5.3. Polarization domain for B-symmetric even polarization of the spin 2 particle.

The domain of possible values of the eight even polarization parameters of a B-symmetric spin 2 particle is fixed by the requirement that the full density matrix, including eventually some ghost parameters, must be positive, i.e. the probabilities of the different pure states present in the statistical mixture must be positive (cf. Part I, A6).

This "polarization domain" can be studied intrinsically, i.e. independently of any concrete system of parametrization, but taking into account the intrinsic metric of the polarization space (cf. Part I, 0 and A6). It is defined in its eigth-dimensional space as the intersection of two hypercylinders, i.e., two domains which are unconfined in some dimensions, these of their "generatrices", and are bounded in the orthogonal dimensions of their "cross section". The generatrices of these two hypercylinders are parallel hyperplanes of five and three dimensions respectively. Two generatrices, one of each cylinder, intersect in one point, but they are not orthogonal. They form an angle χ, the angle between the only two directions, a_1 and a_2, in each generatrix which cannot be chosen orthogonal to each other. The value of this "intersection angle" of the two cylinders is

$$\chi = \text{arc cos}\,\frac{2}{3} \cong 48.2° \quad .$$

The projection of the whole positivity domain on this plane a_1, a_2 is a isosceles triangle represented in Fig. 5.1(P). The three-dimensional cross section of the first hypercylinder is a truncated cone represented in Fig. 5.1(Q). Its revolution axis is a_1. The five-dimensional cross section of the second hypercylinder can be projected in a truncated cone represented in Fig. 5.1(R). It has the same shape as the cone (Q), and its revolution axis is a_2. The orthogonal projection is a circle of center S_o and radius $\overline{S_oS_1}$ represented in Fig. 5.1(S). To each point in the projections (P) and (R) corresponds as section inside this circular projection a certain ellipse with the same center S_o, as represented also in (S).

The projection in (P) defines the Monge's circle of this ellipse, i.e., the concentric circle of radius r, related to the semiaxis s_+ and s_- by $r^2 = s_+^2 + s_-^2$. This Monge's circle is reduced to the point S_o when the projection in (P) lies in the segments $\overline{P_1P_2}$ or $\overline{P_2P_3}$. It reaches the total projection circle S_1S_2 when the projection in (P) lies in P_4, the middle of the segment $\overline{P_1P_3}$. Inside this Monge's circle the ellipse is shrinked to a diameter or swelled out to a circle (of radius $r/\sqrt{2}$) when the projection on (R) is in the cone skin or in its revolution axis respectively. The orientation of the ellipse is given by its azimuth α, which is a half of the azimuth of the projection on (R).

Thus any even, B-symmetric polarization state is represented by two points in the cones (Q) and (R) respectively, whose intersection lies in the triangle (P), and a third point in the corresponding ellipse (S). The unpolarized state is represented by the points Q_o, R_o, and S_o. The degree of polarization of any state is directly given by the distance of the representative point to this "isotropy center". (Recall that a_1 and a_2 are not orthogonal, and use the triangle (P) to measure distances in this plane).

Pure states have degree of polarization 1, and are easily found. They form two connected sets. The first set is represented by the circonference Q_2Q_3 of the base of the cone (Q), and the projections R_1 and S_o in (R) and (S). The second set, by the whole lateral skin of the cone (R), together with Q_1 in (Q), and any of the extremes of the shrinked ellipse in (S).

Thus a pure state of the first set is fixed by its azimuth φ on the cone (Q), which can be measured from c to d. In this transversity quantization, and labelling pure states by their magnetic quantum number, the general expression for this first type of pure states is (up to an arbitrary phase):

$$|\varphi> = \sqrt{1/2}\, e^{i\varphi/2}|1> + \sqrt{1/2}\, e^{-i\varphi/2}|-1> \quad . \qquad (1)$$

A pure state of the second set is fixed by its azimuth ψ on the circle (S) from e to f, or its azimuth 2ψ on the cone (R) from g to h, and the distance x to the axis of this cone, which can be normalized from 0 (for the vertex R_1) to 1 (for the circumference of the base R_2R_3). With the same terminology, the general expression for this second type of pure states is

$$|\chi,\psi\rangle = \sqrt{x/2}\ e^{i\psi}|2\rangle + \sqrt{1-x}\,|0\rangle + \sqrt{x/2}\ e^{-i\psi}|-2\rangle \qquad (2)$$

Pure states of type (1) are evidently orthogonal to pure states of type (2). Two pure states of type (1), $|\varphi_1\rangle$ and $|\varphi_2\rangle$, are orthogonal if

$$\varphi_1 = \varphi_2 \pm \pi \quad ,$$

i.e., if they are represented in the base of the cone (Q) by diametrically opposite points, as Q_2 and Q_3. For pure states of type (2), the orthogonality is a little more disguised. The pure states orthogonal to the vertex R_1 of the cone (R) form the circumference R_2R_3 of its base. The pure states orthogonal to a point R_2 of this circumference form the opposite generatrix R_1R_3, with two different projections on (S) (the extremes of the shrinked ellipse, which coincide only for R_1 and R_2). The pure states orthogonal to a point, R_4, of any generatrix R_1R_2 form the ellipse laying in the plane R_3R_5 which is orthogonal to the corresponding meridian section and has $\overline{R_1R_5} = \overline{R_2R_4}$. To each point in this ellipse corresponds only one projection on (S) (the more distant of the projection chosen for R_4). Two points in this ellipse represent orthogonal states if they are symmetric through the center of the ellipse. The projection of this ellipse on the base of the cone is another ellipse which has as foci the center of the base and the symmetric through this center of the projection of R_4. From this projection, the conditions for three states $|x_j,\psi_j\rangle$ of type (2), to be orthogonal to each other are found to be

$$\Sigma\, x_j = 2 \qquad (j = 1,2,3) \qquad (3a)$$

$$\Sigma\, x_j\ e^{2i\psi_j} = 0 \qquad (3b)$$

$$|\psi_j - \psi_k| \geq \pi/2 \quad (j \neq k) \quad . \qquad (3c)$$

Any even, B-symmetric polarization state can be decomposed in five orthogonal even, B-symmetric pure states. Two of them are of type (1) and have the form $|\varphi>$ and $|-\varphi>$ and the probabilities $\lambda_1 \lambda_2$ which are defined, as functions of the density matrix elements for transversity quantization (in Fig. 5.3 the relation between these density matrix elements and the parameters in Fig. 5.1 is visualized) by :

$$\lambda_1 + \lambda_2 = 2\ {}^T\rho_{11} \tag{4a}$$

$$(\lambda_1 - \lambda_2)\ e^{i\varphi} = 2\ {}^T\rho_{1-1} \quad . \tag{4b}$$

The other three pure states are of type (2) , and have the form $|x_j,\psi_j>$ and the probabilities μ_j defined by equations (3) together with :

$$\Sigma\ \mu_j = {}^T\rho_{oo} + 2\ {}^T\rho_{11} \qquad (j=1,2,3) \tag{5a}$$

$$\Sigma\ \mu_j\ x_j = 2\ {}^T\rho_{22} \tag{5b}$$

$$\Sigma\ \mu_j\ x_j\ e^{2i\psi_j} = 2\ {}^T\rho_{2-2} \tag{5c}$$

$$\Sigma\ \mu_j\ \sqrt{2x_j(1-x_j)}\ e^{i\psi_j} = 2\ {}^T\rho_{20} \quad . \tag{5d}$$

TABLE 5.3. - Positivity conditions for the measurable polarization parameters of B-symmetric spin 2 particle

(a) Convenient terminology

$$a_1 = -\sqrt{\frac{5}{21}}\, T_r\binom{2}{0} - \sqrt{\frac{16}{21}}\, T_r\binom{4}{0} \qquad b_1 = \sqrt{\frac{16}{21}}\, T_r\binom{2}{0} - \sqrt{\frac{5}{21}}\, T_r\binom{4}{0}$$

$$a_2 = \sqrt{\frac{20}{21}}\, T_r\binom{2}{0} + \sqrt{\frac{1}{21}}\, T_r\binom{4}{0} \qquad b_2 = \sqrt{\frac{1}{21}}\, T_r\binom{2}{0} - \sqrt{\frac{20}{21}}\, T_r\binom{4}{0}$$

$$c = \sqrt{\frac{3}{7}}\, T_r\binom{2}{2} - \sqrt{\frac{4}{7}}\, T_r\binom{4}{2}$$

$$d = \sqrt{\frac{3}{7}}\, T_r\binom{2}{-2} - \sqrt{\frac{4}{7}}\, T_r\binom{4}{-2}$$

$$e = \sqrt{\frac{4}{7}}\, T_r\binom{2}{2} + \sqrt{\frac{3}{7}}\, T_r\binom{4}{2}$$

$$f = \sqrt{\frac{4}{7}}\, T_r\binom{2}{-2} + \sqrt{\frac{3}{7}}\, T_r\binom{4}{-2}$$

$$g = T_r\binom{4}{4}$$

$$h = T_r\binom{4}{-4}$$

Positivity conditions

(b) $$-\sqrt{\frac{1}{6}} \le a_1 \qquad -\sqrt{\frac{1}{6}} \le a_2 \qquad a_1 + a_2 \le \sqrt{\frac{1}{24}}$$

(c) $$c^2 + d^2 \le \frac{(1 + \sqrt{6}\, a_1)^2}{10}$$

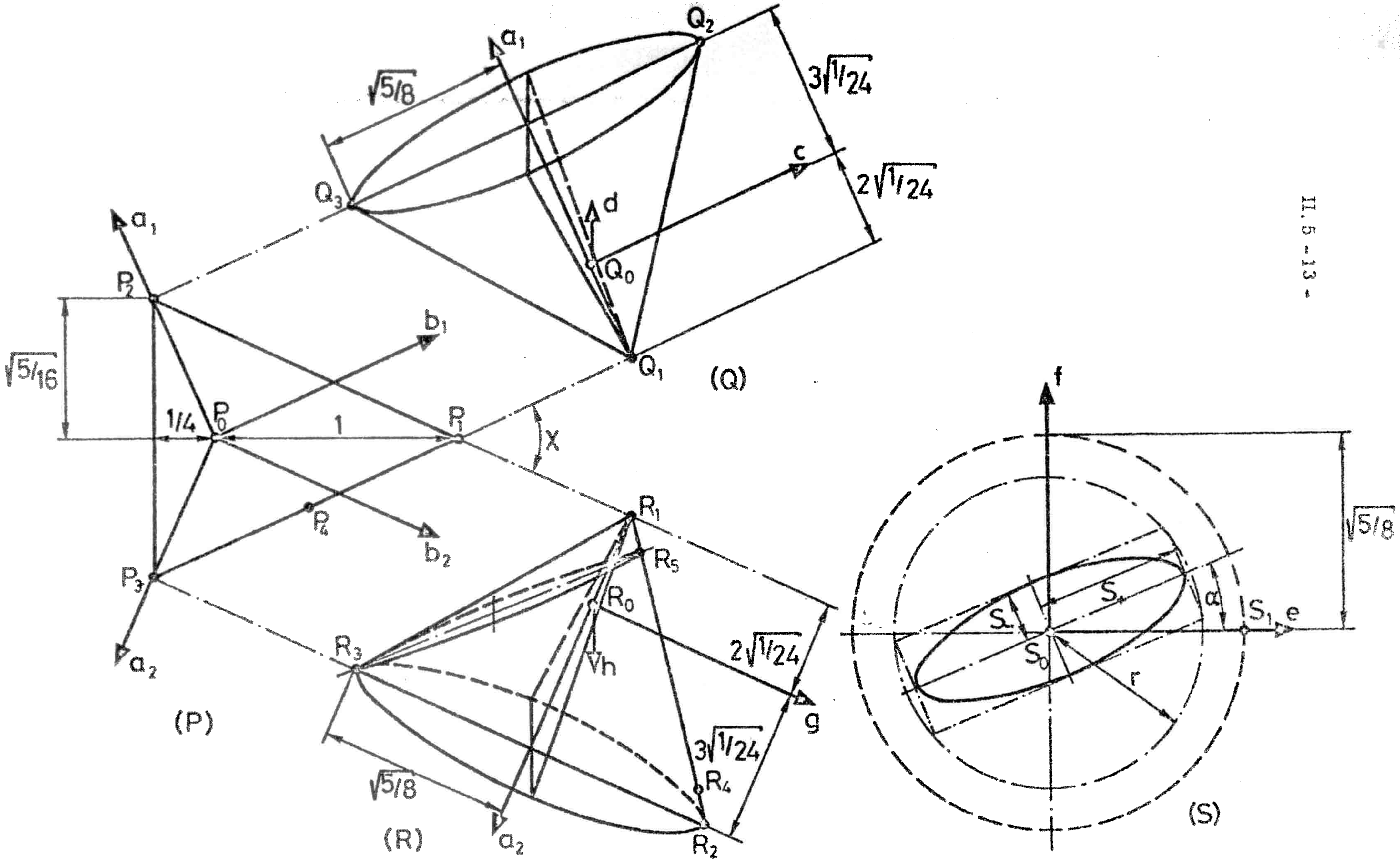
(d) $$(1 + \sqrt{6}\, a_2)(e^2 + f^2) - \sqrt{10}\, g\,(e^2 - f^2) - \sqrt{10}\, h\, 2ef \le$$
$$\le [1 - \sqrt{24}\,(a_1 + a_2)]\left[\frac{(1 + \sqrt{6}\, a_2)^2}{10} - (g^2 + h^2)\right]$$

Decomposition of condition (d)

(d_1) $$g^2 + h^2 \le \frac{(1 + \sqrt{6}\, a_2)^2}{10}$$ (d_2) $$\frac{e'^2}{s_+^2} + \frac{f'^2}{s_-^2} \le 1$$

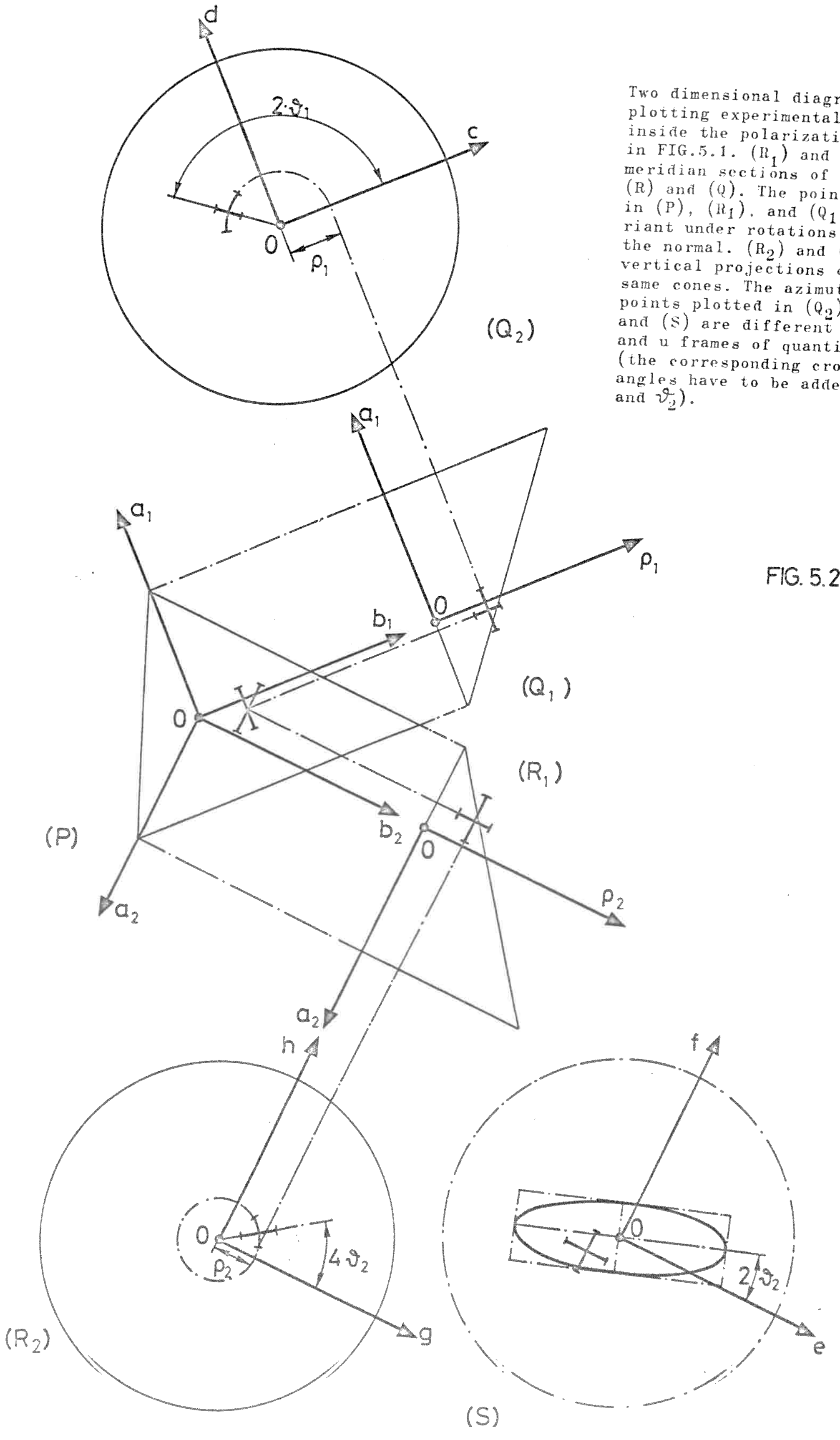
(e) Supplementary terminology

$$e' = e \cos\alpha + f \sin\alpha$$
$$\operatorname{tg} 2\alpha = h/g$$
$$f' = -e \sin\alpha + f \cos\alpha$$
$$s_\pm^2 = \frac{1}{10}[1 - \sqrt{24}\,(a_1 + a_2)]\left[(1 + \sqrt{6}\, a_2) \pm \sqrt{10}\sqrt{g^2 + h^2}\right]$$



Domain of the even, B-symmetric polarization of spin 2 particle.

FIG. 5.1



Two dimensional diagrams for plotting experimental points inside the polarization domain in FIG.5.1. (R_1) and (Q_1) are meridian sections of the cones (R) and (Q). The points plotted in (P), (R_1), and (Q_1) are invariant under rotations around the normal. (R_2) and (Q_2) are vertical projections of the same cones. The azimuth of the points plotted in (Q_2), (R_2), and (S) are different for s, t and u frames of quantization (the corresponding crossing angles have to be added to ϑ_1 and ϑ_2).

FIG. 5.2

5.4. Colinearity Condition and Model Predictions.

The colinearity condition (forward or backward production of the particle in a two-body reaction from unpolarized target and beam) implies that the density matrix is invariant under rotations around the colinearity axis, that is, ρ is diagonal for a helicity quantization. Thus the multipole parameters ${}^{H}r_M^{(L)}$ with $M \neq 0$, must be zero. For the even polarization of the spin 2 particle, only two multipole parameters, ${}^{H}r_o^{(2)}$ and ${}^{H}r_o^{(4)}$, can be different from zero. Therefore the eight-dimensional positivity domain in Fig. 5.1 shrinks to a two-dimensional one : a triangle $C_oC_1C_2$ whose projections on diagrams (P),(Q),(R),(S) are drawn as dotted triangles in Fig. 5.3 (the projection on (S) is a line). The vertices C_o, C_1 and C_2 represent the even states with ${}^{H}\rho_{oo} = 1$, ${}^{H}\rho_{11} = \frac{1}{2}$ and ${}^{H}\rho_{22} = \frac{1}{2}$ respectively. To fix the position of these points, note that the line C_1C_2 out the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ at half their height ; and the relative distances in (R) and (S) are given by

$$\overline{R_3C_o} = 2\,\overline{R_1C_2} = \frac{1}{4}\,\overline{R_1R_3} ,$$

$$\overline{S_oC_o} = 2\,\overline{S_oC_2} = \frac{\sqrt{6}}{8}\,\overline{S_oS_1}$$

Let us now indicate the predictions of the t-channel exchange with fixed quantum numbers, for the polarization of 2^+ particles produced in the following types of reactions :

$$0^-\tfrac{1}{2}^+ \longrightarrow 2^+\tfrac{1}{2}^+ , \qquad (6)$$

$$0^-\tfrac{1}{2}^+ \longrightarrow 2^+J^{\varepsilon} , \qquad (7)$$

where J represents any half-integer spin, and ε any parity (cf. Ader-Capdeville-Cohen Tannoudji-Salin— 68 , Ringland-Thews-68 and Thews-69). Some projections of the subdomains predicted by this model are also visualized in Fig. 5.3. Note that Fig. 5.3 reproduces Fig. 5.1 with the same scale and orientation, but for convenience it has been parametrized by the elements of the density matrix for transversity quantization. To take into account the non normalization of these matrix elements, the absolute maximum of each one of them has been indicated in the figure.

For all reactions of type (7) in the limit of Regge theories (i.e. for small negative fixed t and infinite s) the exchange of "normal parity" (i.e. $\sigma = P(-1)^J = +1$, or $J^P = 1^-, 2^+, 3^-$; 0^+ exchange is forbidden by angular momentum and parity conservation) predicts a polarization subdomain projected on the vertex P_2 of the triangle (P), and therefore on R_1, S_o and the whole base of the cone (Q) . For exchange of "abnormal parity" (i.e. $\sigma = -1$, or $J^P = 0^-$, 1^+ , 2^-...) the predicted subdomain is projected on the side P_1P_3 of the triangle (P) , and therefore on Q_1 , the whole cone (R), and the corresponding ellipses in (S) . But when a "single" trajectory of abnormal parity is exchanged, the projection on (R) must lay on the lateral skin of the cone, and therefore the ellipses in (S) are shrunk to segments.

For reactions of type (6), the exchange of 1^- particle predicts a polarization subdomain projected on the segments $P_2 P_2''$, $Q_2 Q_2'$, $R_1 R_1'$, and on the point S_o . The size of these segments is fixed by

$$\overline{P_2P_2''} / \overline{P_2P_3} = \overline{Q_2Q_2'} / \overline{Q_2Q_1} = \overline{R_1R_1'} / \overline{R_1R_2} = f/2 ,$$

where f is the parameter defined in equations (4a) to (4c) of 2.4. The exchange of 2^+ particle, or higher spin and normal parity particle, predicts a more complicated polarization subdomain, whose projection on (P) is in the triangle $P_2P_2'P_2''$ with

$$\overline{P_2P_2'} / \overline{P_2P_1} = f/2$$

The exchange of 0^- particle predicts the single point whose projections in (P),(Q),(R) and (S) are C_o . The exchange of 1^+ particle predicts a three-dimensional subdomain whose projections on (P) and (Q) are in the triangle $P_3P_3'C_o$, and in the segment $Q_1 Q_1'$, with

$$\overline{P_3P_3'} / \overline{P_3P_2} = \overline{Q_1Q_1'} / \overline{Q_1Q_2} = f/2 .$$

For higher spin and abnormal parity the projection in (P) is in the trapezoidal region $P_3P_1P_1'P_3'$. But if the isoparity of the exchange is well defined and is "normal" ($\tau = G(-1)^{I+J} = +1$), then the projection on (P) is on the segment P_1P_3 , as in the case of exact Regge limit.

Polarization subdomains predicted by t-channel exchange with fixed quantum numbers : $\sigma = P(-1)^J$, $\tau = G(-1)^{J+I}$. The triangles $C_0C_1C_2$ are polarization subdomains imposed by collinearity.

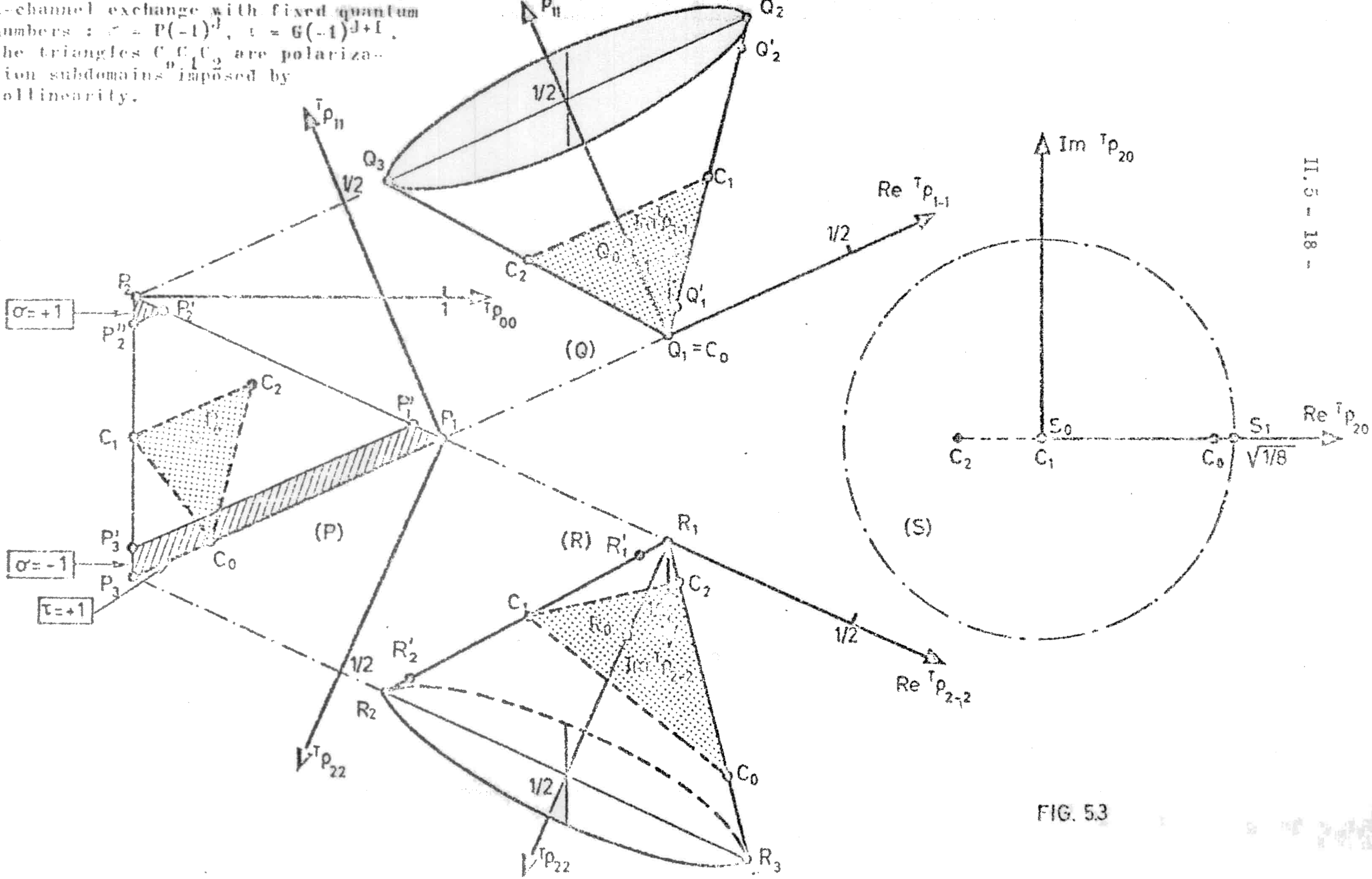


FIG. 5.3

PART II - APPENDIX 1.

THE POLARIZATION VECTOR OF A SPIN-1 PARTICLE.

0. Preliminary remark.

The polarization vector describing a pure state of spin 1 particle is not to be confused with the axial vector describing the density matrix of a spin $\frac{1}{2}$ state or also the dipole part of the density matrix of a state of any spin. We insist that the polarization vector be used only in the description of spin-1 pure polarisation states.

1. The polarization vector of the particle at rest.

In this appendix we exhibit the relationship between the usual "polarization vector" of a pure polarization state of a spin-1 particle and its density matrix of polarization.

Consider first the particle at rest. Its polarization space is a three dimensional Hilbert space $\mathcal{H}_3$ on which the rotation group SO(3) acts through its adjoint representation $D^{(1)}$. It is therefore natural to "identify" $\mathcal{H}_3$ with the space $\bar{\mathcal{E}}_3$, the complexified of the space $\mathcal{E}_3$ in which we live. Thus $\mathcal{H}_3$ has a distinguished real three dimensional subspace. With this correspondence between $\mathcal{H}_3$ and $\bar{\mathcal{E}}_3$, a pure state of polarization for a spin-1 particle at rest is often denoted by the complex vector

$$\vec{\ell} \in \bar{\mathcal{E}}_3 \tag{1}$$

corresponding to

$$|\ell> \in \mathcal{H}_3 \tag{1'}$$

The Hermitian scalar product of two polarization vectors is

$$\vec{\ell}^* . \vec{\ell}' = < \ell | \ell' > \tag{2}$$

where $\vec{\ell}^*$ is the complex conjugate vector of $\vec{\ell}$, i.e. in a <u>real</u> basis of $\mathcal{E}_3$, the coordinates of $\vec{\ell}^*$ are the complex conjugate of those of $\vec{\ell}$. An orthonormal basis for the polarization vectors is defined by

$$\vec{e}_i^{\,*} . \vec{e}_j = \delta_{ij} \quad \text{corresponding to} \quad < i | j > = \delta_{ij} \quad . \tag{3}$$

A frequent example of a non real orthonormal basis is given by :

$$\vec{e}_3 \quad \text{and} \quad \vec{e}_{\pm} = \mp \frac{1}{\sqrt{2}} (\vec{e}_1 \pm i\vec{e}_2) = -\vec{e}_{\mp}^{\,*} \tag{4}$$

where $\vec{e}_1 , \vec{e}_2 , \vec{e}_3$ form a real orthonormal basis $\vec{e}_i = \vec{e}_i^{\,*}$ satisfying (3).

In this real basis the generators J_k of the rotation group are represented by matrices whose elements are :

$$(J_k)^i{}_j = -i \; \epsilon_{ijk} \quad . \tag{5}$$

One can compute the matrix elements of

$$(J_k J_l)^i{}_j = -\epsilon_{imk} \, \epsilon_{mjl} = \delta_l^k \, \delta_j^i - \delta_j^k \, \delta_l^i \quad . \tag{6}$$

One then verifies that the matrix elements of the commutator of two J^k are

$$([J^k , J^\ell])^i{}_j = i \; \epsilon_{k\ell m} (J^m)^i{}_j \quad . \tag{7}$$

One also verifies that

$$\vec{J}^2 = \vec{J} . \vec{J} = 2 \; I \tag{8}$$

and that the vectors $\vec{e}_3$ and $\vec{e}_{\pm}$ correspond to the pure states (1,m) with respectively $m = 0$ and $m = \pm 1$, the eigenvalues of J_3.

To summarize, a pure state of polarization of a spin-1 particle at rest is completely defined by its polarization vector $\vec{\ell}$ normalized to

$$\vec{\ell}^* . \vec{\ell} = 1 \quad . \tag{9}$$

The polarization density matrix of this pure state is

$$\rho = \vec{\ell} \otimes \vec{\ell}^* \tag{10}$$

from (5),

$$\mathrm{tr}\,\rho = \mathrm{tr}\,\vec{\ell} \otimes \vec{\ell}^{*} = \vec{\ell}^{*} . \vec{\ell} = 1 \quad . \tag{11}$$

The tensor $\vec{\ell} \otimes \vec{\ell}^{*}$ is also understood as a linear operator acting on vectors of $\bar{\mathcal{E}}_3$ as follows

$$(\vec{\ell} \otimes \vec{\ell}^{*})\,\vec{a} = \vec{\ell}(\vec{\ell}^{*} . \vec{a}) \tag{12}$$

Remark that polarization vectors which differ by a phase, such as $\vec{\ell}$ and $e^{i\varphi}\vec{\ell}$, represent the same polarization state.

2. Decomposition of the polarization state $\vec{\ell}$ into multipoles.

Let us define the real and imaginary part of the vector $\vec{\ell}$

$$\vec{r} = \mathrm{Re}\,\vec{\ell} = \tfrac{1}{2}(\vec{\ell} + \vec{\ell}^{*}) \quad , \vec{a} = \mathrm{Im}\,\ell = \frac{1}{2i}(\vec{\ell} - \vec{\ell}^{*}) \quad . \tag{13}$$

The decomposition of the tensor $\rho = \vec{\ell} \otimes \vec{\ell}^{*}$ into irreducible rotation tensors is done by taking :

i) its trace :

$$\mathrm{tr}\,\rho = \vec{\ell}^{*} . \vec{\ell} = \vec{r}^{\,2} + \vec{a}^{\,2} = 1 \tag{9}$$

ii) its antisymmetrical part :

$$\tfrac{1}{2}(\vec{\ell} \otimes \vec{\ell}^{*} - \vec{\ell}^{*} \otimes \vec{\ell}) = \rho^{(1)} = -i(\vec{r} \otimes \vec{a} - \vec{a} \otimes \vec{r}) \tag{14}$$

iii) its traceless symmetrical part :

$$\tfrac{1}{2}(\vec{\ell} \otimes \vec{\ell}^{*} + \vec{\ell}^{*} \otimes \vec{\ell}) - \frac{1}{3}I = \rho^{(2)} = \vec{r} \otimes \vec{r} + \vec{a} \otimes \vec{a} - \frac{1}{3}I \quad . \tag{15}$$

That yields the explicit decomposition of ρ into multipoles (see I.A2(67))

$$\rho = \rho^{(0)} + \rho^{(1)} + \rho^{(2)} \quad \text{with} \quad \rho^{(0)} = \frac{1}{3}I \, . \tag{16}$$

Let us choose a <u>real</u> orthonormal basis $\vec{e}_k = \vec{e}_k^{\,*} (k = 1,2,3)$ of $\bar{\mathcal{E}}_3$. We denote by λ^k, ϱ^k, α^k the coordinates of $\vec{\ell}$, $\vec{r}$, $\vec{a}$: †

$$\vec{\ell} = \lambda^k \,\vec{e}_k = (\varrho^k + i\alpha^k)\,\vec{e}_k \tag{17}$$

with

---

† Since $\mathcal{E}_3$ has a Euclidean metric, the corresponding covariant and contravariant components have the same value. We will not distinguish between them but we keep the Einstein summation convention.

$$\sum_k |\lambda^k|^2 = \sum_k ((\beta^k)^2 + (\alpha^k)^2) = 1 \quad . \tag{18}$$

In this basis the density matrix ρ has for matrix elements :

$$\rho^{ij} = \frac{1}{3}\delta^{ij} - i(\beta^i\alpha^j - \beta^j\alpha^i) + \beta^i\beta^j + \alpha^i\alpha^j - \frac{1}{3}\delta^{ij} \tag{19}$$

with the use of (5) and (6), equation (19) can be written :

$$\rho = \frac{1}{3}I + \sigma^k J_k + \chi^{k\ell} J_k J_\ell \tag{19'}$$

where we have denoted by σ^k the coordinates of $\vec{r} \times \vec{a}$, i.e.

$$\beta^i\alpha^j - \beta^j\alpha^i = \epsilon_{ijk}\,\sigma^k \tag{20}$$

and we have defined

$$\chi^{k\ell} = -(\beta^k\beta^\ell + \alpha^k\alpha^\ell - \frac{1}{3}\delta^k_\ell) \quad . \tag{21}$$

Note that

$$\chi^{k\ell} = \chi^{\ell k} \quad , \quad \chi^k_k = 0 \tag{22}$$

and also

$$\chi^{k\ell}\sigma^\ell = \frac{1}{3}\sigma^k \quad . \tag{22'}$$

The real numbers σ^k and $\chi^{k\ell}$ are the coordinates of the dipole and quadrupole polarization.

3. The covariant polarization vector and density matrix.

If the spin-1 particle has an energy-momentum $\underline{p}$, its polarization space $\mathcal{H}_3(\underline{p})$ † can be identified with $\bar{\mathcal{E}}_3(\underline{p})$, the complexified of the three dimensional space orthogonal to $\underline{p}$. When the particle is at rest $\underline{p} = (m,\vec{0})$. The transformation into covariant notations of the formalism established in the rest frame is easy.

We use greek and latin letters for indices with respective values 0, 1, 2, 3 and 1, 2, 3 . Let $n^{(\alpha)}(\underline{p})$ be a chosen tetrad corresponding to the energy-

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† Introduced in I. A1.3.

momentum $\underline{p} = m\,\underline{n}^{(o)}$. To simplify, we consider only real tetrad vectors.

The λ^{i}'s become the complex component of the covariant polarization vector

$$\underline{\ell} = \lambda^{i}\,\underline{n}^{(i)} \tag{23}$$

whose real and imaginary parts are respectively :

$$\mathrm{Re}\,\underline{\ell} = \underline{r} = \tfrac{1}{2}(\underline{\ell}+\underline{\ell}^{*}) = \beta^{i}\,\underline{n}^{(i)} \quad ; \quad \mathrm{Im}\,\underline{\ell} = \underline{a} = \frac{1}{2i}\,(\underline{\ell}-\underline{\ell}^{*}) = \alpha^{i}\,\underline{n}^{(i)} . \tag{24}$$

Because of the relation

$$\underline{n}^{(i)} . \underline{n}^{(j)} = g^{ij} = -\,\delta^{ij} \tag{25}$$

the Hermitian scalar product on $\mathcal{H}_3$ corresponds to

$$< \ell | \ell' > = -\,\underline{\ell}^{*} . \underline{\ell}' \tag{26}$$

and the density matrix ρ is

$$\rho = -\,\underline{\ell} \otimes \underline{\ell}^{*} \quad . \tag{27}$$

So ρ is not only an operator on $\mathcal{E}_3(\underline{p})$, but it is also an operator acting on space-time. We will denote by I_p the identity operator on $\bar{\mathcal{E}}_3(\underline{p})$ and by I_4 the identity operator on space-time :

$$I_p = -\,\delta_{ij}\,\underline{n}^{(i)} \otimes \underline{n}^{(j)}$$

$$I_4 = g_{\alpha\beta}\,\underline{n}^{(\alpha)} \otimes \underline{n}^{(\beta)} \quad . \tag{29}$$

Note that I_p is also the projector from space time onto $\bar{\mathcal{E}}_3(\underline{p})$

$$I_p = I_4 - \frac{1}{m^2}\,\underline{p} \otimes \underline{p} \quad . \tag{30}$$

When we replace J_k by the generators $S^{(k)}(\underline{p})$ of the "rotations" of the little group of $\underline{p}$ that we have introduced in I.A1

$$S^{(\alpha)}(\underline{p}) = -\,\frac{1}{m}\,\underline{W}(\underline{p}).\underline{n}^{(\alpha)} \qquad \text{I.A1(31)}$$

then equation (16) can be written as

$$\rho = \frac{1}{3}\,I_p - \frac{1}{m}\,\underline{s}.\underline{W}(p) + \frac{1}{m^2}\,\underline{W}(\underline{p}).\underline{q}.\underline{W}(\underline{p}) \qquad \text{I.A1(34)}$$

with

$$\underline{s} = +\,\sigma_i\,\underline{n}^{(i)} \tag{31}$$

and $$\underline{\underline{q}} = \chi_{k\ell}\, \underline{n}^{(k)} \otimes \underline{n}^{(\ell)} . \qquad (32)$$

Note that, as required in I.A1 ,

$$\underline{s}\cdot\underline{p} = 0 \ , \ \underline{p}\cdot\underline{\underline{q}} = \underline{\underline{q}}\cdot\underline{p} = 0 \qquad (33)$$

$$\text{tr}\ \underline{\underline{q}} = q^{\mu}_{\mu} = \chi_{k\ell}\, n^{(k)}_{\mu}\ n^{(\ell)}_{\mu} = g_{k\ell}\, \chi^{k\ell} = -\chi^{k}_{k} = 0 . \qquad (34)$$

We also have for the pure state of polarization $\underline{\ell}$ (see (22'))

$$\underline{s}\cdot\underline{\underline{q}} = \underline{\underline{q}}\cdot\underline{s} = -\frac{1}{3}\,\underline{s} \qquad (35)$$

and $$\underline{s}\cdot\underline{\ell} = \underline{s}\cdot\underline{\ell}^{*} = \underline{s}\cdot\underline{r} = \underline{s}\cdot\underline{a} = 0 \qquad (36)$$

$$\det(\underline{p},\underline{s},\underline{r},\underline{a}) = i\,\det(\underline{p},\underline{s},\underline{\ell},\underline{\ell}^{*}) = -m\,\underline{s}^{2} \geq 0 . \qquad (37)$$

4. Covariant positivity conditions.

A general polarization density matrix for a spin-1 particle of energy-momentum $\underline{p}$ is determined by an axial four vector $\underline{s}$ and a traceless symmetric tensor , $\underline{\underline{q}}$, both orthogonal to $\underline{p}$. However, besides equations (33) and (34) , $\underline{s}$ and $\underline{\underline{q}}$ have to satisfy other conditions equivalent to $\rho > 0$. In I.A7 we have shown that these conditions are, since $\text{tr}\ \rho = 1$:

$$1 - \text{tr}\ \rho^{2} \geq 0 \ , \quad 1 - 3\ \text{tr}\ \rho^{2} + 2\ \text{tr}\ \rho^{3} \geq 0 .$$

By a straightforward computation one finds the covariant form of these inequalities :

$$\frac{2}{3} - \text{tr}\ \underline{\underline{q}}^{2} + 2\,\underline{s}^{2} \geq 0 \quad \text{with} \quad \text{tr}\ \underline{\underline{q}}^{2} = q^{\mu}_{\nu}\ q^{\nu}_{\mu} \qquad (38)$$

$$\frac{2}{9} + 2\,\underline{s}^{2} - \text{tr}\ \underline{\underline{q}}^{2} + 2\ \text{tr}\ \underline{\underline{q}}^{3} + 6\ \underline{s}.\underline{\underline{q}}.\underline{s} \geq 0 \quad \text{with}\ \text{tr}\ \underline{\underline{q}}^{3} = q^{\mu}_{\nu} q^{\nu}_{\rho} q^{\rho}_{\nu} . \qquad (39)$$

The polarization degree d_{ρ} is given by

$$d^{2}_{\rho} = \frac{3}{2}\ \text{tr}(\rho - \frac{1}{3} I)^{2} \qquad 0 \leq d_{\rho} \leq 1 . \qquad \text{I.2(5)(5')}$$

Its covariant form is found to be :

$$d^{2}_{\rho} = -3\,\underline{s}^{2} + \frac{3}{2}\ \text{tr}\ \underline{\underline{q}}^{2} = d^{2}_{(1)} + d^{2}_{(2)} \qquad (40)$$

where $d_{(1)} = (-3\,\underline{s}^{2})^{\frac{1}{2}}$ and $d_{(2)} = (\frac{3}{2}\ \text{tr}\underline{\underline{q}}^{2})^{\frac{1}{2}}$ are the degree of odd and of even polarization respectively. For a state without even polarization $\underline{q} = 0$ and equations (39) and (40) yield :

$$0 \leq d_{\rho} = d_{(1)} \leq \sqrt{\frac{1}{3}} \quad . \tag{41}$$

Indeed $\sqrt{1/3}$ is the value of the radius of the positivity cone for B-symmetric states at the level of the symmetry center (see Fig. 2.1)

5. The manifold of pure states of polarization for spin-1 particles.

Every density matrix ρ can be diagonalized by a unitary transformation $\rho \rightsquigarrow u \rho u^{-1}$ with $u \in SU(3)$. However, this diagonalization is not always possible if one restricts u to be a 3×3 unitary matrix representing a "rotation" (= element of the little group of the energy momentum p). For example, for pure states, we will show that the density matrix of nearly all of them cannot be diagonalized. This is equivalent to say that one cannot find any quantization axis such that the polarization state is eigen vector of the "rotations" around this axis, the eigen value being the "magnetic quantum number" $m = -1, 0, 1$.

As we have seen in I.A8 , the manifold of pure states of polarization for spin-1 particles is the homogeneous space $SU(3) : U(2) = P_2(C)$ the complex two dimensional projective plane ; it is the set of normalized vectors $|\ell > \in \mathcal{H}_3$ defined up to a phase.

By the action of the "rotation" group, this manifold is decomposed into three strata

i) a stratum of one orbit, that of the <u>longitudinally polarized states</u>, i.e. states for which there exists a quantization axis such that their magnetic quantum number is $m=0$. Then $\underline{\ell}$ is, up to a phase, a real vector, the unit vector of this quantization axis. For these states

$$\underline{s}^2 = 0 \quad , \quad \mathrm{tr}\, \underline{q}^2 = \frac{2}{3} \quad , \quad \mathrm{tr}\, \underline{q}^3 = \frac{2}{9} \tag{42}$$

the little group of a longitudinally polarized state is $O(2)$, generated by the "rotations" around $\underline{\ell}$ and the "rotations" of π around the axes perpendicular to $\underline{\ell}$ (and $\underline{p}$).

ii) a stratum of one orbit, that of the circularly polarized states, i.e. states for which there exists a quantization axis such that their magnetic quantum number is 1 or -1. Then $\underline{p}/m$, $\sqrt{2}\,\underline{r}$, $\sqrt{2}\,\underline{a}$, $2\underline{s}$ form a right handed tetrad of unit vectors :

$$d_{(1)}^2 = -3\,\underline{s}^2 = \frac{3}{4}\,,\quad d_{(2)}^2 = \frac{3}{2}\,\mathrm{tr}\,\underline{q}^2 = \frac{1}{4}\,,\quad \mathrm{tr}\,\underline{q}^3 = \frac{-1}{36}\,. \tag{43}$$

The corresponding little group is $SO(2)$, the "rotations" around the quantization axis.

iii) the generic stratum containing a one parameter $(d_{(1)})$ family of three dimensional orbits whose little group is the two element group Z_2. Its generator is the "rotation" of π around $\underline{s}$:

$$I + \sum_{\varepsilon=\pm 1} (1+\varepsilon\hat{\underline{a}}.\hat{\underline{r}})^{-1}(\hat{\underline{r}}+\varepsilon\hat{\underline{a}}) \otimes (\hat{\underline{r}}+\varepsilon\hat{\underline{a}}) \quad \text{with} \quad \hat{\underline{r}} = \underline{r}/\sqrt{-r^2},\quad \hat{\underline{a}} = \underline{a}/\sqrt{-a^2}\,; \tag{44}$$

it transforms $\underline{\ell}$ into $-\underline{\ell}$. The parameter $d_{(1)}$ satisfies

$$0 \le -\underline{s}^2 = -\frac{1}{2}\,\mathrm{tr}\underline{q}^2 + \frac{1}{3} = -\mathrm{tr}\,\underline{q}^3 + \frac{2}{9} = \frac{1}{3}\,d_{(1)}^2 \le \frac{1}{4}\,. \tag{45}$$

Since the equations (42), (43) and (45) are covariant, the expressions such that "longitudinally polarized (pure) state" (for $d_{(1)} = 0$) and "circularly polarized (pure) state" ($d_{(1)} = \sqrt{3/2}$) have a covariant meaning for spin-1 particles. Indeed the study we have just made, shows that one can choose a quantization axis such that the corresponding "magnetic quantum number" m is respectively 0, ± 1 for such states. By a "rotation" (i.e. a transformation of the little group of $\underline{p}$ the particle energy-momentum) a longitudinally polarized state and a circularly polarized state for a given quantization axis, stay respectively longitudinally and circularly polarized along the rotated quantization axis.

In the physics literature one can also use the term of "transverse" polarization of a spin-1 particle for describing a pure state with a real -up to a phase- polarization vector $\underline{\ell}$ orthogonal to the quantization axis $\underline{n}$. Of course such a transverse state becomes longitudinal when the quantization axis is "rotated" by $\frac{\pi}{2}$ from $\underline{n}$ to $\underline{\ell}$.

In II 2.3 we have used the language of polarization vectors. The B-symmetric even pure states are represented by the vertex of the cone and the circumference of the base. They are respectively the longitudinal and the transverse polarization states in transversity quantization. †

Note on Covariant Polarization Density Tensor for spin one.

The treatment of this appendix is inspired from Michel-Rouhaninejad Phys. Rev. 122-242 (1961). It shows implicitely that for a spin one particle, we can represent the polarization density matrix by a second rank space time tensor which is hermitian, orthogonal to $\underline{p}$ and has trace one :

$$\underline{\underline{c}} = \underline{\underline{c}}^* \qquad \text{i.e.} \qquad c_{\mu\nu} = \bar{c}_{\nu\mu} \tag{46}$$

$$\underline{p} \cdot \underline{\underline{c}} = 0 = \underline{\underline{c}} \cdot \underline{p} \tag{46'}$$

$$\text{tr}\, \underline{\underline{c}} = 1 \qquad \text{i.e.} \qquad c^{\mu}_{\ \mu} = 1 \tag{46''}$$

In term of the dipole $\underline{s}$ and quadripole $\underline{\underline{q}}$ introduced in (31) and (32)

$$c_{\mu\nu} = \frac{1}{3}\left(g_{\mu\nu} - \frac{1}{m^2} p_{\mu} p_{\nu}\right) + \frac{i}{m} \epsilon_{\mu\nu\rho\sigma} p^{\rho} s^{\sigma} + q_{\mu\nu} \tag{47}$$

or

$$\underline{\underline{c}} = \frac{1}{3} I_p + \frac{i}{2m} (\underline{p} \wedge \underline{s})' + \underline{\underline{q}}$$

where I_p is defined in (32) and $(p \wedge s)'$ is the polar tensor of

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† For spin 1/2 particles the adjective longitudinal and transverse qualify the dipole polarization (which is an axial vector) ; see again our preliminary remark !

$(\underline{p} \wedge \underline{s})_{\mu\nu} = p_\mu s_\nu - s_\mu p_\nu$.

For a pure state of polarization vector $\underline{\ell}$ (with $\underline{\ell}.\underline{\ell}^* = -1$) we have seen that

$$\underline{\underline{c}} = -\ell \otimes \ell^* \qquad (48)$$

The transition probability λ of a reaction involving a spin one particle can be written

$$\lambda = F^{\mu\nu} c_{\mu\nu} \qquad (49)$$

where $F^{\mu\nu}$ depends on the $\underline{p}$'s of the reaction ($F^{\mu\nu}$ can be called the transition function).

One verifies that the density matrix can be written with operators $\underline{W}(p)$ defined in IA.1(30)

$$\rho(p) = \frac{1}{m^2} W(p)^\mu W(p)^\nu (c_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}) = \frac{1}{m^2} \underline{W}(p).(\underline{\underline{c}} - \tfrac{1}{2}I).\underline{W}(p)$$

Appendice A2

CALCUL EXPLICITE DES DOMAINES DE POLARISATION

1 - Particule de spin 1 .

La polarisation d'une particule de spin 1 est décrite par huit paramètres. Mais si la particule est produite dans une réaction B-symétrique, quatre paramètres sont identiquement nuls. Si on utilise une tétrade de transversité les quatre paramètres $r^{(L)}_{M}$ réels non nuls sont les paramètres avec M pair : $r^{(2)}_{2}$, $r^{(2)}_{0}$, $r^{(2)}_{-2}$ et $r^{(1)}_{0}$. Pour simplifier les notations nous les appelons respectivement Z, X, Y et F . Le paramètre F est un paramètre "fantôme" qui n'apparaît pas dans la distribution angulaire d'une désintégration en deux corps car il est de polarisation impaire.

La matrice densité s'écrit en fonction des paramètres $r^{(L)}_{M}$ sous la forme : (Partie I - ch. II - (51))

$$\rho = \frac{1}{2j+1} + \frac{2j}{2j+1} \sum_{L,M} r^{(L)}_{M} Q^{(L)}_{M} . \qquad (1)$$

Pour j = 1 on a :

$$\rho = \frac{1}{3} + \frac{2}{3} (Z\, Q^{(2)}_{0} + X\, Q^{(2)}_{2} + Y\, Q^{(2)}_{-2} + F\, Q^{(1)}_{0}) \qquad (2)$$

et en utilisant la forme explicite des matrices $Q^{(L)}_{M}$ (Partie I - ch. II) on obtient la matrice en damier :

$$\rho = \begin{pmatrix} \frac{1+Z}{3} + \frac{F}{\sqrt{3}} & 0 & \frac{X - iY}{\sqrt{3}} \\ 0 & \frac{1 - 2Z}{3} & 0 \\ \frac{X + iY}{\sqrt{3}} & 0 & \frac{1+Z}{3} - \frac{F}{\sqrt{3}} \end{pmatrix}$$

Les paramètres X, Y, Z, F étant des paramètres $r^{(L)}_{M}$ sont bien orthonormalisé

On vérifie en effet que le degré de polarisation d_ρ dont l'expression générale est (Partie I - ch. I - (48)) :

$$(d_\rho)^2 = \frac{2j+1}{2j} \, \mathrm{Tr}\left[(\rho - \frac{1}{2j+1})^2\right] \tag{3}$$

s'écrit :

$$(d_\rho)^2 = Z^2 + X^2 + Y^2 + F^2 .$$

D'une manière générale le domaine de positivité est défini par les deux conditions suivantes ; (Partie I - ch. II - (14))

a) il ne contient que des matrices ρ telles que

$$\det \rho > 0 \tag{4a}$$

b) il est connexe avec une matrice positive quelconque (par exemple la matrice ρ_o de l'état non polarisé). (4b)

De plus nous rappelons que le domaine est convexe et que par conséquent le contour apparent du domaine sur un plan de symétrie est confondu avec l'intersection par ce plan.

Pour la matrice (2), la condition (4a) s'écrit :

$$\det \rho = \frac{1}{9}\left[1 - 2Z\right]\left[(\frac{1+Z}{3})^2 - (X^2 + Y^2 + F^2)\right] > 0$$

Cette condition sera satisfaite si les quantités entre crochets sont toutes les deux positives ou toutes les deux négatives. Mais pour la matrice ρ_o (X=Y=Z=F=0 les crochets sont positifs. En vertu de (4b) ils le sont pour toutes les matrices positives et le domaine de positivité est donc défini par :

$$Z < \frac{1}{2} \quad , \tag{5a}$$

$$X^2 + Y^2 + F^2 < (\frac{1+Z}{3})^2 \quad . \tag{5b}$$

L'expression (5b) est satisfaite à l'intérieur de deux demi-cônes disconnexes de sommet Z = -1. Mais la condition (4b) précise qu'il ne faut retenir que le demi-cône ouvert vers les $Z > 0$ qui contient ρ_o, et fixe donc la borne inférieure de Z :

$$Z > -1 \tag{5c}$$

La transformation

$$F \longrightarrow -F$$

laisse invariantes les expressions (5). Donc le plan $F = 0$ est plan de symétrie du domaine (On connaissait a priori cette symétrie par rapport au plan des polarisations paires). Le domaine des polarisations paires est le contour apparent du domaine de positivité sur le plan $F = 0$. Ce plan étant plan de symétrie, contour apparent et intersection sont confondus et on obtient donc le domaine de positivité des paramètres de polarisation paire en faisant $F = 0$ dans les expressions (5).

Les conditions

$$-1 < Z < \frac{1}{2}$$

$$X^2 + Y^2 < \left(\frac{1+Z}{3}\right)^2$$

définissent bien le tronc de cône équilatéral et unitaire de la Fig. 2.1.

Les états purs se trouvent à une distance $d_\rho = 1$ du centre d'isotopie, et les états orthogonaux sont distants entre eux de

$$d_{orth.} = \sqrt{\frac{2j+1}{j}} \qquad (6)$$

c'est-à-dire $\sqrt{3}$.

2 - Particule de spin 3/2.

Si on choisit une tétrade de transversité, la polarisation d'une particule de spin 3/2 produite dans une réaction B-symétrique est décrite par les mêmes quatre paramètres que la particule de spin 1 : $r_0^{(2)}, r_2^{(2)}, r_{-2}^{(2)}, r_0^{(1)}$, plus trois paramètres avec $L = 3$: $r_0^{(3)}, r_2^{(3)}, r_{-2}^{(3)}$. Les paramètres avec $L = 1$ et $L = 3$ ne sont pas mesurables par étude de la distribution angulaire de la simple désintégration mais ils sont mesurables dans le cas de désintégration en cascade. Ce ne sont donc pas des "fantômes" au même titre que $r_0^{(1)}$ pour le spin 1. Pour simplifier l'écriture nous posons :

$r_0^{(2)} = Z \qquad r_2^{(2)} = X \qquad r_{-2}^{(2)} = Y$

$r_2^{(3)} = A \qquad r_{-2}^{(3)} = B$

$r_0^{(1)} = \frac{2C + D}{\sqrt{5}} \qquad r_0^{(3)} = \frac{-C + 2D}{\sqrt{5}}$

Les deux dernières relations définissent une nouvelle base dans le plan $r_0^{(1)}, r_0^{(3)}$. Cette nouvelle base est orthonormalisée et on passe de la base $r_0^{(1)}, r_0^{(3)}$ à la base C, D par une rotation d'angle $\alpha = \text{arc} \ \cos\sqrt{4/5} \simeq 27^o$.

Avec ces notations, et en utilisant l'expression (1) et la forme explicite des matrices $Q_M^{(L)}$ pour le spin 3/2 (Partie I - ch. II) on obtient la matrice en damier :

$\rho = \frac{1}{4}$

$1+\sqrt{3}Z+\sqrt{3}(C+D)$	0	$\sqrt{3}(X-iY)+\sqrt{3}(A-iB)$	0
0	$1-\sqrt{3}Z+\sqrt{3}(C-D)$	0	$\sqrt{3}(X-iY)-\sqrt{3}(A-iB)$
$\sqrt{3}(X+iY)+\sqrt{3}(A+iB)$	0	$1-\sqrt{3}Z-\sqrt{3}(C-D)$	0
0	$\sqrt{3}(X+iY)-\sqrt{3}(A+iB)$	0	$1+\sqrt{3}Z-\sqrt{3}(C+D)$

(7)

On vérifie sur cette matrice que les paramètres X, Y, Z sont pairs et que les paramètres A, B, C, D sont impairs : les paramètres pairs sont symétriques par rapport à la deuxième diagonale, les paramètres impairs sont antisymétriques. On vérifie aussi que tous ces paramètres sont orthonormalisés. En calculant le degré de polarisation $(d_\rho)^2$, Eq. (3), on obtient :

$$(d_\rho)^2 = Z^2 + X^2 + Y^2 + A^2 + B^2 + C^2 + D^2$$

Pour calculer le domaine de positivité on utilise les deux conditions (4). Le déterminant de la matrice (7) se calcule facilement, car en permutant les lignes et les colonnes on peut diagonaliser la matrice en deux blocs, et le déterminant se factorise aisément. (Cette propriété est une propriété générale des matrices en damier et constitue un des avantages de la quantification en transversité). La condition (4a) s'écrit :

$$\det \rho = \frac{3^2}{4^4}\left[(\sqrt{1/3} + D)^2 - (Z+C)^2 - (X+A)^2 - (Y+B)^2\right]$$
$$\left[(\sqrt{1/3} - D)^2 - (Z-C)^2 - (X-A)^2 - (Y-B)^2\right] > 0$$

La condition sera satisfaite si les quantités entre crochets sont toutes les deux positives ou toutes les deux négatives. Mais la condition (4b) impose que les deux crochets soient positifs. Le domaine de positivité est donc défini par :

$$(X+A)^2 + (Y+B)^2 + (Z+C)^2 < \left(\sqrt{\frac{1}{3}} + D\right)^2 \qquad (8a)$$

$$(X-A)^2 + (Y-B)^2 + (Z-C)^2 < \left(\sqrt{\frac{1}{3}} - D\right)^2 \qquad (8b)$$

On vérifie facilement que la transformation

$$A \rightarrow -A \qquad B \rightarrow -B \qquad C \rightarrow -C \qquad D \rightarrow -D$$

laisse invariantes les expressions (8) et que par conséquent le trois-plan des paramètres impairs nuls : $A = B = C = D = 0$ est plan de symétrie du domaine à sept dimensions. Le contour apparent du domaine sur ce plan est confondu avec son intersection par ce plan, et le domaine de positivité des paramètres pairs s'écrit :

$$X^2 + Y^2 + Z^2 < \frac{1}{3} \qquad (9a)$$

qui est bien l'équation de la sphère de la Fig. 3.1 .

Le trois-plan $X = Y = Z = D = 0$ est un autre plan de symétrie du domaine à sept dimensions. Le contour apparent du domaine sur ce plan s'écrit :

$$A^2 + B^2 + C^2 < \frac{1}{3} \qquad (9b)$$

c'est le domaine de positivité des trois paramètres impairs A, B, C.

Enfin, la droite $A = B = C = X = Y = Z = 0$ est aussi un axe de symétrie. Le domaine de positivité du quatrième paramètre impair D est donc :

$$|D| < \sqrt{1/3} \qquad (9c)$$

Les projections (9a), (9b) et (9c) sont les sphères (P) et (Q) et le segment (R) de la Fig. 4. 1. Une matrice positive est <u>nécessairement</u> représentée par un point à l'intérieur de chacun de ces domaines mais ce n'est pas une condition <u>suffisante</u>. Les conditions suffisantes de positivité sont données par les inégalités (8) qui peuvent être représentées comme des conditions sur la longueur de la somme et de la différence de deux vecteurs comme cela est discuté dans le paragraphe 4. 3. Pour tout point dans (Q) et (R) on peut trouver un point dans (P) qui satisfasse les relations (8). C'est-à-dire que la projection du domaine de positivité sur l'espace des quatre paramètres impairs est une portion d'un hypercylindre droit de base sphérique (Q) et de hauteur (R).

3 - Particule de spin 2 .

La polarisation d'une particule de spin 2 produite dans une réaction B-symétrique est décrite par douze paramètres. Nous nous intéressons uniquement aux huit paramètres de polarisation paire mesurables par la distribution angulaire de désintégration en deux corps. L'hyper-plan de polarisation paire étant un plan de symétrie du domaine à douze dimensions on obtient le domaine de positivité des paramètres pairs en annulant les paramètres impairs. Nous pourrions utiliser comme dans les cas précédents les huit paramètres $r^{(L)}_{M}$ de transversité (L et M pairs) mais les formules deviendraient trop compliquées. Nous préférons utiliser directement les éléments de matrice densité convenablement normalisés. La matrice densité en damier, paire, peut être écrite de la façon suivante :

$$\rho = \frac{1}{5} \begin{pmatrix} 1+\sqrt{6}\,a_2 & & \sqrt{5}(e-if) & & \sqrt{10}(g-ih) \\ & 1+\sqrt{6}\,a_1 & & \sqrt{10}(c-id) & \\ \sqrt{5}(e+if) & & 1-\sqrt{24}(a_1+a_2) & & \sqrt{5}(e-if) \\ & \sqrt{10}(c+id) & & 1+\sqrt{6}\,a_1 & \\ \sqrt{10}(g+ih) & & \sqrt{5}(e+if) & & 1+\sqrt{6}\,a_2 \end{pmatrix} \qquad (10)$$

La relation entre les paramètres a_1 , a_2 , c, d, e, f, g, h et les paramètres multipolaires $r^{(L)}_{M}$ est donnée dans la Table 5. 3. Les paramètres a_1 et a_2 de la matrice (10) sont normalisés mais ils ne sont pas orthogonaux. Si on introduit les paramètres b_1 ou b_2 par les formules :

$$a_2 = -\frac{2}{3} a_1 - \frac{\sqrt{5}}{3} b_1 \tag{11a}$$

$$a_1 = -\frac{2}{3} a_1 - \frac{\sqrt{5}}{3} b_2 \tag{11b}$$

on vérifie, en calculant la quantité $(\delta_\rho)^2$, Eq. (3), que les huit paramètres $a_1, b_1, c, \ldots, h$, ou les huit paramètres $a_2, b_2, c, \ldots, h$, forment deux ensembles de paramètres orthonormalisés. Cependant, à cause de la forme géométrique du domaine de positivité nous préférons employer les paramètres a_1 et a_2 en gardant en mémoire que ces paramètres sont normalisés, mais que d'après (11a) et (11b) ils forment un angle

$$\chi = \text{arc cos} \frac{2}{3} \cong 48^o$$

La matrice (10) peut être diagonalisée en deux blocs, l'un 2×2 (correspondant aux éléments d'indices pairs), l'autre 3×3 (correspondant aux éléments d'indices impairs), chaque bloc étant symétrique par rapport à la deuxième diagonale. Le déterminant se factorise en deux expressions, l'une du deuxième degré, l'autre du troisième degré. Selon (4a) les deux expressions doivent avoir le même signe et selon (4b) ce signe commun est + . Le domaine de positivité est donc défini par :

$$\frac{(1 + \sqrt{6}\, a_1)^2}{10} - (c^2 + d^2) > 0 \tag{12a}$$

$$\left[1-\sqrt{24}(a_1+a_2)\right]\left[\frac{(1+\sqrt{6}\, a_2)^2}{10} - (g^2+h^2)\right] - (1+\sqrt{6}\, a_2)(e^2+f^2)+\sqrt{10}\, g(e^2-f^2)+2\sqrt{10}\, hef > 0 \tag{12b}$$

La condition (12a) est satisfaite à l'intérieur de deux demi-cônes disconnexes de sommet $a_1 = -\sqrt{\frac{1}{6}}$. La condition (4b) précise qu'il ne faut retenir que le demi-cône ouvert vers $a_1 > 0$ qui contient ρ_o et fixe donc la borne inférieure de a_1 :

$$a_1 > -\sqrt{\frac{1}{6}} \tag{12c}$$

L'expression (12b) est plus compliquée mais on observe qu'elle est invariante pour la transformation

$$e \rightarrow -e \quad , \quad f \rightarrow -f$$

Le plan $e = f = 0$ est un plan de symétrie du domaine, et le contour apparent du domaine sur ce plan est égal à son intersection par ce plan. L'expression qui définit ce contour apparent se factorise et, en utilisant de nouveau la condition (4b), le domaine est défini par :

$$1 - \sqrt{24}\,(a_1 + a_2) > 0 \tag{12d}$$

$$\frac{(1 + \sqrt{6}\, a_2)^2}{10} - (g^2 + h^2) > 0 \tag{12e}$$

La condition (12e) est de la même forme que la condition (12a). Elle définit un demi-cône et une borne inférieure pour a_2 :

$$a_2 > -\sqrt{\frac{1}{6}} \tag{12f}$$

Les conditions (12c, d, f) définissent le domaine convexe des paramètres a_1 et a_2 , c'est un triangle. Les conditions (12f, c) déterminent les bases des troncs de cône définis par les conditions (12a, e).

Enfin pour fixer la limite des paramètres e et f quand les autres paramètres sont donnés à l'intérieur de leur domaine de positivité, il suffit d'utiliser la condition (12b). On obtient une ellipse dont il est facile de déterminer les axes principaux.

L'ensemble de ces résultats, concernant le domaine de polarisation des huit paramètres pairs d'une particule de spin 2 , est rassemblé dans la Table 5.3, et dans la Fig. 5.1.