mathcal{V}$; it is also a symmetry plane of $\mathcal{D}_n$, hence it is a symmetry plane of their intersection : $\mathcal{V} \cap \mathcal{D}$.

Corollary. Of course, instead of (26) we could have used

$$K_{\mathcal{V}} K_{\mathcal{W}} = K_{\mathcal{W}} K_{\mathcal{V}} \tag{26'}$$

in the theorem, since $P_{\mathcal{P}} = \frac{1}{2}(I + K_{\mathcal{P}})$.

We note that $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$ commute. Indeed for one particle states, if we shorten $D^{(j,\eta)}(S_n)$ into D^j we have

$$K_{\mathcal{A}} K_{\mathcal{B}} \rho = (\Gamma^j D^j \rho (D^j)^{-1} (\Gamma^j)^{-1})^T = \bar{\Gamma}^j \bar{D}^j \bar{\rho} (\bar{D}^j)^{-1} (\bar{\Gamma}^j)^{-1} = \Gamma^j D^j \rho^T D^{j-1} \Gamma^{j-1}$$

since ρ and $K_{\mathcal{A}} K_{\mathcal{B}} \rho$ are Hermitian matrices and $\bar{\Gamma}^j = \Gamma^j$, (A 1.50); using equations (A 1.49) : $\bar{D}^j = \Gamma D \Gamma^{-1}$, and (A 1.51) : $\Gamma^{j-1} = (-1)^{2j} \Gamma^j = (\Gamma^j)^T$ we can transform this equation into :

$$K_{\mathcal{A}} K_{\mathcal{B}} \rho = D^j \Gamma^{j-1} \rho^T C^j D^{j-1} = K_{\mathcal{B}} (\Gamma^{j-1} \rho^T \Gamma^j) = K_{\mathcal{B}} (\Gamma^j \rho^T \Gamma^{j-1}) = K_{\mathcal{B}} K_{\mathcal{A}} \rho .$$

For r particle states

$$D^j = \otimes_i D^{(j_i, \eta_i)}(S_n)$$

and Γ^j is to be replaced by

$$\otimes_i \Gamma^{j_i}$$

and the proof still holds.

So theorem 5 tells us that $\mathcal{B}_\cap \mathcal{D}_n$ the domain of B-symmetric density matrices has $\mathcal{A}$, the domain of aligned matrices as symmetry plane and also is a symmetry plane of $\mathcal{A}_\cap \mathcal{D}_n$.

From now on we will prefer the expression even polarization to alignment.

<u>One particle-state, B-symmetry, even polarization and pure states.</u>

As we have seen in A 3, for one particle, B-symmetry imposes to ρ the conditions :

in transversity quantization $\rho^{m}{}_{m'} = (-)^{m-m'} \rho^{m}{}_{m'}$, (27)

in helicity quantization $\rho = \Gamma \rho \Gamma^{-1}$. (28)

For a pure state $\rho = x><x$. Let ξ be the components of x .

In transversity, B-symmetry is equivalent to

either $\xi^{j-1} = \xi^{j-3} = \xi^{j-5} = \ldots = 0$, (29)

or $\xi^{j} = \xi^{j-2} = \xi^{j-4} = \ldots = 0$. (29')

In helicity, B-symmetry is equivalent to $\Gamma|x> = \lambda|x>$ with $\lambda^2 = (-)^{2j}$ since $\Gamma^2 = (-)^{2j} I$ so, explicitly

$$\lambda \xi^{m} = (-)^{j-m} \xi^{-m} \quad (30)$$

As we have just seen, to say that the polarization matrix ρ has only even polarization is equivalent to the condition, for any quantization axis

$$\rho^{T} = \Gamma \rho \Gamma^{-1} = \bar{\rho} . \quad (31)$$

It implies for pure states

$$\Gamma |x> = \lambda|\bar{x}> .$$

If we multiply by Γ both members we obtain

$$(-)^{2j}|x> = \lambda \Gamma |\bar{x}> ,$$

so $\lambda \bar{\lambda} = (-)^{2j}$. Hence :

<u>Theorem 6</u> : For half integer spin, there are no aligned pure states, i. e., pure states with even polarization only.

4. The one particle polarization state with B-symmetry and/or even polarization.

We had noticed in A 3 that the one-particle polarization matrix can be considered as a checker board with :

- black squares if $(m-m')$ is even (this includes the diagonal)
- white squares if $(m-m')$ is odd.

Then, in any quantization axis :

Even polarization any quantization $\Leftrightarrow$
- i) the black square matrix is symmetrical through the second diagonal
- ii) the white square matrix is antisymmetrical through the second diagonal.

We saw in A. 3 :

B-symmetric Transversity quantization $\Leftrightarrow$ white squares have zero.

B-symmetric Helicity quantization $\Leftrightarrow$
- i) the black square matrix is symmetrical through the center
- ii) the white square matrix is antisymmetrical through the center.

By combining the two sets of conditions we find for one particle, B-symmetric, even polarization density matrix ρ

Transversity quantization:
- white squares are zero.
- The black square matrix is symmetrical through the second diagonal.

Helicity quantization:
- The matrix is real = hermitian and symmetrical through the first diagonal.
- The black square matrix is symmetrical through the center and the second diagonal.
- The white square matrix is antisymmetrical through the center and the second diagonal.

When j is half-integer, let us shuffle the indices of both lines and columns (by a unitary transformation on $\mathcal{K}_{2j+1}$) and write them in the order

$$j,\ j-2,\ j-4, \ldots,\ 3-j,\ 1-j,\ -j,\ -j+2,\ \ldots,\ j-1$$

for example

for $j = \frac{3}{2}$, order $\frac{3}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$, $\frac{1}{2}$

$j = \frac{5}{2}$ $\frac{5}{2}$, $\frac{1}{2}$, $-\frac{3}{2}$, $-\frac{5}{2}$, $-\frac{1}{2}$, $\frac{3}{2}$

Then ρ , in transversity quantization is the direct sum of twice the same hermitian positive matrices $(j+\frac{1}{2})$ by $(j+\frac{1}{2})$.

When j is integer, with the order of indices

$j,\ j-2, \ldots, 2-j,\ j\ ;\ j-1,\ j-3, \ldots,\ 1-j$

ρ is again a direct sum of two hermitian positive matrices of size $(j+1) \times (j+1)$ and $(j \times j)$ which both are symmetrical through the second diagonal.

Deuxième Partie

APPLICATIONS A LA POLARISATION D'UNE SEULE PARTICULE DE SPIN DETERMINE

0.- Introduction.

Dans cette deuxième partie nous appliquons notre étude géométrique des domaines de polarisation aux cas les plus simples : étude de la polarisation d'une seule particule de spin 1, 3/2 ou 2 produite dans une réaction à quasi deux corps, conservant la parité, à partir de cible et faisceau non polarisés (réactions B-symétriques). Nous nous intéressons plus particulièrement au domaine des paramètres mesurables par distribution angulaire. En général c'est le domaine des paramètres de polarisation paire. Cependant, pour les résonances de spin 3/2 qui se désintègrent en cascade, la polarisation impaire étant également mesurable, nous l'étudions dans un chapitre spécial.

Les cinq chapitres suivants sont écrits en anglais. Leur rédaction a été conçue de manière que l'expérimentateur intéressé par la polarisation d'une particule de spin déterminé puisse lire directement le chapitre qui le concerne. Il y trouvera la manière de mesurer les paramètres de polarisation, la relation entre les différentes paramétrisations possibles, la géométrie du domaine de polarisation et les régions de ces domaines prédits par certains modèles dynamiques courants.

Le chapitre I concerne le spin 1/2. Il a été introduit ici pour des raisons pédagogiques, afin de montrer que les paramètres multipolaires et les domaines de polarisation ne sont que des généralisations du vecteur de polarisation de Stokes et de la sphère de Poincaré.

La table 0. 1 donne une vue globale du nombre de paramètres de polarisation de divers types. Les nombres encadrés donnent la dimension des domaines de polarisation étudiés dans les cinq chapitres suivants.

Dans ces chapitres d'application nous décrivons les différents domaines de polarisation sans montrer comment on les calcule explicitement. Le calcul explicite de ces domaines est rédigé en français dans l'appendice A 2 placé à la fin de cette deuxième partie. Avant cet appendice nous avons également inclus un appendice, noté II A 1, rédigé en anglais, dans lequel nous étudions la relation entre la matrice densité et le vecteur de polarisation qui décrit un état pur d'une particule de spin 1.

TABLE 0.1. - Number of parameters corresponding to even or odd, B- symmetric or B- antisymmetric polarization, for particles of spin j.

(a) $j = 1/2$

	B-sym.	B-ant.	Total
Even	0	0	0
Odd	[1]	2	3
Total	1	2	3

(b) $j = 1$

	B-sym.	B-ant.	Total
Even	[3]	2	5
Odd	1	2	3
Total	4	4	8

(c) $j = 3/2$

	B-sym.	B-ant.	Total
Even	[3]	2	5
Odd	[4]	6	10
Total	7	8	15

(d) $j = 2$

	B-sym.	B-ant.	Total
Even	[8]	6	14
Odd	4	6	10
Total	12	12	24

(e) j = half-integer

	B-sym.	B-ant.	Total
Even	$(j-\frac{1}{2})(j+\frac{3}{2})$	$(j-\frac{1}{2})(j+\frac{1}{2})$	$(2j-1)(j+1)$
Odd	$(j+\frac{1}{2})^2$	$(j+\frac{1}{2})(j+\frac{3}{2})$	$(2j+1)(j+1)$
Total	$2j(j+1)-\frac{1}{2}$	$2(j+\frac{1}{2})^2$	$4j(j+1)$

(f) j = integer

	B-sym.	B-ant.	Total
Even	$j(j+2)$	$j(j+1)$	$j(2j+3)$
Odd	j^2	$j(j+1)$	$j(2j+1)$
Total	$2j(j+1)$	$2j(j+1)$	$4j(j+1)$

1. Polarization of spin $\frac{1}{2}$ particle.

1.1. Measurement of the polarization of spin $\frac{1}{2}$ particle.

Some spin $\frac{1}{2}$ particles (such as Λ, $\Sigma^{\pm}$, Ξ) decay weakly into one spinless and one spin $\frac{1}{2}$ particles, as indicated in Table 1.1 (a). Angular momentum conservation allows two amplitudes : the parity violating and the parity conserving ones. They define the first asymmetry parameter α in Table 1.1 (c). The angular distribution depends on this asymmetry parameter, and is given by the expression in Table 1.1 (b_1). $\vec{P}$ is the Stokes polarization three-vector, and $\vec{e}$ is a unit three-vector whose polar and azimuthal angles θ and φ fix the direction of the spinless decay product, in any frame in which the decaying particle is at rest. This angular distribution is equivalently given in Table 1.1 (b_2) as function of the multipole parameters $t_M^{(1)}$ and the usual spherical harmonics $Y_M^{(1)}$. The inverse expressions, which supply a method of independent measurement for each component of the Stokes vector, or equivalently for each multipole parameter, are given also in Table 1.1 (d_1) and (d_2). The angular brackets $\langle \ldots \rangle$ indicate experimental mean values of the enclosed expression for the ensemble of events. $N_{\pm z}$ is the number of events with $\cos\theta \lessgtr 0$ and analogously for the other axes. Note that the orthonormalization of the multipole parameters $r_M^{(L)}$ has been performed in such a way, that for spin $\frac{1}{2}$ they are identical to the components of the Stokes vector.

In the case of B-symmetry for the production process (i.e., production in a parity conserving reaction with unpolarized target and beam, cf. I.A.3), the Stokes vector has to be oriented along the normal to the production plane, that is along the axis z in the case of transversity or along the axis y in the case of helicity quantization, as indicated in Table 1.1. (e_1) and 1.1 (f_1). Likewise, the multipole parameters $r_M^{(1)}$ with M either odd or positive must be zero, as indicated in Table 1.1 (e_2) and (f_2).

TABLE 1.1. - Measurement of the polarisation of spin $\frac{1}{2}$ particle

(a) Decay

$$\frac{1}{2} \rightarrow \frac{1}{2} + 0 \qquad \text{(weak interaction)}$$

Angular distribution

(b$_1$)

$$I(\theta,\varphi) = \frac{1}{4\pi}\left[1 + \alpha \vec{P}.\vec{e}(\theta,\varphi)\right]$$

(b$_2$)

$$I(\theta,\varphi) = \frac{1}{4\pi} + \frac{\alpha}{\sqrt{4\pi}} \sum_{M=-1}^{+1} \bar{t}^{(1)}_{M}\, Y^{(1)}_{M}(\theta,\varphi)$$

(c) Asymmetry parameter

$$\alpha = \frac{2\ \mathrm{Re}\ A^{(0)}.\bar{A}^{(1)}}{|A^{(0)}|^2 + |A^{(1)}|^2}$$

(d$_1$) Polarization three-vector

$$P_z = \frac{3}{\alpha} \langle\cos\theta\rangle = \frac{2}{\alpha}\,\frac{N_{+z} - N_{-z}}{N_{+z} + N_{-z}}$$

$$P_x = \frac{3}{\alpha} \langle\sin\theta\cos\varphi\rangle = \frac{2}{\alpha}\,\frac{N_{+x} - N_{-x}}{N_{+x} + N_{-x}}$$

$$P_y = \frac{3}{\alpha} \langle\sin\theta\sin\varphi\rangle = \frac{2}{\alpha}\,\frac{N_{+y} - N_{-y}}{N_{+y} + N_{-y}}$$

(d$_2$) Multipole parameters

$$t^{(1)}_{M} = \frac{\sqrt{4\pi}}{\alpha} \langle Y^{(1)}_{M}(\theta,\varphi)\rangle \qquad M = -1, 0, +1$$

$$r^{(1)}_{0} = \sqrt{3}\ t^{(1)}_{0} = P_z$$

$$r^{(1)}_{1} = -\sqrt{6}\ \mathrm{Re}\ t^{(1)}_{1} = P_x$$

$$r^{(1)}_{-1} = -\sqrt{6}\ \mathrm{Im}\ t^{(1)}_{1} = P_y$$

Condition of B-symmetry in the production process

For transversity quantization

(e$_1$) $\quad {}^{T}P_x = {}^{T}P_y = 0 \qquad$ (e$_2$) $\quad {}^{T}r^{(1)}_{1} = {}^{T}r^{(1)}_{-1} = 0$

For helicity quantization

(f$_1$) $\quad {}^{H}P_z = {}^{H}P_x = 0 \qquad$ (f$_2$) $\quad {}^{H}r^{(1)}_{0} = {}^{H}r^{(1)}_{1} = 0$

1.2. Relation between different polarization parameters of spin $\frac{1}{2}$ particle.

The density matrix of spin $\frac{1}{2}$ particle is conveniently written as function of the Stokes vector and the Pauli matrices σ_x, σ_y, σ_z, see Table 1.2 (a_1). The very elements of the density matrix given in Table 1.2 (a_2) could be used as polarization parameters. But every physicist will prefer to measure the components of the Stokes vector, which are identical to the multipole parameters $r_M^{(1)}$. They are related to the density matrix elements by equations in Table 1.2 (b_1) and (b_2).

For each particle, several pairs of reference frames for transversity and helicity quantization can be intrinsically defined. Each pair of frames is fixed by the normal to the production three-plane and some space or time like direction (like the four-momentum transfer, which is associated with the s, t or u channel of a two-body reaction, cf. I.A.1). Associated transversity and helicity frames are simply related in the standard conventions by a rotation of $\frac{\pi}{2}$ radians around the x-axis. And two different pairs of transversity-helicity frames, say a and b, are related through a rotation around the normal by an angle ψ_{ba} (which usually will be a complicated function of the kinematical invariants s, t, u of a two-body reaction, cf. I.A.1). To these two kinds of frame rotations correspond analogous rotations of the Stokes vector in polarization space. They are explicitly given in Table 1.2 (c_1) and (d_1) and in Table 1.2 (c_2) and (d_2) for the terminology of multipole parameters.

As has been mentioned, in the case of B-symmetric production the components of the Stokes vector along the production plane, and their corresponding multipole parameters, have to be zero. They are the parameters written in the last columns of Table 1.2 (c_1) and (c_2) and in the last line or column of Table 1.2 (b_2). Note that in this case, for transversity quantization, the matrix element ρ_{1-1} is zero and the density matrix in Table 1.2 (a) has a "checker-board pattern" ; while, for helicity quantization, the real part of ρ_{1-1} and the difference $\rho_{11} - \rho_{1-1}$ are zero, and the density matrix without the trace is pure imaginary.

TABLE 1.2. - Relation between different polarization parameters of spin $\frac{1}{2}$ particle

Density matrix for spin $\frac{1}{2}$ particle

(a_1)

$$\rho = \frac{1}{2}[1 + \vec{P}.\vec{\sigma}]$$

(a_2)

$$\rho = \begin{vmatrix} \rho_{11} & \rho_{1-1} \\ \bar{\rho}_{1-1} & \rho_{-1-1} \end{vmatrix} \qquad \rho_{-1-1} = 1 - \rho_{11}$$

Relation between the density matrix in (a_1) or (a_2) and the polarization parameters in Table 1.1.

(b_1)

$$\vec{P} = tr[\rho.\vec{\sigma}] \qquad \text{(BP)}$$

(b_2)

		(BH)
		$r_0^{(1)} = \rho_{11} - \rho_{-1-1}$
(BT)	$r_{-1}^{(1)} = -2\ \mathrm{Im}\ \rho_{1-1}$	$r_1^{(1)} = -2\ \mathrm{Re}\ \rho_{1-1}$

Relation between the polarization parameters for transversity and helicity quantizations

		(B)	
(c_1)	${}^H P_y = {}^T P_z$	${}^H P_x = {}^T P_x$	${}^H P_z = -{}^T P_y$
(c_2)	${}^H r_{-1}^{(1)} = {}^T r_0^{(1)}$	${}^H r_1^{(1)} = {}^T r_1^{(1)}$	${}^H r_0^{(1)} = -{}^T r_{-1}^{(1)}$

Relation between the polarization parameters for two different transversity quantizations (rotation on the normal of angle ϕ_{ba})

(d_1)

$${}^T_b P_z = {}^T_a P_z$$

$${}^T_b P_x = \cos\phi_{ba}\ {}^T_a P_x + \sin\phi_{ba}\ {}^T_a P_y$$

$${}^T_b P_y = -\sin\phi_{ba}\ {}^T_a P_x + \cos\phi_{ba}\ {}^T_a P_y$$

(d_2)

$${}^T_b r_0^{(1)} = {}^T_a r_0^{(1)}$$

$${}^T_b r_1^{(1)} = \cos\phi_{ba}\ {}^T_a r_1^{(1)} + \sin\phi_{ba}\ {}^T_a r_{-1}^{(1)}$$

$${}^T_b r_{-1}^{(1)} = -\sin\phi_{ba}\ {}^T_a r_1^{(1)} + \cos\phi_{ba}\ {}^T_a r_{-1}^{(1)}$$

(BP) For B-symmetry ${}^T P$ is along z, and ${}^H P$ along y.

(BH) For B-symmetry and helicity quantization, the r parameters in this column are zero.

(BT) For B-symmetry and transversity quantization, the r parameters in this line are zero.

(B) For B-symmetry all the parameters in these two columns are zero.

1.3. Polarization domain of spin $\frac{1}{2}$ particle.

As is well known, the domain for the Stokes vector is a unitary sphere, the Poincaré sphere. It is a requirement of the positivity of the density matrix, i.e., a requirement of the fact that the probabilities of the pure states present in the statistical mixture have to be positive.

This "polarization domain" can be studied intrinsically, i.e., independently of any parametrization. It will always be a sphere of radius one, for the conveniently normalised intrinsic metric of the matrix space. (See Fig. 1.1(a)) The unpolarized state is represented by its center, O, and the pure state by its surface. So that the distance to this "isotropy center", O, gives directly the polarization degree. Orthogonal pure states (with magnetic quantum numbers $\pm\frac{1}{2}$ for some frame of quantization) are represented by the extremes of a diameter. Any intermediate point of this diameter represents a mixture of these two pure states. If the corresponding probabilities are materialized as weights at the extremes of the diameter, the intermediate point is their barycenter. This description of a state as mixture of orthogonal pure states is unique for any representative point inside the sphere, except the isotropy center.

A concrete frame of quantization will allow to fix a basis for the polarization parameters, and to write down the inequalities defining this polarization domain. Table 1.3 (a gives the inequalities and Fig. 1.1 (a) indicates the axes of the multipole parameters $r_M^{(1)}$ for any pair of associated transversity and helicity parametrizations. They are related by a rotation of $\frac{\pi}{2}$ radians around OP_1. The points P_3 and P_2 represent, for instance, the pure states with magnetic quantum number $+\frac{1}{2}$ for transversity and helicity quantizations.

In the case of B-symmetry, in which the polarization is necessarily along the normal, the three-dimensional polarization domain is reduced to a one-dimensional one, as explicitly shown in Fig 1.1 (b) and in Table 1.3 (b).

In the next chapters we will study simple generalizations of this one-dimensional domain for the case of higher spin. The use of the orthonormalized multipole parameters will be necessary for this study. They were superfluous for the case of spin $\frac{1}{2}$ (Tables 1.1 to 1.3 would only gain in clarity by crossing out all the expressions labelled by a letter with subindex 2), but they will supply a natural generalization of the Stokes vector for the case of higher spin.

TABLE 1.3. - Positivity conditions for the polarization parameters of spin $\frac{1}{2}$ particle

For the general case	
(a_1)	$\vert\vec{P}\vert \leq 1$
(a_2)	$[r^{(1)}_{1}]^2 + [r^{(1)}_{-1}]^2 + [r^{(1)}_{0}]^2 \leq 1$
For the case of B-symmetry	
(b_1)	$-1 \leq {}^{T}P_z = {}^{H}P_y \leq 1$
(b_2)	$-1 \leq {}^{T}r^{(1)}_{0} = {}^{H}r^{(1)}_{-1} \leq 1$

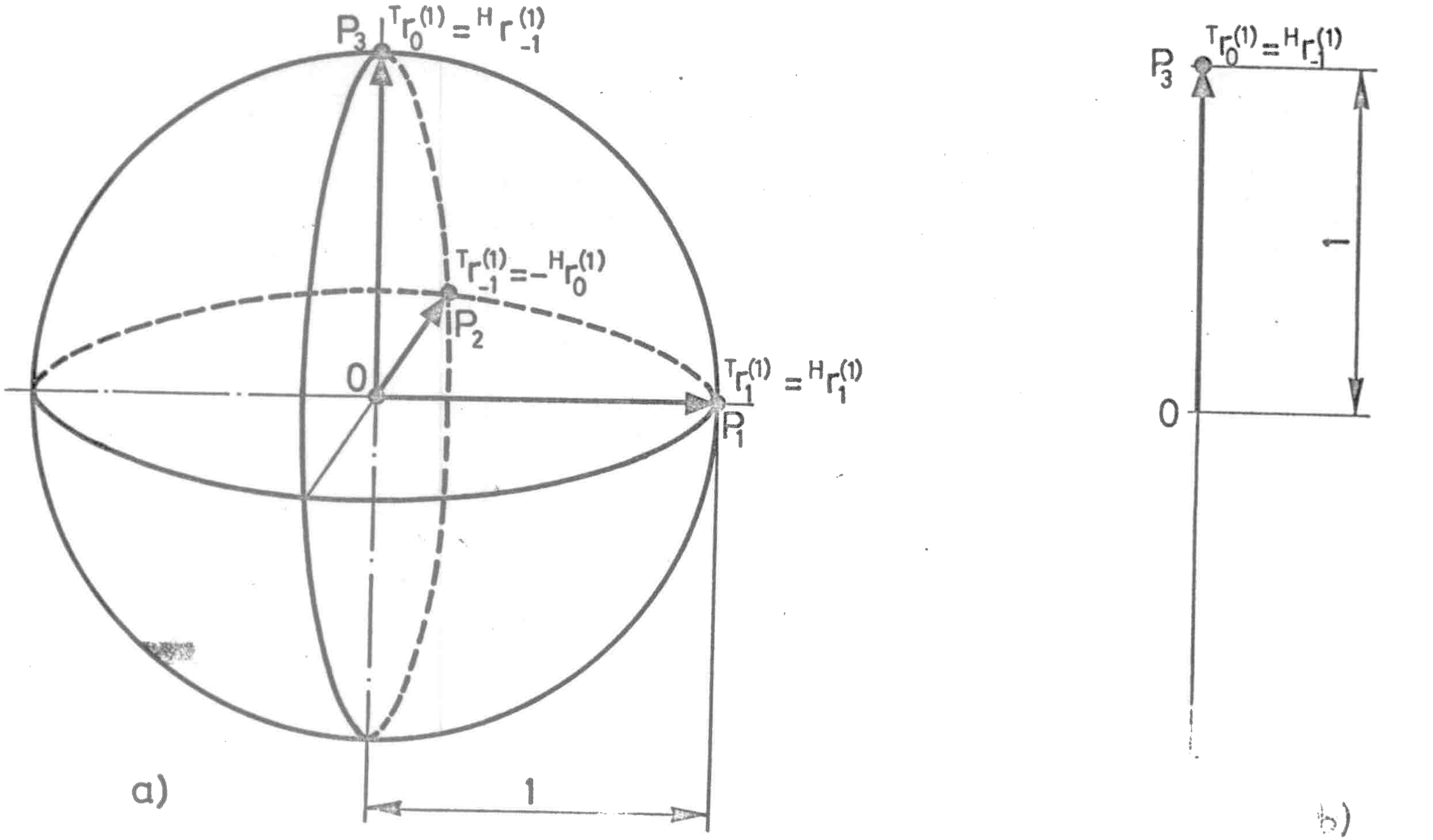

FIG. 1. 1. Polarization domain for spin 1/2 particle : a) in the general case (i. e., the whole Poincaré sphere), and b) in the case of B-symmetric production (i. e., the diameter corresponding to polarization along the normal).

FIG 1.1

2. Polarization of spin 1 particle.

2.1. Measurement of the even polarization of spin 1 particle.

The most common decay of spin 1 particle is the two-body decay intc spinless particles, as indicated in Table 2.1(a). Angular momentum conservation allows one decay amplitude, corresponding to the p-wave. The decay angular distribution depends only on the even multipole parameters $t_M^{(2)}$ or $r_M^{(2)}$, and has the form given in Table 2.1(c) or (d) (cf.I.A.4). The inverse expressions, which suppl a method of independent measurement for each multipole parameter, are given als in Table 2.1(e) and (f). The angular brackets $<\ldots>$ indicate experimental mean values of the enclosed expression for the ensemble of events. $Y_M^{(2)}$ are the usual spherical harmonics. Their arguments θ, φ are the polar and azimuthal angles, fixing for each event the direction of any of the decay products, in any frame in which the spin 1 particle is at rest.

In the case of B-symmetry for the production process of the spin 1 particle (i.e., production in a parity conserving reaction with unpolarized target and beam, cf.I.A.3), two of these five even polarization parameters have to be zero (see Table 0.1(b)). For transversity quantization (i.e., quantization along the normal to the production plane) the two multipole parameters with M odd must be zero, while for helicity quantization (i.e., quantization along any direction inside the production plane) the two $r_M^{(2)}$ multipole parameters with M negative must be zero, as indicated in Table 2.1(g) and (h).

Another well known decay mode of spin 1 particle is the three-body decay of the ω particle into pions, like the type indicated in Table 2.1(b). In general, the description of the angular distribution of a three-body decay is rather intricate (cf.I.A.4). But in this particular case (due to angular momentum and parity conservation, and the concrete values of the intrinsic parity of these four particles), it turns out that only the angular distribution of the normal to the decay three-plane is significant, and is given by exactly the same formulae of Table 2.1(c and (d). So the whole Table 2.1 is also valid for this case of the ω decay. The pola and azimuthal angles θ and φ will now fix for each event the direction of the norma to the decay plane in any frame in which the ω particle is at rest (the orientation of the normal in this case is irrelevant).

We recall that in these decays of vector particle, the three odd polarization parameters, $r_M^{(1)}$ or $t_M^{(1)}$, cannot be measured since they do not appear in the expression of the angular distribution, (see Table 2. 1 (c) and (d)). In the case of B-symmetric production two of these parameters must be zero (see Table 0. 1 (b)), but the third one, ${}^T r_o^{(1)}$ or ${}^H r_{-1}^{(1)}$, is not necessarily zero and cannot be measured. It has to be practically considered as a "ghost parameter". Only rare decay modes, like ρ^o into $\mu^+ \mu^-$, or accurate correlation measurements in some double resonance production at a fixed scattering angle, could supply information about this odd polarization of particles with spin-parity 1^-. In the case of particles 1^+ decaying into three pseudoscalars, the analysis of angular and energy distributions allows also a measurement of the absolute value of this odd polarization (cf. A, 2)

TABLE 2.1. - Measurement of the even polarization of spin 1 particle

Decays

(a) $1 \rightarrow 0 + 0$

(b) $1^- \rightarrow 0^- + 0^- + 0^-$ (strong interaction)

Angular distribution

(c) $$I(\theta,\varphi) = \frac{1}{4\pi} - \sqrt{\frac{1}{2\pi}} \sum_{M=-2}^{+2} \bar{t}_M^{(2)} Y_M^{(2)}(\theta,\varphi)$$

(d) $$I(\theta,\varphi) = \frac{1}{4\pi} [1 - r_0^{(2)} (3 \cos^2\theta - 1) - \sqrt{3} \sin^2\theta \, (r_2^{(2)} \cos 2\varphi + r_{-2}^{(2)} \sin 2\varphi) - \sqrt{3} \sin 2\theta \, (r_1^{(2)} \cos \varphi + r_{-1}^{(2)} \sin \varphi)]$$

Multipole parameters

(e) $$t_M^{(2)} = - \sqrt{2\pi} < Y_M^{(2)}(\theta,\varphi) > \qquad M = -2,-1,\ldots+2$$

(f) $$r_0^{(2)} = \sqrt{\frac{5}{2}} \, t_0^{(2)}$$

$$r_M^{(2)} = (-1)^M \sqrt{5} \, \mathrm{Re} \, t_M^{(2)} \qquad M = 1,2$$

$$r_{-M}^{(2)} = (-1)^M \sqrt{5} \, \mathrm{Im} \, t_M^{(2)}$$

Condition of B-symmetry in the production process

(g) For transversity quantization

$$^T r_1^{(2)} = {}^T r_{-1}^{(2)} = 0$$

(h) For helicity quantization

$$^H r_{-2}^{(2)} = {}^H r_{-1}^{(2)} = 0$$

2.2. Relation between different even polarization parameters of spin 1 particle.

For anyone reference frame, the even part of the density matrix of spin 1 particle has the form indicated in Table 2.2 (a) (cf. I. A. 6.). The very elements of this even density matrix can be used as polarization parameters. They are related to the measurable multipole parameters of Table 2.1 by the relations given in Table 2.2 (b).

For each particle, several pairs of reference frames for transversity and helicity quantization can be intrinsically defined. Each pair of frames is fixed by the normal to the production three-plane and some space or timelike direction (like the four-momentum transfer, which is associated with the s, t, or u channel of a two-body reaction, cf. I. A. 1). Associated transversity and helicity frames are simply related in the standard conventions by a rotation of $\frac{\pi}{2}$ radians around the x-axis. The corresponding linear transformation on the multipole parameters is given in Table 2.2 (c), where the left T and H superscripts refer to transversity and helicity quantizations.

Two different pairs of transversity-helicity frames, say a and b, are related through a rotation around the normal by an angle ψ_{ba} (which usually will be a complicated function of the kinematical invariants s, t, u of a two-body reaction, cf. I. A. 1). The corresponding linear transformation on the multipole parameters is very simple for transversity quantization. It is also given in Table 2.2 (d), where the left subscripts a, b, label any transversity frames.

As has been mentioned, in the case of B-symmetric production the transversity multipole parameters with M odd or the helicity multipole parameters with M negative must be zero. They are the parameters written in the second column of Table 2.2 (c), and in the last line or column of Table 2.2 (b). Note that in this case for transversity quantization the matrix element ρ_{10} is zero, and the density matrix in Table 2.2 (a) has a "checker-board pattern", while for helicity quantization the imaginary parts of ρ_{10} and ρ_{1-1} are zero, and the density matrix is real.