Invariance in Quantum Mechanics and Group Extension

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I. Introduction

Group theory was the main theme of the Istanbul Summer School. The subject of group extensions has never been taught in a physics summer school. It seemed to me a good opportunity to do it, but I have to explain why this subject can be of interest for physicists.

We physicists have to consider several kind of invariance: relativistic invariance, gauge invariance, strong coupling invariance, . . . (read the title of chapters of some other sets of lectures). How are related the different invariance groups? This is a fundamental question to answer.

Too often physicists consider them separately (that is, they consider their direct product) because they do not know of other solutions. Let us show on a simple example how this attitude can be misleading. If one considers separately invariance under the space-time translations $\mathscr{T}$ (with infinitesimal operators P^λ) and the connected homogeneous Lorentz group $\mathscr{L}_0$ (infinitesimal operators $M^{\mu\nu}$) one finds three linearly independent invariants: $\mathbf{P}^2 = p^\lambda p_\lambda$ the square of the mass, $M^{\mu\nu}M_{\mu\nu}$ and $\epsilon_{\lambda\mu\nu\rho}M^{\lambda\mu}M^{\nu\rho}$. However relativistic invariance has to be described by $\mathscr{P}_0$, the connected inhomogeneous Lorentz group, also called Poincaré group. The group $\mathscr{P}_0$ is a semi-direct product of $\mathscr{L}_0$ by $\mathscr{T}$. The group $\mathscr{P}_0$ yields only two invariants: $\mathbf{P}^2$ and $\mathbf{W}^2 = W^\lambda W_\lambda$, with $W_\lambda = \epsilon_{\lambda\mu\nu\rho}\mathbf{P}^\mu M^{\nu\rho}$; this last invariant is related to the spin of the particle. It is strange to see that many papers in relativistic quantum mechanics used to consider invariance under $\mathscr{T}$ and $\mathscr{L}_0$ separately. Historically, the emphasis on the role of the inhomogeneous Lorentz group $\mathscr{P}_0$ in relativistic quantum mechanics is mainly due to Wigner. We shall refrain to quote him in order to avoid the accusation of cult of personality; however we shall have often to refer to his paper "On unitary representations of the inhomogeneous Lorentz group", *Ann. Math.*, **40**, 149 (1939). We will refer to it simply as Wigner "F" (F for fundamental).†

These lectures do not presuppose much knowledge of group theory (see the appendix of this Introduction). After two chapters (II and III) written from physical motivations, the mathematical problem of group extensions is developed (IV). This problem is:

Given two groups A and B, find all groups E such that A is invariant subgroup of E and B is the quotient E/A.

If A is abelian and if it is moreover required: $A \subset$ center of E, the group E is called a central extension of B by A. These notes contain the characterization of central extensions of the connected Poincaré group $\mathscr{P}_0$ by an arbitrary abelian group. The solutions presented are rather trivial. Unhappily the non-existence of other solutions is not proven (see V).

The corresponding extensions for the complete Poincaré group, including P, C and T, are presented in VI. The classical mathematical theory of group extensions allows to generalize the above results to extensions by a non-abelian group K. This is explained in VII.

The preparation of these lectures has been done in collaboration with F. Lurçat. We intend to publish together another version of the same subject. Here, these notes are mainly intended to incite the reader to become acquainted with the classical mathematical literature on the subject. The possible physical applications are not systematically and thoroughly investigated.

F. Lurçat and I are grateful for their help in different ways, to H. Epstein, J. Lascoux, many other physicists and few French mathematicians, especially J.-P. Serre.

F. Lurçat has not seen the actual text of these notes. Mr. Jaffe has seen part of it and I am indebted to him for his brave attempt to correct the worst sentences of my dull and ungrammatical English.

† As a point of history we also want to reproduce here Wigner's acknowledgment: Wigner, "F," p. 156: "The subject of this paper was suggested to me as early as 1928 by P. A. M. Dirac."

Appendix

We just want here to remind to the reader some concepts and results of group theory, and to accustom him to the mathematical vocabulary.

1. Mappings (for Sets)

Let G and G' be two sets and f be a mapping of G into G'; that is f is a function defined upon G, with range in G' (for each $x \in G$, $f(x)$ is one element of G'). The range of f is denoted by $f(G)$ or by $Im\,f$. More generally, let X' be the set of values of $f(x)$ for all $x \in X \subset G$. We write $X' = f(X)$ or $X = f^{-1}(X')$. Remark that f^{-1} is not, in general, a mapping of G' into G. Indeed, for any $x' \in G'$, $f^{-1}(x')$ is not generally an element of G, but a subset of G, which may be the empty subset.

The relation $f(x) = f(y)$ is an equivalence relation for the elements of G. For each $x' \in G'$, the $f^{-1}(x')$ are equivalence classes. The set $\underset{\sim}{Q}$ of these equivalence classes is called the quotient of the set G by the equivalence relation $f(x) = f(y)$.

If $Im\,f = G'$, the mapping f is surjective. If $f(x) = f(y) \Rightarrow x = y$, the mapping is "injective". A one-to-one mapping is both surjective and injective.

2. Some Vocabulary Exercises on Groups

Given a subset X of the set of elements of a group G, we define two subsets of G:

The centralizer $\mathscr{C}(X)$ is the set of all $y \in G$ such that for every $x \in X$, $yxy^{-1} = x$.

The normalizer $\mathscr{N}(X)$ is the set of all $y \in G$ such that $yXy^{-1}, = X$. Hence $\mathscr{C}(X) \subset \mathscr{N}(X)$.

To help eventually the reader to assimilate these two new words, we use them in sentences which are either theorems easy to prove or rewording of definitions of well-known concepts. Of course, if X has only one element x, $\mathscr{C}(x) = \mathscr{N}(x)$.

$\mathscr{C}(X)$ and $\mathscr{N}(X)$ are subgroups of G and $\mathscr{N}(G) = G$.

$\mathscr{C}(G)$ is the center of G. If $\mathscr{C}(G) = G$ the group is abelian.

If H is a subgroup of G and if $\mathscr{N}(H) = G$, the group H is said to be invariant subgroup of G (some authors use "normal" instead of invariant). As an example $\mathscr{N}[\mathscr{C}(G)] = G$; the center of G is an invariant subgroup of G.

Theorem. $\mathscr{C}(X)$ is an invariant subgroup of $\mathscr{N}(X)$.

To avoid confusion, for algebra, we shall use the word "commutant" instead of centralizer (see Chapter III).

3. Homomorphisms (for Groups)

A mapping f of the group G into the group G' is a homomorphism if it preserves the group law: for every $x, y \in G$, $f(x)f(y) = f(xy)$. Let e and e' be the unit elements of respectively G and G'; $e' \in f(e)$. Images and inverse images of subgroups are subgroups. Moreover, if H' is an invariant subgroup of G', $f^{-1}(H')$ is invariant subgroup of G. As a particular case, $f^{-1}(e')$ is an invariant subgroup of G called the kernel of f and denoted by $\operatorname{Ker} f$.

If $\operatorname{Ker} f = e$, f is injective and $\operatorname{Im} f \approx G$ (we use $\approx$ for isomorphic, that is: there exists a one-to-one homomorphism). If $\operatorname{Ker} f$ has other elements than e, there is a natural group law on the set $\underset{\sim}{Q}$ of equivalence classes $f^{-1}(x')$. We say that $\underset{\sim}{Q}$ is the quotient group $G/\operatorname{Ker} f$. One has the fundamental isomorphism:

$$\operatorname{Im} f \approx \mathrm{G}/\operatorname{Ker} f$$

We denote by $\operatorname{Hom}(G, A)$ the set of homomorphisms of G into A. This set is not empty: it contains at least the trivial homomorphism $f(G) = e'$. If A is abelian (and noted additively), for every $x \in G$ and every pair f_1, f_2 of homomorphisms, $f_1(x) + f_2(x)$ is a well-defined element of A that we denote by $(f_1 + f_2)(x)$. It is easy to check that A abelian $\Rightarrow f_1 + f_2$ is a homomorphism. Hence, when A is abelian, $\operatorname{Hom}(G, A)$ is an abelian group. Its zero element is the trivial homomorphism.

If a group has no invariant subgroup, except e and G itself, it is simple. If G is non-abelian and simple, if A is abelian, then $\operatorname{Hom}(G, A) = 0$ (that is, it has only the element zero). We also recall a more precise result: given a group G, its elements of the form $xyx^{-1}y^{-1}$ are called commutators; they generate the subgroup G' of G, called the derived group of G. Then, one proves: if A abelian and $G' = G$, then $\operatorname{Hom}(G, A) = 0$.

We shall need often the notion of *direct product* of groups. Given two groups G_1, G_2, the direct product $G = G_1 \otimes G_2$ has for elements the pairs (x_1, x_2), which are the elements of the set product of the sets G_1 and G_2, and for multiplication law: $(x_1, x_2)(y_1, y_2) = (x_1y_1, x_2y_2)$. We can define the direct product of a finite number of groups; the operation $\otimes$ is associative. When the group laws are noted additively, we shall use the synonymous expression "direct sum" and the sign $\oplus$. However, for an infinite set of groups, these two expressions are given two different meanings in mathematical literature.

2. Lie Groups With the Same Lie Algebra

Application: the relation between isospin and hypercharge.

We shall review the general method to determine all connected Lie groups which have a given Lie algebra, and then work out in detail a simple example which corresponds to the minimum symmetry of strong interactions.

All physical schemes for strong coupling contain three observables, T_1, T_2 and T_3, whose Hermitian operators satisfy the commutation relations

$$[T_i, T_j] = i\epsilon_{ijk}T_k. \tag{1}$$

An equivalent set of relations is

$$[T_\pm, T_3] = \mp T_\pm, \ [T+, T-] = 2T_3 \tag{1'}$$

where

$$T_\pm = T_1 \pm iT_2. \tag{2}$$

In addition, the commutation relations with the electric charge operator Q are

$$[T_3, Q] = 0 \text{ and } [T_\pm, Q] = \mp T_\pm. \tag{3}$$

Expressions (1′) and (3) define a four-dimensional Lie algebra over the field of real numbers. If we set

$$Y = Q - T_3. \tag{4}$$

Then relations (3) are equivalent to

$$[T_i, Y] = 0. \tag{5}$$

This shows that the four-dimensional Lie algebra $\mathscr{L}$ defined by (1) or (1′) and (3) or (5) is a direct sum

$$\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}(\mathrm{SU}_2) \tag{6}$$

where $\mathscr{L}_1$ is the one-dimensional algebra. (It is unique up to an isomorphism.)

Does this mean that the isospin **T** and hypercharge Y are separately conserved with no relations between them? If we consider the Lie algebra alone, we have the answer yes by basic principle of invariance in quantum physics. However, if we do not restrict ourselves to invariance under infinitesimal transformations, then relations are possible. A poll among physicists (made by reading their relevant articles) revealed that a large majority favor invariance under finite transformations, while a few physicists are either against this approach or are "undecided". The reason that finite transformations are considered necessary is that they also allow the consideration of discrete operations. Thus we are led to consider a global invariance group. In our particular example we have finite hypercharge gauge transformation and finite isospin transformations. Another example to support this point of view will be discussed in Chapter III.

Since there are several Lie groups which have the same Lie algebra, the physicist is forced to choose the relevant one. Each time he writes down a finite transformation, the physicist makes a choice at least implicitly. In our example, which of the five groups should we take to describe isospin and hypercharge conservation? My preference is to list all the possibilities before making a choice. As an example of the general method, we start by stating two theorems. Corresponding to the physical situation, we shall only consider finite dimensional Lie algebras over the field of real numbers.

Theorem 1. For every such algebra $\mathscr{L}$, there exists one and only one connected group which is simply connected and has $\mathscr{L}$ as its Lie algebra.

This group is called the universal covering group $\bar{\mathscr{g}}$ of the Lie algebra $\mathscr{L}$.

(Simply connected means that all closed curves in $\bar{\mathscr{g}}$ obtained by continuous mappings of a circle into $\bar{\mathscr{g}}$ can be contracted by continuous deformation to a point. See Prof. Speiser's lectures.)

Theorem 2. All connected Lie groups $\bar{\mathscr{g}}$ which have $\mathscr{L}$ as Lie algebra are the quotient of the universal covering group $\bar{\mathscr{g}}$ of $\mathscr{L}$, by one of its invariant discrete subgroups D (i.e., D is discrete for the topology induced by that of $\bar{g}$).

One then shows that $D \subset$ Center of $\bar{g}$. Indeed, let $a \in D$ and $x \in \bar{g}$, then xax^{-1} is a continuous function of x; since D is invariant subgroup, the range of this function is in D, and since D is discrete, this function has a fixed value:

Taking $x = 1$, it is found to be a. Hence, for all $x \in \bar{g}$, $xa = ax$.

These two theorems will guide us for the study of our example where $\mathscr{L}$ is defined in (6). However, an answer to the general problem is not that simple. In fact two isomorphic discrete groups D_1 and D_2 may yield non-isomorphic quotients $\bar{g}/D_1$ and $\bar{g}/D_2$.

The simplest application of these theorems is to the one-dimensional Lie algebra $\mathscr{L}_1$. Its universal covering group is R, the additive group of real numbers. Let $\alpha_0 \in R$. It generates a group $Z(\alpha_0)$ which is isomorphic to Z, the additive group of integers. (The elements of $Z(\alpha_0)$ are $n\alpha_0$.) The quotient $R/Z = U_1$ is given by the homomorphism $R \xrightarrow{f} U_1$, where $f'(\alpha) = \exp[2i\pi\alpha/\alpha_0]$. The group operation in U_1, the one-dimensional unitary group, is written multiplicatively and $f'(n\alpha_0) = 1$. Since the choice of α_0 is irrelevant, we will usually choose $\alpha_0 = 2\pi$ and define the homomorphism

$$\alpha \in R \xrightarrow{f} U_1, \ni f(\alpha) \text{ with } f(\alpha) = e^{i\alpha} \tag{7}$$

Has R other discrete subgroups?

Let $\alpha_1 \notin Z(\alpha_0)$. If α_1/α_0 is rational $\alpha_1/p = \alpha_0/q = \alpha'$ where $p, q \in Z$. Then $Z(\alpha_1)$ and $Z_1(\alpha_0)$ are subgroups of $Z(\alpha')$, and the quotient $R/Z(\alpha')$ yields again U_1, up to an isomorphism.

If α_1/α_0 is irrational, the elements of the group generated by α_0 and α_1 are $m\alpha_0 + n\alpha_1$ where $m, n \in Z$. This group is isomorphic to $Z \oplus Z$. But it is not discrete in R. Indeed, given $\alpha \in R$ and $\epsilon > 0$, it is always possible to find m and n such that

$$|m\alpha_0 + n\alpha_1 - \alpha| < \mathscr{E}$$

i.e., the group $Z(\alpha_0) \oplus Z(\alpha_1)$ is open and dense in R. (Its closure is R itself.)

To summarize:

There are only two, non-isomorphic, one (real) parameter Lie groups, the groups R and U_1.

We can now study our example $\mathscr{L}$ given by (6). Let us first recall some results well-known by physicists. The covering group of the Lie algebra generated by (1) is SU_2 (the group of unitary 2×2 matrices with determinant $= 1$).

We shall denote its elements:

$$\sigma(\mathbf{n}, \theta) = \exp(-i\theta\mathbf{n} \cdot \boldsymbol{\tau}/2) \tag{8}$$

where θ is a real number modulo 4π; $\mathbf{n} \cdot \boldsymbol{\tau}$ is a short hand for

$$\sum_{i=1}^{3} \mathbf{n}_i \boldsymbol{\tau}_i$$

where τ_i are the three Pauli matrices, and n_i are three real numbers such that $\Sigma_i n_i^2 = \mathbf{n} \cdot \mathbf{n} = 1$.

The group law is the matrix multiplication:

$$\sigma_1(\mathbf{n}_1, \theta_1)\sigma_2(\mathbf{n}_2, \theta_2) = (\sigma_1\sigma_2)(\mathbf{n}, \theta) = \sigma(\mathbf{n}, \theta) \tag{9}$$

The center Z_2 of SU_2 has two elements

$$\sigma(\mathbf{n}, 0) = 1 \quad \text{and} \quad \sigma(\mathbf{n}, 2\pi) = -1$$

The quotient SU_2/Z_2 is isomorphic to SO_3 (the group of 3×3 real orthogonal matrices with determinant $= 1$), that is the rotation group in three dimensions.

The universal covering group of the four-dimensional Lie algebra defined by (6) (or by (1) and (5)) is the direct product

$$\bar{\mathscr{g}} = R \otimes SU_2 \tag{10}$$

with the multiplication law

$$(\alpha_1, \sigma_1)(\alpha_2, \sigma_2) = (\alpha_1 + \alpha_2, \sigma_1\sigma_2) \tag{10'}$$

and the unit element (0,1).

Table 1 gives the complete list of the connected groups $\bar{\mathscr{g}}$ which have $\mathscr{L}$ as Lie algebra. They are obtained by a quotient $\bar{\mathscr{g}}/D$ when D is a discrete subgroup of the center $R \otimes Z_2$ of $\bar{\mathscr{g}}$.

TABLE 1

$\mathcal{g} = \bar{\mathcal{g}}/D$	D isomorphic to	D generated by	Group law of
$\mathcal{g} = R \times SU_2$	$\{1\}$ = one element	$(0, 1)$	$(\alpha_1, \sigma_1)(\alpha_2, \sigma_2) = (\alpha_1 + \alpha_2, \sigma_1\sigma_2)$
$R \times SO_3$	$\{0\} \times Z_2 \approx Z_2$	$(0, -1)$	$(\alpha_1, \epsilon_1 \sigma_1)(\alpha_2, \epsilon_2 \sigma_2) = (\alpha_1 + \alpha_2, \epsilon\sigma_1\sigma_2)$
$U_1 \times SU_2$	$Z \times \{1\} \approx Z$	$(2\pi, 1)$	$(e^{i\alpha_1}, \sigma_1)(e^{i\alpha_2}, \sigma_2) = (e^{i(\alpha_1+\alpha_2}, \sigma_1\sigma_2)$
$U_1 \times SO_3$	$Z \times Z_2$	$(2\pi, 1)$ and $(0, -1)$	$(e^{i\alpha_1}, \epsilon_1\sigma_1)(e^{i\alpha_2}, \epsilon_2\sigma_2) = (e^{i(\alpha_1+\alpha_2)}, \epsilon\sigma_1\sigma_2)$
U_2	Z	$(\pi, -1)$	$e^{i\alpha_1}\sigma_1 e^{i\alpha_2}\sigma_2 = e^{i(\alpha_1+\alpha_2)}\sigma_1\sigma_2$

The ϵ are arbitrary signs

This table gives the complete list, up to an isomorphism, of the connected groups $\mathcal{g}$, which have $\mathscr{L}$ as Lie algebra.

For all the lines of Table 1, except the last, D is a direct product $D = D_1 \otimes D_2$ and we obtain the quotient $\bar{\mathcal{g}}/D$ by the elementary theorem.

Theorem 3. $\bar{\mathcal{g}}/D \equiv (\mathcal{g}_1 \otimes \mathcal{g}_2)/(\mathscr{D}_1 \otimes \mathscr{D}_2) = (\mathcal{g}_1/\mathscr{D}_1) \otimes (\mathcal{g}_2/\mathscr{D}_2)$.

Let us study in more detail the last line. The group D is generated by $(\pi, -1)$; its elements are

$$(\pi, -1)^k = [k\pi, (-1)^k] \tag{11}$$

when k is an integer.

Consider now the homomorphism g of $\bar{\mathcal{g}}$ upon U_2 defined by

$$\begin{aligned} \bar{\mathcal{g}} \ni [\alpha, \sigma(\mathbf{n}, \theta)] &\xrightarrow{g} \exp(i\alpha - i\theta\mathbf{n}.\boldsymbol{\tau}/2) \\ &= e^{i\alpha}\left(\cos\frac{\theta}{2} - i\boldsymbol{\tau}.\mathbf{n}\sin\frac{\theta}{2}\right). \end{aligned} \tag{12}$$

The elements that g maps on 1 are defined by

$$\sin\frac{\theta}{2} = 0,\ e^{i\alpha}\cos\frac{\theta}{2} = 1, \quad \text{hence } \theta = 2k\pi,\ \alpha = k\pi,$$

$$\text{i.e. } [k\pi, (-1)^k] \in \bar{\mathcal{g}}$$

and they are also the elements of D.

Hence $$\bar{\mathcal{G}} \xrightarrow{g} \mathscr{L}/D = U_2. \tag{13}$$

Of course, the matrices $e^{i\alpha}$, multiples of the unit, form the center (U_1) of U_2, and we have the relation

$$U_2/U_1 = SO_3. \tag{14}$$

If one does not normalize α_0 to 2π, it is easy to check that there are no other (non-isomorphic) groups solution of our problem.

We now want to study the list of unitary, irreducible representations of the different groups $\mathcal{G}$ which have the same Lie algebra $\mathscr{L}$. (For indeed in the physical formalism the invariance groups will act through their unitary representations.)

Let us emphasize that a representation is a group homomorphism, so it might not be faithful. For instance, the irreducible representations of SU_2 are usually labelled by the integer $2j \geq 0$. When j is integer, the representations are not faithful, but they are faithful representations of SO_3. The representations of SU_2 with $2j$ odd are *not* representations of SO_3.

The unitary irreducible representations of R are given by $\alpha \to e^{ir\alpha}$ where r is a real number. They are never faithful, and they are representations of U_1 only when r is integer.

Hence we can write down the four first lines of Table 2.

TABLE 2

List of Inequivalent, Unitary, Irreducible Representations of the Group $\mathcal{G}$ of Table 1

(The representations are labelled by two numbers r and j which satisfy the conditions)

$R \times SU_2$	r real	$2j$ integer ≥ 0
$R \times SO_3$	r real	j integer ≥ 0
$U_1 \times SU_2$	r integer	$2j$ integer ≥ 0
$U_1 \times SO_3$	r integer	j integer ≥ 0
U_2	r integer	$2j$ integer ≥ 0 and $(-1)^{2j+r} = 1$

For the last line, the group U_2, we have a relation between j and r; indeed, in the first line, the element $(\pi, -1)$ of $\bar{\mathcal{G}}$ is represented by $e^{i\pi r}(-1)^{2j}$ in the representation r, j. This representation is a representation of U_2 only if the element $(\pi, -1)$ is represented by 1, that is

$$\exp(i\pi r)(-1)^{2j} = (-1)^{2j+r} = 1. \tag{15}$$

In the physical interpretation of our example j is the isospin t and r is the hypercharge y.

Of course a choice of $\alpha_0 \neq 2\pi$ would also have given the set of integers r, and the same relation (15). If we want the possibility for the hypercharge of physical states to be any integer, the identification must be $y = r$.

The famous relation between the electric charge q, the third component t_3 of isospin, the baryonic charge b and the strangeness s

$$q = t_3 + \frac{b+s}{2} \tag{16}$$

can be written with $b + s = y$

$$2q = 2t_3 + y. \tag{17}$$

It implies

$$2t + y \equiv \text{modulo } 2, \text{ i.e.} \tag{18}$$

$$(-1)^{2t+y} = 1 \tag{18'}$$

which is exactly the relation (15).

Many papers have required that the strong coupling invariance group contain SU_2 (isospin) and U_1 (hypercharge) as a direct product (third line of Tables 1 and 2). (For references see other lecturers.) It seems more appropriate to require only that it must contain U_2 as a subgroup.

Let us recall how U_2 will appear naturally in any field formalism. The requirement that the Lagrangean be invariant when the field of isospin t is multiplied either by the unitary matrices which correspond to the representation $j = t$ of SU_2 or by the first kind of gauge transformation $e^{i\alpha y}$, where y is the hypercharge of the field, is equivalent to choosing the ordinary multiplication as the multiplication law of the total invariance group. Table 1 shows that it yields U_2 (fifth line) and not the direct product of line 3. Indeed the products $e^{i\alpha}\sigma$ and $(-e^{i\alpha})(-\sigma)$ are identified; this yields U_2 as quotient:

$$U_2 \simeq (U_1 \otimes SU_2)/Z_2 \tag{19}$$

where the two elements of Z_2 are $(1, 1)$ and $(-1, -1)$.

This long chapter will probably seem trivial to the physicist, so I would like to end it by a question I am unable to answer.

I could have considered another Lie algebra isomorphic to $\mathscr{L}$, but generated by the isospin T and the baryonic charge B. The commutation relations are indeed (1) and

$$[\mathbf{T}, B] = 0. \tag{20}$$

If I had considered only non-strange particles in the nucleon and π fields, I could have taken U_2 as invariance group for the strong coupling part of the Lagrangian. But if one considers all known particles, there are no relations between baryonic charge and isospin. Hence one has to consider the direct product

$$U_1(B) \times SU_2$$

as an invariance group, and one must not multiply together a baryonic charge gauge transformation and an isospin transformation applied to the same field.

The same reasoning applies to strangeness and isospin.

To summarize, among the six operators T, B, S, Y there are five linearly independent ones ($Y = B + S$), which form a five-dimensional Lie algebra

$$\mathscr{L}_1 \oplus \mathscr{L}_1 \oplus \mathscr{L}(SU_2)$$

It seems that the invariance group of strong coupling must contain the group

$$U_1 \otimes U_2$$

where U_1 corresponds to the baryonic charge and U_2 to the isospin and the hypercharge. In particular, this is the case of the two fashionable examples wrongly ascribed, in physics literature, to the group SU_3 for strong coupling invariance: the "Sakata model" and the "eight fold way". The strong coupling invariance group of these two modes are respectively:

For the Sakata model: U_3 which does contain $U_1 \otimes U_2$; for the eight-foldway: $U_1 \otimes (SU_3/Z_3)$; the second factor is SU_3 divided by its center; it does contain U_2. In this last model, baryonic charge conservation is a completely independent conservation law.

Why is there no relation between baryonic charge and the other strong coupling quantum numbers t and y?

This lecture was given as a seminar at CERN in December 1961, and at Bruxelles (Centre interuniversitaire de Physique nucléaire) in February 1962.

3. A Mathematical Framework for Quantum Mechanics

(This chapter is based on work done with F. Lurçat. Two preliminary short notes have been published; F. Lurçat and L. Michel, *Nuovo Cimento* **21**, 574 (1961), and *Comptes-Rendus of the Conference of Aix-en-Provence*, p. 183.)

We are aware that an axiomatic approach is rarely the road to discovery in physics and that the mathematical framework presented here is still so general that it cannot yield many physical results. However, such a framework contains just what we need to study invariance in quantum mechanics and to shed some light on the points we want to emphasize. On the other hand, we do not want to be dogmatic and we will choose this mathematical structure only as a possible one, not as an exact or definitive choice. So first we shall try to give the axioms in a physical language; only afterwards will we attempt to make a mathematical translation.

1. Axiom a

"In quantum mechanics, observables are represented by hermitian operators acting in a Hilbert space."

Among the observables are the density operators R_α which describe the physical states. The R_α are positive definite and bounded (see below), and for convenience we normalize them by:

$$Tr\ R_\alpha = 1.$$

The set $\mathcal{O}$ of observables can be used to generate an algebra; it is this algebra which we shall use as a handy mathematical tool.

Like nearly all physicists, we choose an algebra over the field of complex numbers. In the beginning of his book, Dirac justifies this choice, but there are tentative reasons to choose either the field of real numbers, e.g., Stückelberg and his school, or the field of quaternions, e.g., D. Finkelstein, J. M. Jauch, S. Schiminovich, D. Speiser, *Jour. Maths. Phys.*, **3**, 207 (1962) for the latest reference.

Chosen the field of the complex numbers, there are different mathematical techniques to derive an algebra from $\mathcal{O}$; they produce non-isomorphic algebra. The existence of several such possibilities is irrelevant for our study of invariance. We shall choose the "von Neumann algebra". (We could not take an algebra we don't know! At least we can learn about the von Neumann algebra from the excellent book by Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien* (Paris, 1957), quoted below as Dixmier).

We shall not deal with the mathematical difficulty of considering $\mathcal{N}$, the von Neumann algebra generated by the set $\mathcal{O}$ as canonical; as we said, we consider the observables as given by hermitian operators on $\mathcal{H}$. But there remains a difficulty: some operators of $\mathcal{O}$ are unbounded; in other words, their norm is not finite.

Let us recall that the norm of a linear operator A is the $\mathrm{Sup}(\|Ax\|/\|x\|)$ for all $x \in \mathcal{H}$. By definition, a bounded operator has a finite norm. (Two examples of operators with norm 1 are the unitary operators $U^* = U^{-1}$ and the hermitian projectors $P^2 = P = P^*$.) The sum and the product of bounded operators are bounded operators.

Unbounded operators are not defined everywhere in $\mathcal{H}$, so the definition of their sum and their product is not obvious.

We could refer to a paper by J. M. Jauch (*Helv. Phys. Acta*, **33**, 711 (1960)) for the construction of $\mathcal{N}$ from $\mathcal{O}$. However, it is useful to give here some details, . . . and some definitions!

Definition. $\mathcal{A}$ is a *-algebra of bounded operators (over the field of complex numbers $\mathcal{C}$) if $D_1, D_2 \in \mathcal{A}$, $\alpha \in \mathcal{C} \Rightarrow \alpha D_1, D_1 + D_2, D_1D_2, D_1^* \in \mathcal{A}$. (The symbol D^* means the hermitian conjugate of D.)

Definition. Let $\mathcal{A}$ be a *-algebra of bounded operators on $\mathcal{H}$; let $\mathcal{A}$ contain the unit operator. Its closure for any of the four topologies: weak, strong, ultra-weak, ultra-strong, is "the" von Neumann algebra generated by $\mathcal{A}$ (Dixmier, p. 44).

Instead of A, an unbounded hermitian operator of the set $\mathcal{O}$, we shall consider the set of its spectral projectors P_λ. You all know that the proper values of $A = A^*$ are real, and that the proper vectors corresponding to an isolated proper value λ, form a subspace

$P_\lambda \mathscr{H}$ of $\mathscr{H}$ (P_λ is a hermitian projector: $P_\lambda = P_\lambda^* = P_\lambda^2$). Furthermore $\lambda \neq \mu \Rightarrow P_\lambda P_\mu = 0$. The set $s(A)$ of values of λ is called the spectrum of A. If this set is discrete,

$$A = \sum_\lambda \lambda P_\lambda$$

is called the spectral decomposition of A.

When $s(A)$ is not discrete, Σ must be replaced by a Stieljes integral:

$$A = \int \lambda dE_\lambda$$

where the E_λ are hermitian projectors such that

$$\lambda < \lambda' \Rightarrow E_\lambda \mathscr{H} \subset E_{\lambda'} \mathscr{H}.$$

The commutation of bound operators is well defined, as usual; we extend it to unbounded hermitian operators.

Definition. A bounded operator B commutes with an unbounded hermitian operators A, if it commutes with all projectors (P_λ or E_λ) of the spectral decomposition of A.

Again we have to use some preliminary definitions and lemmas.

Definition. Let $\mathscr{M}$ be a set of operators on $\mathscr{H}$; $\mathscr{M}'$ the *commutant* of $\mathscr{M}$ is the set of all *bounded* operators on $\mathscr{H}$, which commute with all operators of $\mathscr{M}$.

One can define $(\mathscr{M}')' = \mathscr{M}''$ and so on . . ., and can easily verify that $(\mathscr{M}'')' = (\mathscr{M}')''$, which we denote by $\mathscr{M}'''$.

Lemma 1. If $\mathscr{M}$ is a set of bounded operators, $\mathscr{M} \subset \mathscr{M}''$.

Lemma 2. If $\mathscr{M}$ and $\mathscr{N}$ are sets of bounded operators,

$$\mathscr{M} \subset \mathscr{N} \Rightarrow \mathscr{N}' \subset \mathscr{M}'$$

Let $\mathscr{M}$ be an arbitrary set of operators. From lemma 1, we have $\mathscr{M}' \subset \mathscr{M}'''$ and $\mathscr{M}'' \subset \mathscr{M}^{\text{iv}}$, but lemma 2 applied to $\mathscr{M}' \subset \mathscr{M}'''$ yields $\mathscr{M}^{\text{iv}} \subset \mathscr{M}''$. Hence $\mathscr{M}'' = \mathscr{M}^{\text{iv}} = \mathscr{M}^{\text{vi}}$ and also $\mathscr{M}''' = \mathscr{M}^{\text{v}} = \ldots$.

Definition (Dixmier, p. 2). $\mathscr{A}$ is a von Neumann algebra if it is a *-algebra and $\mathscr{A}'' = \mathscr{A}$.

Given any set $\mathscr{O}$ of hermitian operators, it is easy to check that $\mathscr{O}''$ is a *-algebra. Then it is a von Neumann algebra. We say that it is "the" von Neumann algebra generated by $\mathscr{O}$. (Note, however, that unbounded operators of $\mathcal{O}$ do not belong to $\mathcal{O}''$; only their spectral projectors do).

We will call $\mathcal{N}$ *the von Neumann algebra generated by the set* $\mathcal{O}$ *of the observables.*

The center $\mathcal{Z}$ of $\mathcal{N}$ is $\mathcal{N} \cap \mathcal{N}'$. If $\mathcal{N}'$ abelian, then $\mathcal{N}' \subset \mathcal{N}'' = \mathcal{N}$, hence $\mathcal{Z} = \mathcal{N}'$. Dixmier, p. 120, calls an algebra $\mathcal{N}$ whose commutant $\mathcal{N}'$ is abelian, a *discrete algebra.*

We can call $\mathcal{N}_+$ the set of hermitian positive operators of $\mathcal{N}$. It is a convex set which contains all density operators of state mixtures. The set "$\mathcal{N} \cap$ the envelope of $\mathcal{N}_+$" contains all density operators of pure states, i.e., one-dimensional projectors:

$$P = P^* = P^2, \quad TrP = 1$$

Now that we have constructed the handy mathematical tool $\mathcal{N}$ from the set of observables, we must make a physical assumption giving some properties to $\mathcal{N}$.

2. Axiom b

"There exists a complete set of commuting observables."

We refer to J. M. Jauch (*Helv. Phys. Acta*, **33,** 711 (1960)) for the translation of this physical requirement into the mathematical axiom:

Axiom b again

"There exists a subalgebra $\mathcal{A}$ of $\mathcal{N}$ such that $\mathcal{A}' = \mathcal{A}$."

Such an algebra is a maximal abelian subalgebra. By lemma 2, $\mathcal{N}' \subset \mathcal{A}' = \mathcal{A} \subset \mathcal{N}$, i.e., $\mathcal{N}' \subset \mathcal{N}$ and the algebra is discrete.

In a subsequent paper, J. M. Jauch and B. Misra (*Helv. Phys. Acta*, **34,** 699 (1961)) draw the following conclusions: for every $X \in \mathcal{O}$ there exists $\mathcal{A}(X)$ such that $\mathcal{P}(X) \in \mathcal{A}(X) = \mathcal{A}'(X) \in \mathcal{N}$ (where $\mathcal{P}(X)$ = set of spectral projectors of X). That is, in physical translation, every observable can be a member of a complete set of commuting observables.

Furthermore, the intersection of the $\mathcal{A}(X)$ for all X is $\mathcal{N}'$. Hence if we let $\hat{\mathcal{O}} \subset \mathcal{O}$ be the subset of observables which generates $\mathcal{N}'$, then $\hat{\mathcal{O}}' = \mathcal{N}$. The observables of $\hat{\mathcal{O}}$ commute with all observables; every observable of $\hat{\mathcal{O}}$ belongs to each complete set of commuting observables.

The important physical question is therefore: what are the observables of $\hat{\mathcal{O}}$? Jauch and Misra show that in electrodynamics, the electric charge Q belongs to $\hat{\mathcal{O}}$. The general answer we propose here will be chosen as axiom c.

Axiom c

"The set $\hat{\mathcal{O}}$ of observables which commute with all observables is generated by the charges."

We will take B, Q, L for the operators representing baryonic, electric and leptonic charges (or L_1, L_2, . . . if there are more charges). The spectral decomposition of $\mathcal{N}'$ yields a direct sum of Hilbert spaces:

$$\mathcal{H} = \sum_{\lambda} P_{\lambda}\mathcal{H} = \oplus_{\lambda}\mathcal{H}_{\lambda}$$

For mathematical convenience we take the sum to be discrete. This is the case if axiom c is true; then the label λ is given by the set of values b, q, l . . .of the charges. Let P_{λ} be the hermitian projector on $\mathcal{H}_{\lambda}$. The P_{λ} generate $\mathcal{N}'$. Let $|x\rangle \in \mathcal{H}$; the projector $|x\rangle\langle x|\, \|x\|^{-2}$ commutes with all P_{λ} if and only if $|x\rangle$ belongs to one $\mathcal{H}_{\lambda}$. If it does not commute, it cannot be an observable.

Hence, not all vectors of $\mathcal{H}$ represent physical states; only those of the $\mathcal{H}_{\lambda}$'s do. The $\mathcal{H}_{\lambda}$ are called "coherent" subspaces; two state vectors belonging to two different $\mathcal{H}_{\lambda}$ are said separated by a "superselection rule". This notion first appeared in G. C. Wick, A. S. Wightman, E. P. Wigner, *Phys. Rev.*, **88**, 101 (1952), where it was deduced from Lorentz invariance.

3. Relativistic Invariance

It is now time to introduce the relativity group of the theory (inhomogeneous Lorentz group or Poincaré group, Galilean group . . .). By definition, the relativity group transforms the set of observables $\mathcal{O}$ into itself, and no such transformation leaves all the observables invariant. The mathematical translation is straightforward.

Axiom d

"The relativity group of the theory is isomorphic to a subgroup of the group of automorphisms of $\mathcal{O}$."

One could deduce general properties from this axiom. To be more specific, let us take as relativity group $\mathcal{P}_0$, the connected Poincaré (= inhomogeneous Lorentz) group. Let us also assume axiom c. Since charges are invariant under $\mathcal{P}_0$, *every* $A \in \mathcal{N}'$, the center of $\mathcal{N}$, is invariant under the automorphisms which correspond to $\mathcal{P}_0$. Then (Dixmier, pp. 255-256), one can prove that these automorphisms are inner automorphisms of $\mathcal{N}$. That is, they can be represented by operators of $\mathcal{N}$ such that:

$$\forall A \in \mathcal{N},\ A \xrightarrow{x} UAU^{-1},\ U \in \mathcal{N},\ x \in \mathcal{P}_0 \tag{2}$$

Furthermore, from $x(A^*) = [x(A)]^*$, one shows that U is unitary.

Physically, this proves that the connected Poincaré transformations (hence the energy, momentum and angular momentum) are observables. Had we admitted it as an axiom c′, instead of axiom c, we would have found that all of $\hat{\mathcal{O}}$ (the observables generating $\mathcal{N}'$, that is those which commute with all of $\mathcal{O}$) are invariant under Poincaré transformations. We leave to the reader to reformulate our final conclusions in this case.

Given an $x \in \mathcal{P}_0$, what characterizes the set of U which represent the automorphism $A \xrightarrow{x} x(A)$? Let U_1 and U_2 be two such representatives, then $U_1U_2^{-1}$ and $U_2U_1^{-1} \in \mathcal{N}'$; that is $U_2 = U_1\Omega$ where Ω is an arbitrary unitary operator $\in \mathcal{N}'$, the center of $\mathcal{N}$.

Hence the set of Ω form an abelian group $\mathcal{A}$. It is easy to see that the set of all U for all $x \in \mathcal{P}$, form a group $\mathcal{U}$, homomorphic to the Poincaré group $\mathcal{P}_0$.

$$\mathcal{U} \xrightarrow{f} \mathcal{P}_0$$

The kernel of f is $\mathcal{A}$, i.e., $f(\mathcal{A}) = 1 \in \mathcal{P}_0$. In other words, we have the relation:

$$\mathcal{P}_0 = \mathcal{U}/\mathcal{A} \tag{3}$$

Since $\mathcal{A} \subset$ center of $\mathcal{N}$ and $\mathcal{U} \subset \mathcal{N}$,

$$\mathcal{A} \subset \text{center of } \mathcal{U}. \tag{4}$$

We shall summarize these two statements (3) and (4) by

$$\text{"}\mathcal{U} \text{ is a central extension of } \mathcal{P}_0 \text{ by } \mathcal{A}\text{"}. \tag{5}$$

We shall call $\mathscr{P}_0$ the relativity group of the theory and $\mathscr{U}$ the invariance group of the formalism. $\mathscr{A}$ is a kind of gauge group.

Now the problem is: given an arbitrary abelian group $\mathscr{A}$, find all the central extensions of $\mathscr{P}_0$ by $\mathscr{A}$. This problem is solved in Chapter V. I will now only give the results.

Let us call $\text{Ext}(\mathscr{P}_0, \mathscr{A})$ the set of *all* groups which are central extensions of $\mathscr{P}_0$ by $\mathscr{A}$. There is a one-to-one mapping of this set onto the set of elements of order 2 in $\mathscr{A}$ (i.e., the square roots of 1, if $\mathscr{A}$ is noted multiplicatively). These elements form a group called ${}_2\mathscr{A}$, and as we shall see, we can put a group law on $\text{Ext}(\mathscr{P}_0, \mathscr{A})$, which is isomorphic to ${}_2\mathscr{A}$. Indeed, the rest of the lectures will mainly be devoted to the extension problem. We shall also give a precise definition of equivalence for extensions; accurately $\text{Ext}(\mathscr{P}_0, \mathscr{A})$ is the set of equivalence classes.

In fact, the different possible groups are easily described once we are given $\omega \in {}_2\mathscr{A}$. The corresponding $\mathscr{U}$ is

$$\mathscr{U}_\omega = (\mathscr{A} \otimes \bar{\mathscr{P}}_0)/Z_2 \tag{6}$$

when $\bar{\mathscr{P}}_0$ is the universal covering group of $\mathscr{P}_0$. Its center has two elements, the unit (0 for translation, 1 for SL_2) and what we physicists call the "rotation of 2π", i.e., $\epsilon = (0, -1)$. The two elements of the group Z_2 in formula (6) are $(1 \in \mathscr{A}, 1 \in \bar{\mathscr{P}}_0)$ and (ω, ϵ).

The irreducible unitary representations of $\mathscr{A} \otimes \bar{\mathscr{P}}_0$ are the product of those of $\mathscr{A}$ and those of $\bar{\mathscr{P}}_0$. We shall use those of $\bar{\mathscr{P}}_0$, which correspond to real mass. These have all been given by E. Wigner: "F".

For these representations, the rotation of 2π is represented by $(-1)^{2j}$ where j is the spin of the particle. So in order that a representation of $\mathscr{A} \otimes \bar{\mathscr{P}}_0$ be a representation of $\mathscr{U}$, the element ω must be represented by $(-1)^{2j}$. Since ω is a function of the charges (by axiom c), this yields one, and only one, relation between charge and spin.

We can ask whether such a relation exists in nature? The answer is yes.

$$(-1)^{2j} = (-1)^{b+l} \tag{7}$$

$[(-1)^{2j} = (-1)^{b-l}$ is the same relation$]$.

By taking

$$\omega = (-1)^{B+L} \tag{8}$$

we fix the group $\mathscr{U}$ uniquely.

The group $\mathscr{U}$ must be included as an invariance group in every theory of elementary particles. Note that $\mathscr{P}_0$ is not a subgroup of $\mathscr{U}$, but only a quotient.

Note also that the irreducible unitary representations of $\mathscr{U}$ are irreducible unitary representations up to a phase of $\mathscr{P}_0$, and that they are representations of $\bar{\mathscr{P}}_0$. This is in agreement with Wigner "F".

We can say that axioms a, b, c, d predict the existence of one and only one relation between charge and spin. While the choice of possible relations is very large ($_2\mathscr{A}$) it is possible to inject more physics into the mathematical framework outlined above by adding a new axiom related to the existence of asymptotic states and the possibility of forming their tensor products. Alternatively it might be related to the property of the unitary representation up to a phase $\mathscr{D}$ of $\mathscr{P}_0$ to contain all irreducible representations of $\mathscr{D} \otimes_s \mathscr{D}$ (where the tensor product is symmetrized or antisymmetrized accordingly to statistics, that is to the sign of $(-1)^{2j}$). Then it is possible to reduce the possible set of extensions to the eight squares root of the group:

$$\exp[i(\alpha_q Q + \alpha_b B + \alpha_l L)].$$

One of them does yield relation (7). (See F. Lurçat and Michel, as quoted in the beginning of the chapter.)

4. Non-relativistic Invariance

The situation in non-relativistic mechanics is more complicated. The relativity group is the Galilean group. Its irreducible unitary representations have been determined by E. Inönü and E. P. Wigner (*Nuovo Cimento*, **9,** 705 (1952)). Under more restrictive hypotheses; $\mathscr{A}$ is the one parameter group $\mathscr{S}_1$ and the extension is assumed to be a Lie group (by "continuity of the phases"). V. Bargmann has shown (*Ann. Math.*, **59,** 1 (1954)) that the Galilean group has already many more extensions. Indeed in non-relativistic physics, the mass is a superselection rule. So, instead of a discrete sum $\mathscr{H} = \oplus_\lambda \mathscr{H}_\lambda$, we have to consider $\mathscr{H}$ as a direct integral of Hilbert space and, in the strict sense, no vector of $\mathscr{H}$ can represent a state vector!

4. The Extensions of a Group G by an Abelian Group K

The appendix I and this lecture (Chapter IV) have been prepared from more detailed notes written with F. Lurçat. Chapter IV does not contain any original results, but is an exposition of a classical mathematical theory. See, for instance, A. G. Kurosh, *The Theory of Groups*, 2nd edition, Chapter XII (Moscow, 1952) (English translation New York, 1955). The aim is to help physicists who want to read the current mathematical literature on this and related subjects by introducing them to the symbols, vocabulary, and concepts used in such papers. While these concepts have not yet been integrated into the physics literature, we are sure that they will be used more and more in physics.

1. The Language of Exact Sequences and Commutative Diagrams

A sequence of two homomorphisms

$$G \xrightarrow{f} G' \xrightarrow{f'} g'' \tag{1}$$

defines an homomorphism G onto G' which is denoted $f'{}_0 f$. (The composition law $_0$ is associative.)

Definition. A sequence of homomorphisms

$$\ldots \longrightarrow G_n \xrightarrow{fn} G_{n+1} \xrightarrow{fn+1} G_{n+2} \xrightarrow{fn+2} G_{n+3} \xrightarrow{fn+3} \ldots \tag{2}$$

such that for all n

$$\mathrm{Im} f_n = \mathrm{Ker}\, f_{n+1} \tag{3}$$

is an *exact* sequence.

From now on, all sequences of homomorphisms we shall write with the arrows on a same straight line, will be exact, except if otherwise stated.

Example.

$1 \to G \xrightarrow{f} G'$ means $\mathrm{Ker} f = 1$ (f is an injective homomorphism that is G is isomorphic to $\mathrm{Im} f$, a subgroup of G').

$G \to G' \to 1$ means $\mathrm{Im} f = G'$ (f is a surjective homomorphism, or homomorphism onto).

$1 \to G \xrightarrow{f} G' \to 1$ means f is an isomorphism (denoted $\approx$).
This also implies that $1 \leftarrow G \leftarrow G' \leftarrow 1$.

We have recalled to the reader (in Appendix I) that in the homomorphism $G \xrightarrow{f} G'$, $\operatorname{Ker} f$ is invariant subgroup of G and $\operatorname{Im} f$ is the quotient $G/\operatorname{Ker} f$. In exact sequence language:

$$1 \to \operatorname{Ker} f \to G \xrightarrow{f} \operatorname{Im} f \to 1 \tag{4}$$

and more generally

$$1 \to A \to B \to C \to 1 \text{ means the quotient } C = B/A \tag{5}$$

If $G = G'$, homomorphisms and isomorphisms are called respectively endomorphisms and automorphisms.
The automorphisms of G form a group Aut G.
We remind the reader of the existence of the following exact sequences, for any group G (or we ask him to accept them as definitions of the group of inner automorphisms $\mathscr{I}(G)$ and the group of automorphism classes $\mathscr{A}(G)$!).

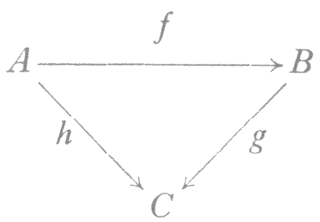
$$1 \to \mathscr{C}(G) \to G \to \mathscr{I}(G) \to 1 \tag{6}$$

$$1 \to \mathscr{I}(G) \to \operatorname{Aut} G \to \mathscr{A}(G) \to 1 \tag{6'}$$

Here $\mathscr{C}(G)$ means the center of G. We have already met (6) in Eq. III(3). Of course, if G is abelian, $\mathscr{I}(G) = 1$ and (6) and (6′) reduce to $\mathscr{C}(G) \approx G$ and Aut $G \approx A(G)$.

Commutative Diagrams (for Groups)

A commutative diagram is a set of groups and homomorphisms between them, such that all possible compositions of mappings which define a homomorphism between two groups of the diagrams, define the same homomorphism.
Examples. The simple example of commutative diagram is $G \rightleftarrows G'$. It is equivalent to $G \approx G'$ (isomorphism).
The second simplest example is:

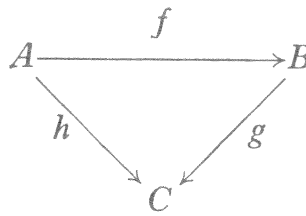


is commutative $\Leftrightarrow h = g_0 f$.

From now on, all diagrams we shall write will be commutative. Furthermore, a sequence of homomorphisms is exact if only and only if the corresponding arrows in the diagram lie on a straight line. Note that this is not a universal convention in literature.

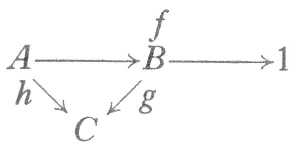
The reader can play with diagrams by proving the following lemmata. (Proofs of 1 to 4 are given in an appendix of this chapter.)

Lemma 1.



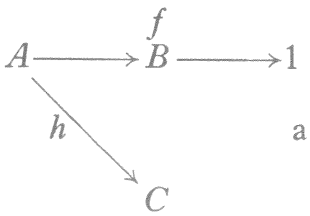
$\Rightarrow 1 \to \operatorname{Ker} f \to \operatorname{Ker} h \to \operatorname{Ker} g$

Lemma 2.

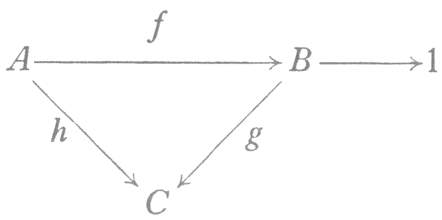


$\Rightarrow 1 \to \operatorname{Ker} f \to \operatorname{Ker} h \to \operatorname{Ker} g \to 1$

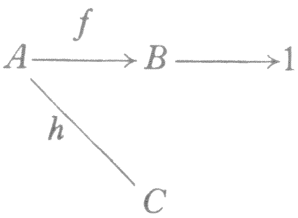
Lemma 3.



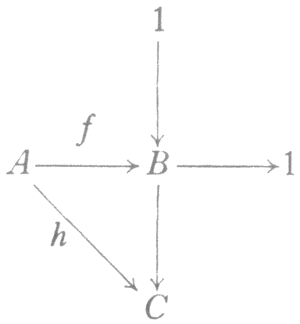
and $\operatorname{Ker} f \subset \operatorname{Ker} h \Rightarrow$



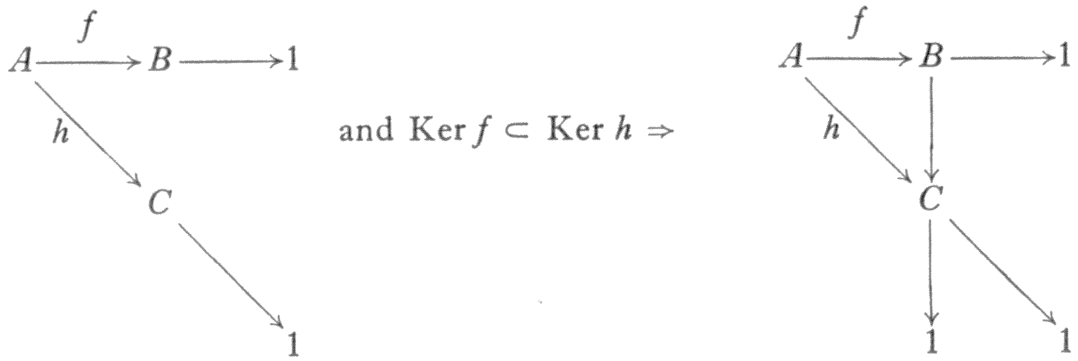
Lemma 3′.



and $\operatorname{Ker} f = \operatorname{Ker} h \Rightarrow$

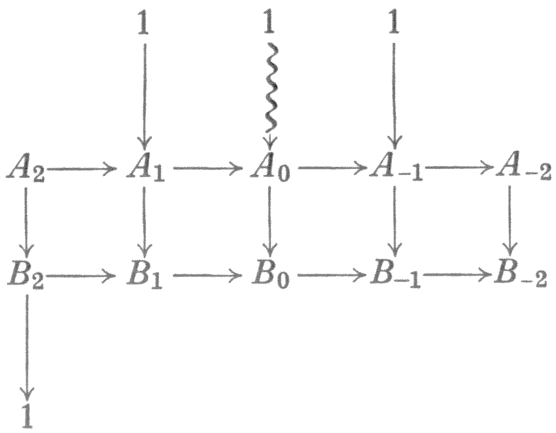


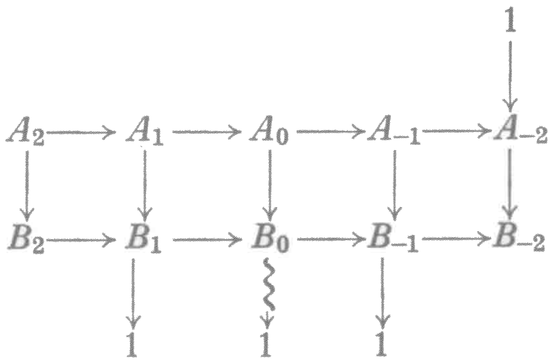
Lemma 4.



We shall make the convention that homomorphisms in a commutative diagram which are a consequence of the rest of the diagram, will be indicated by wiggly arrows: ⇝. Examples:

Lemmata 5 *and* 5′.





In lemma 5, the fact that the kernel of the homomorphism $A_0 \to B_0$ is 1 is a consequence of the diagram. Lemmata 5 and 5′ are known as the "5 lemmata". (For a proof, see for instance H. Cartan and S. Eilenberg, *Homological Algebra* (Princeton, 1956, p. 7).)

Note that 5′ can be obtained from 5 merely by reversing every arrow and vice versa. (This duality of reversing maps is general and fruitful.) Of course, commutative diagrams are nothing more than a handy way to condense mathematical information.

2. The Problem of Group Extensions

Now that we possess this picturesque short-hand, let us go back to the problem of extension. "E is an extension of G by K", can be written:

$$1 \to K \xrightarrow{i} E \xrightarrow{s} G \to 1 \tag{7}$$

To say that $i(K)$ is an invariant subgroup of E means that the inner automorphisms of E induce automorphisms on K. (That is, there exists a homomorphism $E \xrightarrow{f} \text{Aut}\, K$.)

To $x_1, x_2 \in E$ belonging to the same coset of K (i.e., $x_1 = x_2\alpha'$ where $\alpha' \in i(K)$) there correspond two automorphisms of K which differ by an inner automorphism. (Here $i(K) \ni \xi \to \alpha'\xi'\alpha'^{-1}$.) This correspondence between cosets of K (or elements of G) and elements of $\mathscr{A}(K)$ is a homomorphism. (Apply lemma 3, since Ker $s \subset$ Ker $s'{}_0f$, it yields g.) To summarize by a diagram, if (7) is given, then (8)

$$\begin{array}{ccccccccc}
 & & 1 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \mathscr{C}(K) & & & & & & \\
 & & \downarrow & & & & & & \\
1 & \longrightarrow & K & \xrightarrow{i} & E & \xrightarrow{s} & G & \longrightarrow & 1 \\
 & & \downarrow & & \wr\downarrow f & & \wr\downarrow g & & \\
1 & \longrightarrow & \mathscr{I}(K) & \xrightarrow{i'} & \text{Aut}\, K & \xrightarrow{s'} & \mathscr{A}(K) & \longrightarrow & 1 \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array} \tag{8}$$

In the last chapter (Chapter VIII) we shall indicate how to solve the following problem: given two groups G and K and a homomorphism $G \xrightarrow{g} \mathscr{A}(K)$, find all extensions E of G by K such that (8) is satisfied. This problem is summarized by diagram (9):

$$
\begin{array}{ccccccccc}
 & & 1 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \mathscr{C}(K) & & & & & & \\
 & & \downarrow & & & & & & \\
1 & \longrightarrow & K & \longrightarrow & E & \rightsquigarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \wr\downarrow & & \downarrow g & & \\
1 & \rightarrow & \mathscr{I}(K) & \longrightarrow & \mathrm{Aut}\, K & \rightarrow & \mathscr{A}(K) & \longrightarrow & 1 \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}
\tag{9}
$$

As we shall show, this problem may have no solutions for a given g. In this chapter, we shall study the easier case of abelian K. This problem is summarized by:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \overset{i}{\rightsquigarrow} & E & \overset{s}{\rightsquigarrow} & G & \longrightarrow & 1 \\
 & & & & & & \downarrow g & & \\
 & & & & & & \mathrm{Aut}\, K & &
\end{array}
\tag{10}
$$

That is, given a group G, an abelian group K, and a homomorphism $G \xrightarrow{g} \mathrm{Aut}\, K$, find an extension E of G by K, such that the inner automorphism of E corresponding to $x \in E$ induces on K the automorphism $g_{\circ}s(x)$.

This problem always has at least one solution (the trivial extension) which is called the semi-direct product and which we will construct. The direct product is the particular case of a semi-direct corresponding to $g = 0$, the trivial homomorphism. In this chapter we use Roman letters for elements of G and we denote G multiplicatively; we use Greek letters of elements of K and denote K additively. The elements of E are elements of the product of sets K and G, that is there are the pairs (α, a) and the group law of the trivial extensions is

$$(\alpha, a)(\beta, b) = (\alpha + a\beta, ab) \tag{11}$$

where $a\beta$ is the transform of β by the K-automorphism $g(a)$.

We leave to the reader to check that this law is associative, that (0, 1) is the unit element, that

$$(\alpha, a)^{-1} = (-a^{-1}\alpha, a^{-1}) \tag{12}$$

that $i(K)$ is an invariant subgroup, and that the inner automorphism of E corresponding to (α, a) induces on K the automorphism $g(a)$. We shall use the notation:

$$i(\alpha) = (\alpha, 1) = \alpha' \tag{13}$$

with

$$i(\alpha + \beta) = \alpha'\beta'. \tag{14}$$

Note that the set of elements of the form $(0, a)$ is a subgroup of E isomorphic to G. So the semi-direct product E satisfies:

$$\left.\begin{array}{l}\text{Not commutative} \\ \text{Two exact sequences:} \quad 1 \to K \to E \underset{k}{\overset{s}{\rightleftarrows}} G \rightleftarrows 1 \\ \text{with } s_0\, k = \mathrm{I}\ (\mathrm{I} = \text{the identity automorphism of } G).\end{array}\right\} \tag{15}$$

When the exact sequence (7) has the property (15), mathematicians say that it "splits". We will write $E = K \times G$.

If G is the trivial homomorphism ($\operatorname{Im} g = 1$), then the group law of the trivial extension is:

$$(\alpha, a)(\beta, b) = (\alpha + \beta, ab). \tag{16}$$

Then E is the direct product $E = K \otimes G$. It satisfies:

$$\left.\begin{array}{l}\text{not commutative: } 1 \rightleftharpoons K \underset{s}{\overset{i}{\rightleftharpoons}} E \underset{i'}{\overset{s'}{\rightleftharpoons}} G \rightleftharpoons 1 \\ \text{and } s_0 i = I \qquad s'_0 i' = I'\end{array}\right\} \tag{17}$$

We leave to the reader to prove that (15) and (17) can be taken as definitions of the semi-direct and the direct product.

3. Structure of an Extension, System of Factors

Let us consider an extension E which satisfies (10). Any $x \in E$ belongs to a coset of $i(K)$ which is labelled by an element $a \in G$. To specify x, we must also give its position inside the coset. For this we choose an element in each coset which we shall call $k(a)$. Then there is $\alpha' \in i(K)$ such that $x = \alpha' k(a)$. Note that $k(a)$ is not in general a homomorphism; it is only a mapping of G on E such that:

$$s_0 k = I = \text{identity on } G \tag{18}$$

(i.e., $\forall a \in G,\ s_0 k(a) = a$).

Such a mapping, which is not a homomorphism, will be denoted by a dotted arrow

$$1 \longrightarrow K \xrightarrow{\;i\;} E \underset{k}{\overset{s}{\rightleftarrows}} G \longrightarrow 1 \qquad s_0 k = I,\qquad G \xrightarrow{\;g\;} \text{Aut}\, G \tag{19}$$

We can also use the language of fiber bundles for E. The group G is the base and the cosets are the fibers. The set of $k(a)$ for $a \in G$ is a section.

For a given mapping k satisfying (18), $k(a)k(b)$ and $k(ab)$ are in the same coset (the same fiber). Indeed, since s is a homomorphism

$$s_0 k(ab) = ab \text{ and } s[k(a)k(b)] = [s_0 k(a)][s_0 k(b)] = ab.$$

This defines a function $\omega'(a, b)$ on the set $G \times G$, with values in K such that

$$k(a)k(b) = \omega'(a, b)k(ab). \tag{20}$$

It is called a factor system.

The group law in E is:

$$\alpha' k(a)\beta' k(b) = \alpha' \,.\, a\beta' \,.\, \omega'(a, b)k(ab). \tag{21}$$

For convenience only, we shall furthermore require for k that:

$$k(1) = 1 \in E. \tag{22}$$

So, from (21), putting $\alpha = \beta = 0$ and either a, b or both $= 1$,

$$\omega(a, 1) = \omega(1, b) = \omega(1, 1) = 0. \tag{23}$$

A factor system satisfying (23) is said to be "normalized". The law (21) is associative if and only if:

$$\omega(a, b) + \omega(a, b, c) - a\omega(b, c) - \omega(a, bc) = 0 \tag{24}$$

(To find this condition, compute in the two different ways the product $k(a)k(b)k(c)$.)

Let us choose now another mapping $\hat{k}(a)$ satisfying (18) and (22). It can be written:

$$\hat{k}(a) = \phi'(a)k(a) \tag{25}$$

where $\phi(a) \in K$.

Then we obtain a new factor system $\hat{\omega}(a, b)$ such that

$$\hat{\omega}(a, b) = \omega(a, b) + \theta(a, b) \tag{26}$$

with

$$\theta(a, b) = \phi(a) + a\phi(b) - \phi(ab). \tag{27}$$

Note that $\theta(a, b)$ satisfies (24) which characterizes the factor systems. Any factor system which can be written in the form (27) is said to be *trivial*.

Hence the extension E defines a factor system up to a trivial factor system. Reciprocally, given a normalized factor system $\omega(a, b)$ defined on the set $G \times G$, with values in K, and given the homomorphism $g\colon G \overset{g}{\to} \mathrm{Aut}\, K$, we can form a group E whose elements are pairs (α, a) with the composition law is

$$(\alpha, a)(\beta, b) = [\alpha + a\beta + \omega(a, b), ab]. \tag{28}$$

The relation (24) makes this law associative; the unit is $(0, 1)$, the inverse is given by:

$$(\alpha, a)^{-1} = [-a^{-1}\alpha - a^{-1}\omega(a, a^{-1}), a^{-1}] = [-a^{-1}\alpha - \omega(a^{-1}, a), a^{-1}] \tag{29}$$

since

$$\omega(a, a^{-1}) = a\omega(a^{-1}, a). \tag{30}$$

The inner automorphism of E corresponding to (α, a) induces on K the automorphism $g(a)$.

Two factors systems which differ by a trivial factor system yields two isomorphic extensions. These extensions will be said to be equivalent. As a particular case, if the given factor system is trivial, the extension is equivalent to the trivial one ($E = K \times G$, or when $\mathrm{Im}\, g = 1$, $K \otimes G$). Since by a change of the mapping k, a trivial factor system can be reduced to $\theta(a, b) = 0$, then the group law (28) becomes identical to (11).