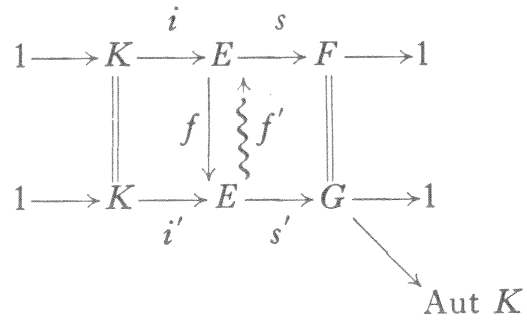
We will not prove here this other possibility of the definition of equivalent extensions, although this definition is valid not only for groups, but for many other mathematical structures:

Definition. E and E' are equivalent extensions if the commutative diagram exists (where = means identity transformation).



(31)

In agreement with our convention, the existence of f' and therefore the property of f to be an isomorphism, is a consequence of the rest of the diagram.

So the old problem: "Given $g \xrightarrow{g} \mathrm{Aut}\, K$, find all inequivalent extensions E of G by K, satisfying (10)" has become: "Find all normalized factor systems, up to a trivial factor system."

This problem is a classical one for mathematicians, but to help the physicist benefit from the mathematical literature, we first have to explain to him that, by following this lecture, he was just doing "cohomology". (As the master of philosophy explained to Molière's "Bourgeois Gentilhomme", when he spoke he was making prose.)

4. Cohomology

Let us consider the following sequence of abelian groups (denoted additively). We call it a "complex" and it is:

not exact sequence

$$0 \xrightarrow{\delta_{-1}} C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} C^2 \xrightarrow{\delta_2} C^3 \longrightarrow \dots \xrightarrow{\delta_{n.1}} C^n \xrightarrow{\delta_n} C \xrightarrow{n+1} \dots \rightarrow \qquad (32)$$

where the homomorphisms δ are such that

$$\mathrm{Im}(\delta_{n+1}{}^0\delta_n) = 0. \qquad (33)$$

That is

$$\text{Im}\,\delta_n \subset \text{Ker}\,\delta_{n+1}. \tag{34}$$

Let $Z^n = \text{Ker}\,\delta_n$ and $B^n = \text{Im}\delta_{n-1}$.

The groups

$$H^n = Z^n/B^n \tag{35}$$

are called the cohomology groups of the complex.

If we had for all n, $H^n = 0$, the sequence (32) would have been exact. Thus the cohomology group measures the "lack of exactness" of the sequence.

We can also define $C = \bigoplus^{\infty}{}_{n=0}C^n$ and respectively we can form Z, B and H with $\delta = \bigoplus_n \delta_n$ one sees that $\delta^2 = 0$, hence $B = \text{Im Sc Ker}\,\delta = Z$. Then $H = Z/B$.

The elements of C^n are called n-cochains, those of Z^n, n-cocycles, and those of B^n, n-coboundaries. We define the abelian groups:

$$C^n(G, K)$$

to be the n-variable functions α_n defined on G with values in K, with the further condition (so called "normalization") that $\alpha_n = 0$ if at least one variable is 1. Since the sum of these functions $\alpha_n(x_1 \ldots x_n)$ is defined by the group law in K, they form a group. Furthermore, since G acts on K, we can make it act on $C^n(G, K)$. Now we define the homomorphism δ as follows:

$$\delta_n[\alpha_n(x_1 \ldots x_n)] = (\delta_n\alpha_n)(x_1 \ldots x_{n+1}) = x_1\alpha_n(x_2, \ldots x_{n+1})$$
$$+ \sum_{k=1}^{n} (-1)^k\alpha_n(x_1, \ldots, x_kx_{k+1}, \ldots x_{n+1}) + (-1)^{n+1}\alpha_n(x_1 \ldots x_n). \tag{36}$$

Let us consider the cases $n = 0$, 1 and 2.

The $C^0(G, K)$ are constant functions of G into K. That is they are elements of K. Thus we set

$$C^0(G, K) = K. \tag{37}$$

Since $x \in G$ induces an automorphism on K, $x\,C^0$ can be defined. Now:

$$(\delta_0\alpha_0)(x) = x\alpha_0 - \alpha_0.$$

Hence

$$\operatorname{Ker} \delta_0 = K^G$$

the fixed elements of K under G.

By definition we put $B^0 = 0$. (Im $\delta_{-1} = 0$.) Therefore

$$H^0(G, K) = K^G. \tag{38}$$

Let us write $\delta_n \alpha_n$ for the case $n = 1$ and 2.

$$\delta_1[\alpha_1(x)] = (\delta_1\alpha_1)(x, y) = x\alpha(y) - \alpha(x, y) + \alpha(x) \tag{39}$$

$$\begin{aligned} \delta_2[\alpha_2(x, y)] &= (\delta_2\alpha_2)(x, y, z) \\ &= x\alpha(y, z) - \alpha(xy, z) + \alpha(x, yz) - \alpha(x, y). \end{aligned} \tag{40}$$

We see that relations (24), characteristic of a factor system, can be written:

$$(\delta\omega)(a, b, c) = 0. \tag{41}$$

In other words, a factor system is a 2-cocycle.

Relation (27), characteristic of a trivial factor system can be written:

$$\theta(a, b) = (\delta\phi)(a, b) \tag{42}$$

that is a trivial factor system is a 2-coboundary.

Since the set of inequivalent group extensions of G by K is the set of factor systems defined up to trivial factor systems, we have for a given $G \xrightarrow{g} \operatorname{Aut} K$:

$$\operatorname{Ext}(G, K) = H^2(G, K). \tag{43}$$

Cohomology theory was born in mathematics for the study of algebraic topological properties of topological spaces. But the same mechanism could be used for different mathematical objects. For groups, the first papers on cohomology theory are by S. Eilenberg and S. Maclane, 1947, "Cohomology Theory in Abstract Groups," I and II, *Ann. Math.*, **48**, 57, 326. I strongly advise you to look at them, and also at the review of S. Eilenberg, 1949, "Topological Methods in Abstract Algebra, Cohomology Theory of Groups", *Bull. Ann. Math. Soc.*, **55**, 3. Cohomology theory and, more generally, the related methods in algebraic topology (hology, homotopy,...)

can be applied to so many mathematical objects, that they have become a full-fledged theory of their own. This theory was expounded for the first time in book form by H. Cartan and S. Eilenberg, *Homological Algebra* (Princeton, 1956). See also Chap. 1, "Algèbre homologique" of Godement's *Théorie des Faisceaux* (Paris, 1958).

There are methods (reduction theorems) for computing $H^n(G, K)$ in terms of $H^{n'}(G, K')$ with $n' < n$, but with K' a much larger group. For abelian finite groups, there is a general constructive method for the computation of the H^n, "the method of the free acyclic complex". In this approach the cohomology groups $H^n(G, K)$ are finally given by those of a complex such as (32). Here we shall only give some results.

Let us consider the case that G is a finite cyclic group of order p; that is, there is $x \in G$ such that $x^p = 1$ and the elements of G are $1, x, x^2, \ldots, x^{p-1}$. ($x$ is a generator of G.) Let $\alpha \in K$. We define the "norm" of $N\alpha$ by:

$$N\alpha = \sum_{k=0}^{p-1} x^k\alpha \tag{44}$$

N is an endomorphism of K, that is $N \in \mathrm{Hom}(K, K)$.

Let $D\alpha = (\alpha - x\alpha)$. It is $\in \mathrm{Hom}(K, K)$ with $\mathrm{Ker}\, D = K^G$. One sees easily $DN\alpha = 0 = ND\alpha$, that is $D_0N = 0$ and $N_0D = 0$ where 0 is the trivial homomorphism $\in \mathrm{Hom}(K, K)$. Hence $\mathrm{Im}\, N \subset \mathrm{Ker}\, D$ and $\mathrm{Im}\, D \subset \mathrm{Ker}\, N$.

One proves the following theorem (the result is independent of the choice of x). See for instance S. Eilenberg and MacLane in I, s. 16, or H. Cartan and S. Eilenberg, Chapter XII, p. 250.

$$H^n(G, K) = \begin{cases} \mathrm{Ker}\, D/\mathrm{Im}\, N, & n \text{ even} \geq 2 \\ \mathrm{Ker}\, N/\mathrm{Im}\, D, & n \text{ odd} \end{cases} \tag{45}$$

In particular, if $K^G = K$ (G acts trivially on K), then $\mathrm{Im}\, D = 0$ and $N\alpha = p\alpha$ and the traditional notation for $\mathrm{Im}\, N$ is pK, in other words it is the subgroup of elements of K divisible by p. On the other hand, $\mathrm{Ker}\, N$ is the subgroup ${}_pK$ of the elements $\alpha \in K$ such that $p\alpha = 0$. Then

$$H^n(G, K) = \begin{cases} K/pK = K_p & \text{for even } n \geqslant 2 \\ {}_pK & \text{for odd } n \end{cases} \tag{46}$$

We can find this very easily for $p = 2$; then G has two elements x and $x^2 = 1$. A 2-cochain of G is given by the value $\omega(x, x)$, since $\omega(1, 1) = \omega(1, x) = \omega(x, 1) = 0$.

It is always a 2-cocycle ($\delta\omega = 0$) and it is a 2-coboundary only if there exists $\phi(x)$ such that

$$\omega(x, x) = 2\phi(x), \ (\phi(x^2) = 0).$$

That is $\omega \in 2K$. Then:

$$H^2(G, K) = K/2K = K_2 \tag{47}$$

Of course, if every element of K is divisible by 2, $K_2 = 0$.

Another possibility for the two element group G to act on K is that $\forall \alpha \in K$, $x\alpha = -\alpha$. Then $N\alpha = 0$ and $D\alpha = 2\alpha$, which correspond respectively to D and N of the previous case.

We just give here some results that we shall use later for the discrete operation of the Poincaré group.

(a) $$G = Z_2 = [1, x].$$

We consider two possibilities for $G \to \operatorname{Aut} K$.

(1) $$K^G = K = H^0(G, K), H^{2p+1}(G, K) = {}_2K \qquad H^{2p}(G, K) = K_2 \tag{48}$$

(2) $$x\alpha = -\alpha, H^0(G, K) = K^G = {}_2K, H^{2p+1}(G, K) = K_2 \qquad H^{2p}(G, K) = {}_2K \tag{48'}$$

(b) $$G = Z_2 \times Z_2 = [1, X, Y, Z], 1 = X^2 = Y^2, XY = YX = Z \tag{49}$$

(1) $$K^G = K = K = H^0(G, K), H^1(G, K) = {}_2K \oplus {}_2K, H^2(G, K) = K_2 \oplus {}_2K \oplus K_2 \tag{49'}$$

(2) $$X\alpha = \alpha, Y\alpha = -\alpha, \text{ then } Z\alpha = -\alpha; \ K^G = {}_2K = H^0(G, K)$$
$$H^1(G, K) = {}_2K \oplus K_2 \quad H^2(G, K) = {}_2K \oplus {}_2K \oplus K_2 \tag{49''}$$
$$H^3(G, K) = {}_2K \oplus {}_2K \oplus K_2 \oplus K_2.$$

Quaternion Group $\underset{\sim}{Q}$

It is the 8-element group generated by Pauli matrices. It often occurs in group theoretical problems. For example, there exist nonabelian groups in which every nonabelian subgroup is an invariant subgroup. All these groups contain the quaternion group as an invariant subgroup. The cohomology of Q is also cyclic.

Examples: If Q does not act on K,

$$H^0(Q, K) = K = K^Q,\ H^{4p+1} = {}_2K + {}_2K,\ H^{4p+2} = K_4$$
$$H^{4p+3} = {}_4K \text{ and } H^{4p} = K_4.$$

Note that when

$$K^G = K.$$

Then

$$H^1(G, K) = \text{Hom}\,(G, K). \tag{50}$$

If G acts on K and K', it acts in a natural way on $K \oplus K'$. By repeated use of theorem 3 of Chapter II, it is easy to prove that:

Theorem 1

$$H^2(G, K \oplus K') = H^2(G, K) \oplus H^2(G, K').$$

Theorem 2. If G is a finite group with k elements, for all

$$\alpha \in H^n(G, K),\ k\alpha = 0$$

Proof: Let ω_n be an arbitrary n-cocycle. We define:

$$\psi_{n-1}(x_1, x_2 \ldots, x_{n-1}) = \sum_{x_n \in G} \omega_n(x_1, \ldots, x_n).$$

Now from (36), we find that:

$$0 = \sum_{x_{n+1} \in G} (\delta\omega_n)(x_1, \ldots, x_{n+1})$$
$$= (\delta\psi_{n-1})(x_1, \ldots, x_n) + (-1)^{n+1}k\omega(x_1 \ldots x_n)$$

i.e.,

$$\delta\omega_n = 0 \Rightarrow k\omega_n \Rightarrow (-1)^n\delta\psi_{n-1}, \text{ so } kZ^n \subset B^n \text{ hence } kH^n = 0.$$

We want to close this chapter by a study of the group law of $H^2(G, K)$. If $\omega_1(a, b)$ and $\omega_2(a, b)$ are factors systems for two extensions E_1 and E_2, $\omega_1(a, b) + \omega_2(a, b)$ is also a factor system.

Let E be the extension corresponding to it, we will note $\vee$ the relation

$$E = E_1 \vee E_2 = E_2 \vee E_1$$

which is called "extension product" and is group law of $H^2(G, K)$. We shall now give a canonical definition of this product.

Let E_1, E_2 be two extensions verifying (51):

$$1 \longrightarrow K \xrightarrow{i_i} E_i \xrightarrow{s_i} G \longrightarrow 1 \quad i = 1,2 \qquad G \xrightarrow{g} \text{Aut } K \tag{51}$$

whose elements are

$$\hat{a}_i, \hat{b}_i \ldots$$

Let us consider the following groups:

$E_1 \times E_2$ with elements $(\hat{a}_1, \hat{b}_2)$.

F with elements $(\hat{a}_1, \hat{a}_2)$ such that $s_1(\hat{a}_1) = s_2(\hat{a}_2) = a \in G$.

$K \times K$ with elements (α', β'), $K \times K \subset F$ since $s_1(\alpha') = s_2(\beta') = 0$.

$\tilde{K}$ with elements (α', α'^{-1}).

We have the inclusions:

$$\tilde{K} \subset K \times K \subset F \subset E_1 \times E_2 \tag{52}$$

$K \times K$ is an invariant subgroup of $E_1 \times E_2$, hence is also invariant subgroup of F.

$$F/(K \times K) = (\hat{a}_1 K, \hat{a}_2 K) = (\text{coset of } a, \text{coset of } a) \approx G$$
$$K \times K/\tilde{K} \ni \{(\beta', \gamma')(\alpha', \alpha'^{-1})\} = \{(\beta'\alpha', \gamma'\alpha'^{-1})\}$$
$$= \text{set of these elements for all } \alpha \in K$$

i.e., this elements of $K \times K/\tilde{K}$ is the class of all pairs (μ', ν') of elements of K, such that

$$\mu'\nu' = \beta'\gamma' = \text{constant} \tag{53}$$

or

$$\beta + \gamma = \text{constant}$$
$$K \times K/\tilde{K} \approx K$$

We denote:

$$K \times K/\tilde{K} = K^*.$$

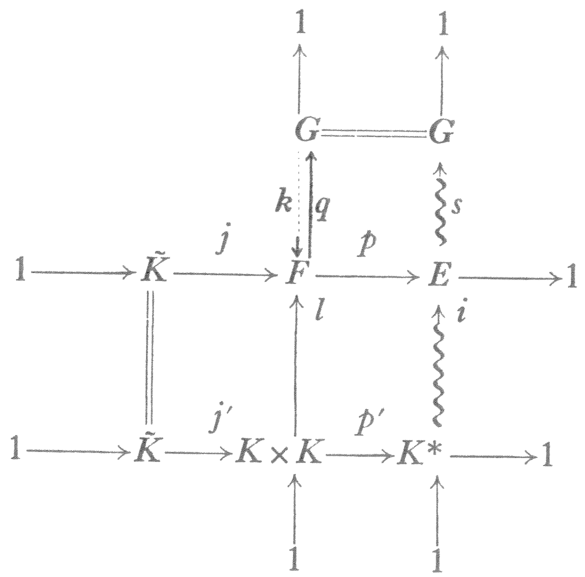
Now $\tilde{K}$ is invariant subgroup of F, indeed

$$[(\hat{a}_1, \hat{a}_2)(\alpha', \alpha'^{-1})(\hat{a}_1^{-1}, \hat{a}_2^{-1})] = [a\alpha', (a\alpha')^{-1}] \in K \tag{54}$$

We define:

$$E = F/\tilde{K}$$

and summarize all these results in the following diagram. Commutativity is easily verified and the consequences are written in wiggly arrows.



From $\text{Ker}(p_0, 1) = \text{Ker}\, p' = j'(\tilde{K})$ and lemma 3′, one obtains the injection i.

From $j(\tilde{K}) = \text{Ker}\, p \subset l(K \times K) = \text{Ker}\, q$, lemma 4 yields s.

Hence E is an extension of G by K. We will compute one of its factor systems.

Let k be a mapping of G into F such that $q_0 k =$ identity on G. Then $p_0 k$ is a mapping of G into E, such that

$$s_0(p_0 k) = q_0 k = \text{identity on } G.$$

We can write k in the fashion:

$$k(a) = [k_1(a), k_2(a)].$$

Hence the factor system of E is:

$$\begin{aligned} \omega'(a, b) &= p[k(a)k(b)k(ab)^{-1}] \\ &= p[k_1(a), k_2(a)]p[k_1(b), k_2(b)]p[k_1(ab)^{-1}, k_2(ab)^{-1}] \\ &= p[\omega_1'(a, b), \omega_2'(a, b)] \end{aligned}$$

and by (53) and the fact that the diagram is commutative:

$$\omega'(a, b) = i[\omega_1(a, b) + \omega_2(a, b)]$$

which proves the relation:

$$E = F/\tilde{K} = E_1 \vee E_2.$$

Appendix

We give here the proofs of lemmata 1 to 4.

Lemma 1.

$$h = g_0 f.$$

If $a \in \operatorname{Ker} f$, $0 = g_0 f(a)$, hence $\operatorname{Ker} f \subset \operatorname{Ker} h$.

Let $\bar{f}$ the restriction of f on $\operatorname{Ker} h$: that is $\bar{f} \in \operatorname{Hom}(\operatorname{Ker} h, B)$ and for every $a \in \operatorname{Ker} h$, $\bar{f}(a) = f(a)$. We have:

$$\operatorname{Ker} \bar{f} = \operatorname{Ker} f \subset \operatorname{Ker} h. \tag{1}$$

Furthermore:

$$h(\operatorname{Ker} h) = 0 \Rightarrow g_0 \bar{f} = 0$$

hence

$$\operatorname{Im} \bar{f} \subset \operatorname{Ker} g. \tag{2}$$

The exact sequence:

$$1 \to \operatorname{Ker} f \to \operatorname{Ker} h \xrightarrow{\bar{f}} \operatorname{Ker} g$$

is equivalent to (1) and (2).

Lemma 2. Let $b \in \operatorname{Ker} g$. The homomorphism f is surjective. Hence $f^{-1}(G)$ is not empty. Let $a \in f^{-1}(G)$; then $h(a) = g(b) = 1$, hence $a \subset \operatorname{Ker} h$ and $b = f(a)$. Hence $\operatorname{Im} \bar{f} = \operatorname{Ker} g$.

Lemma 3 is a little less trivial. We have

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \longrightarrow & 1 \\ & \searrow^{h} & & & \\ & & C & & \end{array}$$

and $\operatorname{Ker} f \subset \operatorname{Ker} h$.

The homomorphism f is surjective, for every $b \in B$, $f^{-1}(b)$ is not empty. We choose a and $a' \in f^{-1}(b)$. Then

$$a^{-1}a' \in \operatorname{Ker} f \subset \operatorname{Ker} h.$$

Hence

$$h(a) = h(a') = h[f^{-1}(b)] = c.$$

We write $c = g(b)$. It is a mapping $B \xrightarrow{g} C$ such that $h = g_0 f$. Is g a homomorphism?

$$g(b)g(b') = h(a)h(a') = h(aa')$$

where

$$f(a) = b, \quad f(a') = b'.$$

Hence

$$f(aa') = bb'.$$

From $h = g_0 f$,

$$g(b)g(b') = g_0 f(aa') = g(bb').$$

Lemma 3′. From lemma 2, $1 \to \operatorname{Ker} f \to \operatorname{Ker} h \xrightarrow{\bar{f}} \operatorname{Ker} g \to 1$. From $\operatorname{Ker} f = \operatorname{Ker} h$, we deduce $\operatorname{Im} \bar{f} = 1$. Since $\bar{f}$ is surjective. $\operatorname{Ker} g = 1$.

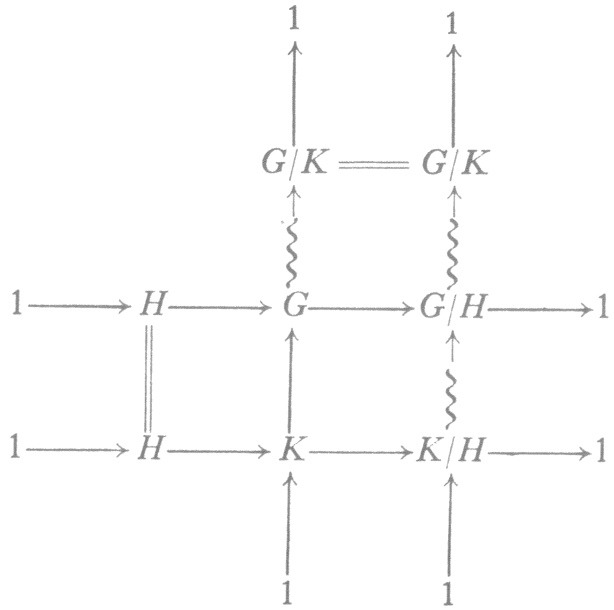
Lemma 4 is an obvious consequence of lemma 3. Indeed

$$h = g_0 f \Rightarrow \operatorname{Im} h = \operatorname{Im} g.$$

Remark

We used lemmata 3 and 4 to complete the last diagram of this chapter. Doing so we proved a very simple and well known theorem:

Theorem 3: in diagram language:



or in plain language:

Theorem 3

Let H, G, K be groups such that H < K < G and H invariant subgroup of G. Then:

$$\frac{G/H}{K/H} \approx \frac{G}{K}$$

Since the reader has not been taught to reduce fractions (i.e. theorem 3 for the group Z) in the language diagram, I feel sure he rightly finds pompous by now the extensive use of diagrams in this chapter.

Indeed this chapter was written as a "first step book on diagrams". By reading the present mathematical literature, one gets a sense of the good use of this language.

5. General Extensions of the Connected Poincaré Group $\mathscr{P}_0$

1. Definition of the Groups $\mathscr{L}_0$, $\mathscr{P}_0$, and of their Universal Coverings $\bar{\mathscr{L}}_0$, $\bar{\mathscr{P}}_0$

1*a*. *The group* $\mathscr{L}$ and $\mathscr{L}_0$. We denote four-vectors by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \dots$, their time component by a^0, their space part by $\mathbf{a}$, their scalar product by $\mathfrak{a} \cdot \mathfrak{b} = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$, their square by $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}^2$. Vectors with $\mathfrak{a}^2 > 0$, $\mathfrak{a}^2 = 0$, $\mathfrak{a}^2 < 0$ are respectively called time like, light like, space like.

We define for all vectors $\mathfrak{a}$, the function $\eta(\mathfrak{a})$ with values 1, 0, -1 by

$$\eta(0) = 0;\ \mathfrak{a}^2 < 0,\ \eta(\mathfrak{a}) = 0;\ \mathfrak{a}^2 \geqslant 0,\ \mathfrak{a} \neq 0,\ \eta(\mathfrak{a}) = a^0/|a^0| \quad (1)$$

The homogeneous Lorentz group $\mathscr{L}$ is the set of linear transformations which leaves invariant the scalar product of any pair of vectors. $\mathscr{L}$ is not connected. Let $\mathscr{L}_0$ be the connected component which contains the identity. It is a 6 real parameter Lie group. We shall denote its elements by Λ, M, ...

The parity operation P transforms

$$\mathfrak{a} = (a^0, \mathbf{a}) \text{ into } P\mathfrak{a} = (a^0, -\mathbf{a});$$

The time reversal T transforms $\mathfrak{a}$ into

$$T\mathfrak{a} = (-a^0, \mathbf{a});$$

So

$$P^2 = T^2 = 1, \ PT = TP \text{ and } PT\mathfrak{a} = -\mathfrak{a}$$

Then $\mathscr{L}$ is the union of the four connected sheets:

$$\mathscr{L} = \mathscr{L}_0 \cup P\mathscr{L}_0 \cup T\mathscr{L}_0 \cup PT\mathscr{L}_0 \tag{2}$$

The four-element group 1, P, T, PT is isomorphic to $Z_2 \otimes Z_2$, so $\mathscr{L}$ is the semi-direct product:

$$\mathscr{L} = \mathscr{L}_0 \times (Z_2 \otimes Z_2) \tag{3}$$

The function $\eta(\mathfrak{a})$ is invariant by $\mathscr{L}_0 \cup P\mathscr{L}_0$ (that is $\eta(\Lambda\mathfrak{a}) = \eta(\mathfrak{a})$, but is not invariant by $\mathscr{L}$.

1*b*. *The group* $\mathscr{P}$ *and* $\mathscr{P}_0$. The Poincaré group $\mathscr{P}$, or inhomogeneous Lorentz group, is the group of inhomogeneous linear transformations, which leaves invariant $(\mathfrak{a} - \mathfrak{b})^2$. The connected component is the semi-direct product:

$$\mathscr{P}_0 = \mathscr{T} \times \mathscr{L}_0 \tag{4}$$

where $\mathscr{T}$ is the groups of translations. Its elements are in a one-to-one correspondence with the four-vectors and we use the same notation $\mathfrak{a}$, $\mathfrak{b}$, . . . for them. So the group law of $\mathscr{P}_0$ is (see IV.11):

$$(\mathfrak{a}, \Lambda)(\mathfrak{b}, M) = (\mathfrak{a} + \Lambda\mathfrak{b}, \Lambda M) \tag{5}$$

where $\Lambda\mathfrak{b}$ corresponds to

$$(\Lambda\mathfrak{b}, 1) = (0, \Lambda)(\mathfrak{b}, 1)(0, \Lambda)^{-1} \tag{5'}$$

$\mathscr{P}$ is the complete Poincaré group. The commutative diagram (6) exhibits the relations between these different groups.

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{P}_0 & \longrightarrow & \mathscr{L}_0 & \longrightarrow & 1 \\
 & & \downarrow\uparrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{P} & \longrightarrow & \mathscr{L} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Z_2 \times Z_2 & \rightleftarrows & Z_2 \times Z_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{6}$$

(See also Prof. Wigner's lectures.)

1*c*. *Symmetries with respect to hyperplanes*. Let $\mathfrak{n}$ such that $\mathfrak{n}^2 \neq 0$. We define the symmetry with respect to the hyperplane orthogonal to $\mathfrak{n}$ by:

$$\sum_{\mathfrak{n}} = 1 - 2\frac{\mathfrak{n} \otimes \mathfrak{n}}{\mathfrak{n}^2} = \sum_{-\mathfrak{n}} \tag{7}$$

where $\otimes$ means here the tensor product of vectors. This symmetry transforms the vector $\mathfrak{x}$ into:

$$\mathfrak{x}' = \sum_{\mathfrak{n}} \mathfrak{x} = \mathfrak{x} - 2(\mathfrak{n} \cdot \mathfrak{x})\mathfrak{n}(\mathfrak{n}^2)^{-1} \tag{8}$$

The square of $\Sigma_{\mathfrak{n}}$ is the identity transformation.

Lemma 1.

$$\mathfrak{n}_1 \cdot \mathfrak{n}_2 = 0 \Rightarrow \sum_{\mathfrak{n}_1} \sum_{\mathfrak{n}_2} = \sum_{\mathfrak{n}_2} \sum_{\mathfrak{n}_1}$$

Lemma 2.

$$\mathfrak{n}_1^2 \cdot \mathfrak{n}_2^2 > 0 \Rightarrow \sum_{\mathfrak{n}_1} \sum_{\mathfrak{n}_2} \in \mathscr{L}_0.$$

We will also use:

$$\sum_{\mathfrak{n}} \mathfrak{n} = -\mathfrak{n} \tag{9}$$

$$\mathfrak{n} \cdot \mathfrak{a} = 0 \Leftrightarrow \sum_{\mathfrak{n}} \mathfrak{a} = \mathfrak{a}. \tag{10}$$

If

$$\mathfrak{a}^2 = \mathfrak{b}^2, (\mathfrak{a} - \mathfrak{b})^2 \neq 0: \sum_{\mathfrak{a}-\mathfrak{b}} \mathfrak{a} = \mathfrak{b}, \sum_{\mathfrak{a}-\mathfrak{b}} \mathfrak{b} = \mathfrak{a} \tag{11}$$

$$(\mathfrak{a} + \mathfrak{b})^2 \neq 0: \sum_{\mathfrak{a}+\mathfrak{b}} \mathfrak{a} = -\mathfrak{b}, \sum_{\mathfrak{a}+\mathfrak{b}} \mathfrak{b} = -\mathfrak{a} \tag{12}$$

Lemma 3. If $\mathfrak{a}^2 = \mathfrak{b}^2$ and $(\mathfrak{a} - \mathfrak{b})^2 < 0$, there exists $\Lambda \in \mathscr{L}_0$ such that $\Lambda\mathfrak{a} = \mathfrak{b}$ and $\Lambda\mathfrak{b} = \mathfrak{a}$.

Indeed we can find $\mathfrak{n}$ such that $\mathfrak{n}^2 = -1$, $\mathfrak{n}\cdot\mathfrak{a} = \mathfrak{n}\cdot\mathfrak{b} = 0$. Then from lemma 1, lemma 2 and Eqs. (10) and (12):

$$\Lambda = \sum_{\mathfrak{n}} \sum_{\mathfrak{a}-\mathfrak{b}}$$

The following pairs of vectors satisfy the assumption of this lemma.

$$\mathfrak{a}^2 = \mathfrak{b}^2 > 0, \quad \eta(\mathfrak{a}) = \eta(\mathfrak{b}) \tag{13}$$

$$\mathfrak{a}^2 = \mathfrak{b}^2 = 0, \quad \eta(\mathfrak{a}) = \eta(\mathfrak{b}) \text{ and } \mathfrak{a}.\mathfrak{b} \neq 0 \tag{13'}$$

Theorem of Cartan. Every element of $\mathscr{L}$ is the product of at most 4 symmetries. Every element of $\mathscr{L}_0$ is the product of 2 or 4 symmetries.

For a proof, see for instance E. Cartan, *La théorie des Spineurs* (Hermann, Paris, 1937).

1*d. The group* $\bar{\mathscr{L}}_0$ *universal covering of* $\mathscr{L}_0$. As explained in Prof. Wigner's lectures, we can build two one-to-one mappings between complex four-vectors and two-by-two matrices:

$$\mathfrak{a} \leftrightarrow \tilde{a} = a^0 + \mathbf{a}\,.\,\boldsymbol{\tau} \text{ and } \mathfrak{a} \leftrightarrow \underset{\sim}{a} = a^0 - \mathbf{a}\,.\,\boldsymbol{\tau} \tag{14}$$

where $\boldsymbol{\tau}$ are the three Pauli matrices.

$$\mathfrak{a} \text{ real} \Leftrightarrow \tilde{a} = \tilde{a}^*, \ \underset{\sim}{a} = \underset{\sim}{a}^* \tag{15}$$

$$\mathfrak{a}^2 = \tilde{a}\underset{\sim}{a} = \underset{\sim}{a}\tilde{a} = \det \ \tilde{a} = \det \ \underset{\sim}{a} \tag{16}$$

$$\mathfrak{a}\,.\,\mathfrak{b} = \tfrac{1}{2}(\tilde{a}\underset{\sim}{b} + \tilde{b}\underset{\sim}{a}) = \tfrac{1}{2}(\underset{\sim}{a}\tilde{b} + \underset{\sim}{b}\tilde{a}) \tag{17}$$

From $\mathfrak{a}' = \Sigma_{\mathfrak{n}}\mathfrak{a}$, we obtain:

$$\tilde{a}' = -\tilde{n}\underset{\sim}{a}\underset{\sim}{n}^{-1} \text{ and } \underset{\sim}{a}' = -\underset{\sim}{n}\tilde{a}\tilde{n}^{-1} \tag{18}$$

and from

$$\mathfrak{a}'' = \sum_{\mathfrak{n}_2} \sum_{\mathfrak{n}_1} \mathfrak{a}$$

we obtain

$$\tilde{a}'' = \tilde{n}_2\underset{\sim}{n}_1\tilde{x}(\underset{\sim}{n}_2\tilde{n}_1)^{-1} = \tilde{n}_2\underset{\sim}{n}_1\tilde{x}\underset{\sim}{n}_1\tilde{n}_2(\mathfrak{n}_1{}^2\mathfrak{n}_2{}^2)^{-1} \tag{19}$$

When $\mathfrak{n}_1{}^2\mathfrak{n}_2{}^2 \geq 0$, we define $\sqrt{\mathfrak{n}_1{}^2\mathfrak{n}_2{}^2}$ the positive square root of $\mathfrak{n}_1{}^2\mathfrak{n}_2{}^2$ and we define:

$$B = \tilde{n}_2\underset{\sim}{n}_1/\sqrt{\mathfrak{n}_1{}^2\mathfrak{n}_2{}^2} \tag{20}$$

$$\det B = \det \tilde{n}_2 \det \underset{\sim}{n}_1/\mathfrak{n}_1{}^2\mathfrak{n}_2{}^2 = 1 \tag{21}$$

So (19) can be written:

$$\tilde{a}'' = B\tilde{a}B^* \tag{22}$$

since $\tilde{n}^* = \tilde{n}$.

From $\Sigma_{\mathfrak{n}} = \Sigma_{-\mathfrak{n}}$, we see that B and $-B$ represent the same $\Lambda \in \mathscr{L}_0$. We call $\bar{\mathscr{L}}_0$ the group of the B, that is the group of two-by-two matrices with determinant $= 1$. It is easy to check that the correspondence $B \to \Lambda$ is a homomorphism $\bar{\mathscr{L}}_0 \xrightarrow{s} \mathscr{L}_0$, and Cartan's theorem shows that s is a surjective. It is also easy to compute Ker s; it contains the two matrices 1 and -1. The exact sequence (23) summarizes the situation:

$$1 \to Z_2 \to \bar{\mathscr{L}}_0 \xrightarrow{s} \mathscr{L}_0 \to 1 \tag{23}$$

Topologically, $\bar{\mathscr{L}}_0$ is a simply connected Lie group, which is the universal covering of $\mathscr{L}_0$ (see Prof. Speiser's lectures). Z_2 is called the first homotopy group of $\mathscr{L}_0$.

1*e*. *The universal covering* $\bar{\mathscr{P}}_0$ of $\mathscr{P}_0$. With the definition

$$B\mathfrak{a} = s(B)\mathfrak{a} \tag{24}$$

we can make $\bar{\mathscr{L}}_0$ act on $\mathscr{T}$. Then $\bar{\mathscr{P}}_0$ is the semi-direct product $\bar{\mathscr{P}}_0 = \mathscr{T} \times \bar{\mathscr{L}}_0$. The commutative diagram exhibits the relations between the groups $\mathscr{L}_0$, $\mathscr{P}_0$, $\bar{\mathscr{L}}_0$, $\bar{\mathscr{P}}_0$:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Z_2 & \rightleftarrows & Z_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathscr{T} & \longrightarrow & \bar{\mathscr{P}}_0 & \longrightarrow & \bar{\mathscr{L}}_0 & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{P}_0 & \longrightarrow & \mathscr{L}_0 & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{25}$$

2. What is known on $H^2(\mathscr{P}_0, \mathscr{A})$?

Wigner "F" has computed $H^2(\mathscr{P}_0, U_1) = Z_2$ where U_1 is the one-dimensional unitary group (group of phases). It is obvious from

Wigner "F" that the same proof is valid for the extensions of $\mathscr{P}_0$ by R the additive group of real numbers: $H^2(\mathscr{P}_0, R) = Z_2$. By theorem IV.1 the computation of $H^2(\mathscr{P}_0, \mathscr{A})$ can be extended to an abelian n parameter Lie group $\mathscr{A}$ which is a direct sum :

$$\mathscr{A} = \oplus_{i=1}{}^{n}\mathscr{A}_i \tag{26}$$

where each $\mathscr{A}_i$ is a one-parameter Lie group, hence isomorphic to either U_1 or R. This might be the only interesting physical case.

Wigner, in his fundamental paper, was in fact interested by the unitary representations up to a factor of $\mathscr{P}_0$ and he showed (s 5.A of Wigner "F") that this factor could be made a continuous function of the elements of $\mathscr{P}_0$. Our problem here is different: what can be said on $H^2(\mathscr{P}_0, \mathscr{A})$ for an arbitrary abelian group $\mathscr{A}$? So we want to avoid the use of any topological property for $\mathscr{A}$.

One can prove, purely *algebraically*,

Theorem 1: $H^2(\mathscr{P}_0, \mathscr{A}) = H^2(\mathscr{L}_0, \mathscr{A})$

Theorem 1′: $H^2(\bar{\mathscr{P}}_0, \mathscr{A}) = H^2(\bar{\mathscr{L}}_0, \mathscr{A})$

(their proofs might be published by Lurçat and I).

Also by an algebraic proof, in the next paragraph we shall compute $H^2(\mathscr{L}_0, \mathscr{A})$ if $H^2(\bar{\mathscr{L}}_0, \mathscr{A}) = 0$ is assumed.

So our question becomes: What is known on $H^2(\bar{\mathscr{L}}_0, \mathscr{A})$? Let us recall first a theorem on abelian groups:

Definition. An element x of an abelian group A, noted additively, is divisible if, for every integer n, there exists $y \in A$ such that $x = ny$. The element 0 is divisible.

Definition. An abelian group A is said to be divisible if all its elements are divisible.

Definition. An abelian group K is said to be reduced if it has no divisible elements except 0. *Example*. Every finite group is reduced.

Theorem. Given an abelian group A, it has a largest divisible subgroup D; furthermore $A = D \oplus K$ where K is a reduced group.

One can prove for a reducible group K that $H^2(\mathscr{L}_0, K) = 0$. Theorem IV.1 yields from

$$A = D \oplus K \tag{27}$$

$$H^2(\bar{\mathscr{L}}_0, A) = H^2(\bar{\mathscr{L}}_0, D) \tag{28}$$

and from the next paragraph:

$$H^2(\mathscr{P}_0, A) = {}_2K \oplus H^2(\mathscr{L}_0, D) \tag{29}$$

We do not know, yet, if there are no algebraic central extensions of $\bar{\mathscr{L}}_0$ by a divisible group D. However, we shall compute $H^2(\mathscr{L}_0, A)$ only with the assumption that $H^2(\bar{\mathscr{L}}_0, A) = 0$ which is true if enough topological requirements are made on A.

3. Computation of $H^2(\mathscr{L}_0, A)$ for Central Extensions

This computation is based on the assumption

$$H^2(\bar{\mathscr{L}}_0, \mathscr{A}) = 0 \tag{30}$$

We shall denote by $A, B, \ldots$, the elements of $\mathscr{L}_0$, by $\bar{A}, \bar{B}, \ldots$ those of $\bar{\mathscr{L}}_0$. By the homomorphism $\bar{\mathscr{L}}_0 \xrightarrow{s'} \mathscr{L}_0$, $A = s'(\bar{A})$.

Let us consider an extension $\mathscr{E}_0$ of $\mathscr{L}_0$ by $\mathscr{A}$, with the multiplication law:

$$(\alpha, A)(\beta, B) = [\alpha + \beta + \omega(A, B), AB]. \tag{31}$$

We can define from $\mathscr{E}_0$ an extension $\bar{\mathscr{E}}_0$ of $\bar{\mathscr{L}}_0$ by $\mathscr{A}$ by the law:

$$(\alpha, \bar{A})(\beta, \bar{B}) = [\alpha + \beta + \omega(\bar{A}, \bar{B}), AB] \tag{32}$$

where

$$\omega(\bar{A}, \bar{B}) = \omega[s'(\bar{A}), s'(\bar{B})] = \omega(A, B) \tag{33}$$

The mapping p:

$$(\alpha, \bar{A}) \to (\alpha, A) = p(\alpha, \bar{A})$$

is a homomorphism. Its restriction to $\mathscr{A} \subset \bar{\mathscr{E}}_0$ is the identity.

Since $H^2(\bar{\mathscr{L}}_0, \mathscr{A}) = 0$, $\bar{\mathscr{E}}_0 = \mathscr{A} \otimes \bar{\mathscr{L}}_0$ and $\omega(A, B)$ is a coboundary:

$$\omega(\bar{A}, \bar{B}) = (\delta\phi)(\bar{A}, \bar{B}) = \phi(\bar{A}) - \phi(\bar{A}\bar{B}) + \phi(\bar{B}) \tag{34}$$

Given $\omega(\bar{A}, \bar{B})$, the 1-cochain is unique. Let ϕ_1 and ϕ_2 be such that $\delta\phi_1 = \delta\phi_2 = \omega$; then $\delta(\phi_1 - \phi_2) = 0$, that is

$$\phi_1 - \phi_2 \in \mathrm{Hom}\,(\bar{\mathscr{L}}_0, \mathscr{A}) = 0, \text{ so } \phi_1 - \phi_2 = 0.$$

The injective homomorphism of $\bar{\mathscr{L}}_0$ into $\bar{\mathscr{E}}_0$ is:

$$j(\bar{A}) = [-\phi(\bar{A}), \bar{A}] \tag{35}$$

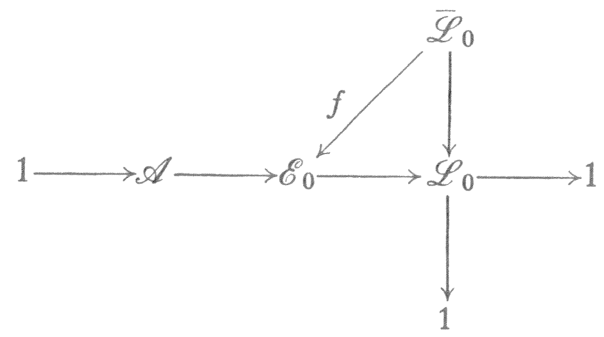
Indeed,

$$[-\phi(\bar{A}), \bar{A}][-\phi(\bar{B}), \bar{B}] = [-\phi(\bar{A}) - \phi(\bar{B}) + \omega(\bar{A}, \bar{B}), \bar{A}\bar{B}]$$
$$= [-\phi(\bar{A}\bar{B}), \bar{A}\bar{B}]$$

This j, allows us to define a homomorphism $\bar{\mathscr{L}}_0 \xrightarrow{f} \mathscr{E}_0$ by $f = p_0 j$:

$$f(\bar{A}) = (- \phi(\bar{A}), A, \tag{36}$$

We can now prove that there are no other homomorphisms f such that the diagram



is commutative.

Indeed, let us suppose there exist two such homomorphisms f_1 and f_2. From

$$g[f_1(\bar{A})\, f_2(\bar{A}^{-1})] = g_0 f_1\ (\bar{A}) \cdot y_0 f_2(\bar{A}^{-1}) = S'(\bar{A})\, S'(\bar{A}^{-1}) = 1$$

we see that

$$h(\bar{A}) = f_1(\bar{A}) f_2(\bar{A}^{-1}) \in i\,(\mathscr{A})$$

Let us prove that $h(\bar{A})$ is a homomorphism,

$$\bar{\mathscr{L}}_0 \overset{h}{\rightarrow} i\,(A)$$

Indeed

$$h\,(\bar{A}\bar{B}) = f_1\,(\bar{A}\bar{B})\, f_2\,(\bar{B}^{-1}\bar{A}^{-1}) = f_1\,(\bar{A})\, h\,(B)\, f_2\,(\bar{A}^{-1})$$

and since $\mathscr{E}_0$ is a central extension, i.e. $i(\mathscr{A}) = l(\mathscr{E}_0)$,

$$h\,(\bar{A}\bar{B}) = f_1\,(\bar{A})\, f_2\,(\bar{A}^{-1})\, h\,(B) = h\,(A)\, h\,(B)$$

However Hom $(\bar{\mathscr{L}}_0, \mathscr{A}) = 0$ implies $h = 0$, so

$$f_1\,(\bar{A})\, f_2\,(\bar{A}^{-1}) = 1 \text{ or } f_1 = f_2$$

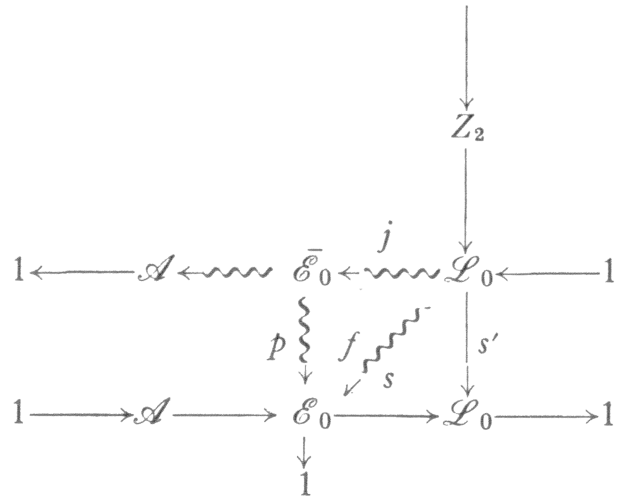
Since f is unique, $f(\bar{\mathscr{L}}_0)$ is invariant by all inner automorphisms of $\mathscr{E}_0$ (this can also be checked by direct computation); that is $f(\bar{\mathscr{L}}_0)$ is invariant subgroup of $\mathscr{E}_0$.

$$f(-1) = [-\phi(-1), 1] \in \text{center of } \mathscr{E}_0 \qquad (37)$$

and from $f(-1)^2 = f(1) = (0, 1)$, we have:

$$-2\phi(-1) = 0. \qquad (38)$$

We summarize the situation by the commutative diagram:



In a unique way, by the homomorphism f, we have defined for the extension $\mathscr{E}_0$, an element $\phi(-1)$ of ${}_2\mathscr{A}$. This mapping $f' : H^2(\mathscr{E}_0, \mathscr{A}) \xrightarrow{f'} {}_2\mathscr{A}$ is a homomorphism. Indeed, let us consider two extensions $\mathscr{E}_0{}^{(1)}$, $\mathscr{E}_0{}^{(2)}$ with their factor systems ω_1 and ω_2 and the corresponding $\phi_1(-1)$, $\phi_2(-1) \in {}_2\mathscr{A}$. To their product $\mathscr{E}_0{}^{(1)} \vee \mathscr{E}_0{}^{(2)}$ corresponds the factor system $\omega_1 + \omega_2$ and therefore the element $\phi_1(-1) + \phi_1(-1)[\in {}_2\mathscr{A}]$.

We now show that f' is surjective (upon); that is given $\epsilon \in {}_2\mathscr{A}$, we built an extension $\mathscr{E}_0(\epsilon)$:

$$1 \to Z_2 \to \mathscr{A} \otimes \bar{\mathscr{L}}_0 \xrightarrow{p'} \mathscr{E}_0(\epsilon) \to 1 \qquad (39)$$

where Ker p' contains $(0, 1)$ and $(\epsilon, -1) \in \mathscr{A} \times \bar{\mathscr{L}}_0$.

$p'(\mathscr{A})$ is identified with $\mathscr{A} \subset \mathscr{E}_0(\epsilon)$. If $\epsilon \neq 0$,

$$p'(\bar{\mathscr{L}}_0) \approx \bar{\mathscr{L}}_0$$

and

$$p'(-1) \equiv p'(0, -1) = p'(\epsilon, 1)$$

identified with $\epsilon \in \mathscr{A}$. So the mapping $\epsilon \to \mathscr{E}_0(\epsilon)$ is the inverse mapping of $f' : H^2(\mathscr{L}_0, \mathscr{A}) \xrightarrow{f'} {}_2\mathscr{A}$, and we have proven

$$H^2(\mathscr{L}_0, \mathscr{A}) \approx {}_2\mathscr{A}.$$

By theorem 1,

$$H^2(\mathscr{L}_0, \mathscr{A}) \approx H^2(\mathscr{P}_0, \mathscr{A}) \approx {}_2\mathscr{A}.$$

We have characterized all central extensions of the connected Poincaré group by an arbitrary abelian group $\mathscr{A}$.

Given $\epsilon \in {}_2\mathscr{A}$, the corresponding extension is:

$$(\mathscr{A} \times \bar{\mathscr{P}}_0)/Z_2$$

where

$$Z_2;\ \{(0;\ 0, 1) \text{ and } (\epsilon;\ 0, -1)\}.$$

6. Extension of the Complete Poincaré Group. Discrete Invariances P, C and T

1.

We begin this chapter by a theorem. Let $\mathscr{P}'$ be the semi-direct product $\mathscr{P}' = \mathscr{P}_0 \times G$ where G is a finite group: $G = \mathscr{P}'/\mathscr{P}_0$. Given an arbitrary abelian group $\mathscr{A}$ and a homomorphism g':

$$\mathscr{P}' \xrightarrow{g'} \text{Aut } \mathscr{A} \tag{1}$$

with

$$\mathscr{P}_0 \subset \text{Ker } g' \tag{2}$$

Theorem 1. For a given g', $H^2(\mathscr{P}', \mathscr{A}) = {}_2\mathscr{A}^G \otimes H^2(G, \mathscr{A})$.

Upon our request, this theorem has been kindly tailored by J. P. Serre for F. Lurçat and I.

Let us remark first that g' defines (because of (2))a homomorphism:

$$G \xrightarrow{g} \text{Aut } \mathscr{A}. \tag{3}$$

Conversely, given (3), we can define (1). Indeed, let $Q_1, Q_2 \ldots$ the elements of G and (a, A, Q) of $\mathscr{P}'$. We define

$$g'(a, A, Q) = g(Q) \tag{4}$$

For the reader we want to see things explicitly written, we shall write the group law of $\mathscr{P}'$:

$$(a, A, Q_1)(b, B, Q_2) = (a + Q_1Ab, AQ_1BQ_1{}^{-1}, Q_1Q_2) \tag{5}$$

(The physical interpretation of Q is $P, T, C, \ldots$).

Let us consider an extension $\mathscr{E} \in H^2(\mathscr{P}', \mathscr{A})$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathscr{A} & \longrightarrow & \mathscr{E} & \xrightarrow{s} & \mathscr{P}' & \longrightarrow & 1 \\ & & & & & & & \searrow^{g'} & \\ & & & & & & & & \text{Aut } \mathscr{A} \end{array} \tag{6}$$

Its group law is:

$$(\alpha, a, A, Q_1)(\beta, b, B, Q_2) \tag{7}$$

$$=[\alpha + Q_1\beta + \omega(a, A, Q_1; b, B, Q_2),\ a + Q_1Ab,\ AQ_1BQ_1^{-1},\ Q_1Q_2]$$

Let $\mathscr{E}_0 = s^{-1}(\mathscr{P}_0)$. It is the subgroup of elements $(\alpha, a, A, 1)$. It is an invariant subgroup of $\mathscr{E}$. It is a central extension.

$$1 \to \mathscr{A} \to \mathscr{E}_0 \xrightarrow{s} \mathscr{P}_0 \to 1 \tag{8}$$

We can repeat for this extension what we did in s VI.5. Its multiplication law is:

$$(\alpha, a, B)(\beta, b, B) = [\alpha + \beta + \omega(a, A; b, B), a + Ab, AB] \tag{9}$$

We define from $\mathscr{E}_0$ an extension of $\bar{\mathscr{P}}_0$ by $\mathscr{A}$, by the factor system:

$$\omega(a, \bar{A};\ b, \bar{B}) = \omega(a, A;\ b, B) \tag{10}$$

From theorems VI.1′ and assumption V.30

$$H^2(\bar{\mathscr{P}}_0, \mathscr{A}) = H^2(\bar{\mathscr{L}}_0, \mathscr{A}) = 0.$$

We know that $\omega(a\bar{A}, b\bar{B})$ is a coboundary:

$$\omega(a, \bar{A}, b\bar{B}) = (\delta\phi) \tag{11}$$

and $\mathrm{Hom}(\mathscr{P}_0, \mathscr{A}) = 0$ implies that ϕ is uniquely defined by ω. This defines uniquely the homomorphism f.

$$\bar{\mathscr{P}}_0 \xrightarrow{f} \mathscr{E}_0 \quad f(a, \bar{A}) = [-\phi(a, \bar{A}), a, \bar{A}] \tag{12}$$

Given an inner automorphism θ of $\mathscr{E}$, and $\bar{\theta}$ the corresponding automorphism of $\bar{\mathscr{P}}_0$, the unicity of f implies (the incredulous reader can check it by explicit computation):

$$f_0\theta = \bar{\theta}_0 f \tag{13}$$

$f(\bar{\mathscr{P}}_0) = \mathrm{Im}\, f$ is invariant subgroup of $\mathscr{E}$.

As in s V.3, we are interested by the image of $(0, -1) \in$ center of $\bar{\mathscr{P}}_0$. It is in the center of $\mathscr{E}_0$. Let us apply (13) when θ is the inner automorphism of $\mathscr{E}$ generated by $(0, 0, 1, Q)$. Then

$$f_0\bar{\theta}(0, -1) = f(0, -1) = [-\phi(0, -1), 0, 1, 1]$$

$$\theta_0 f(0, -1) = [-Q\phi(0, -1), 0, 1, 1].$$

Hence

$$Q\phi(0, -1) = \phi(0, -1) \tag{14}$$

or

$$\phi(0, -1) \in {}_2\mathscr{A}^G \tag{14'}$$

With the method used for f' in s V.3, we can prove that the mapping

$$H^2(\mathscr{P}', \mathscr{A}) \ni \mathscr{E} \xrightarrow{l} \phi(0, -1) \in {}_2\mathscr{A}^G$$

is a homomorphism.

Note that we still have the isomorphism:

$$H^2(\mathscr{P}_0, \mathscr{A}) \approx {}_2\mathscr{A}$$

but not all extension $\mathscr{E}_0 \in H^2(\mathscr{P}_0, \mathscr{A})$ can be subgroup of the extension $\mathscr{E} \in H^2(\mathscr{P}', \mathscr{A})$. It must correspond, by the homomorphism f', to an element of the subgroup ${}_2\mathscr{A}^G$ of ${}_2\mathscr{A}$.

What is the kernel of l? It contains the extension $\mathscr{E}$ such that the subgroup $\mathscr{E}_0 = \mathscr{A} \otimes \mathscr{P}_0$. So the group law (7) of $\mathscr{E}$ is more simply:

$$(\alpha, a, A, Q_1)(\beta, b, B, Q_2)$$
$$= [\alpha + Q_1\beta + \omega(Q_1,Q_2), a + Q_1Ab, A\, Q_1BQ_1^{-1}, Q_1Q_2]. \tag{7'}$$

Indeed $f(\bar{\mathscr{P}}_0)$ is the invariant subgroup $(0, q, A, 1)$ and the quotient $\mathscr{E}/f(\bar{\mathscr{P}}_0)$ has for group law:

$$(\alpha, Q_1)(\beta, Q_2) = [\alpha + Q_1\beta + \omega(Q_1, Q_2), Q_1Q_2]. \tag{15}$$

It is an extension $\in H^2(G, \mathscr{A})$.

Conversely, given the group law (15) of an extension of G by $\mathscr{A}$, we can form an extension $\mathscr{E} \in H^2(\mathscr{P}', \mathscr{A})$ by the group law (7') obtained from (15). Hence we have proven the exactness of the sequence:

$$0 \to H^2(G, \mathscr{A}) \to H^2(\mathscr{P}', \mathscr{A}) \xrightarrow{l} {}_2\mathscr{A}^G. \tag{16}$$

To prove that l is "upon" or surjective, we use a method similar to that used in s V.3. Indeed given $\epsilon \in {}_2\mathscr{A}^G$, we built:

$$\mathscr{E}_0(\epsilon) = (\mathscr{A} \otimes \bar{\mathscr{P}}_0)/Z_2(\epsilon)$$

where $Z_2(\epsilon)$ is the two-element group generated by $(\epsilon; 0, -1)$. Since G acts on $\mathscr{A}$ and $\bar{\mathscr{P}}_0$, it acts on $\mathscr{E}_0(\epsilon)$. Consider the semi-direct product $\mathscr{E}_0 \times G$. It is an extension of $\mathscr{P}'$ by $\mathscr{A}$ which is mapped by 1 on $\mathscr{E} \in {}_2\mathscr{A}$. So we have found a mapping

$$_2\mathscr{A}^G \xrightarrow{k} H^2(\mathscr{P}', \mathscr{A})$$

such that

$$l_0 k = \text{identity on } {}_2\mathscr{A}^G.$$

We can check that k is an isomorphism, by computing

$$\mathscr{E}_0(\epsilon_1) \vee \mathscr{E}_0(\epsilon_2)$$

that we find isomorphic to:

$$(\mathscr{A} \otimes \mathscr{P}_0)/Z_2(\epsilon_1 + \epsilon_2).$$

Hence $H^2(\mathscr{P}', \mathscr{A})$ is the semi-direct product of ${}_2\mathscr{A}^G$ by $H^2(G, \mathscr{A})$. Since it is abelian, it is even the direct product.

2. Application to Time Reversal T

We consider for G the two-element group $\{1, T\}$. The time reversal T acts on $\bar{\mathscr{P}}_0$ by the automorphism:

$$\bar{\mathscr{P}}_0 \ni (a, \bar{A}) \xrightarrow{T} (a_T, \bar{A}^{*-1}) \tag{17}$$

where

$$a_T = (-a^0, \mathbf{a})$$

The group $\mathscr{P}'$ we consider in this paragraph is the semi-direct product $\mathscr{P}_0 \times G$ where $G : \{1, T\}$.

As we have seen in Chapter III, the time reversal T induce an automorphism on $\mathscr{N}$, the algebra of observable. In order to preserve the sign of the energy, this automorphism is an antiautomorphism, that is it contains a *-conjugation in $\mathscr{N}$. Since T commute with the charges Q, B, L, the only effect of T on the center $\mathscr{N}'$ of $\mathscr{N}$ is the hermitian conjugation.

The group $\mathscr{A}$ is the group of the unitary operators in $\mathscr{N}$. Any element $\Omega \in \mathscr{A}$ is transformed by the antiautomorphism T into $\Omega^* = \Omega^{-1}$.

So $\mathscr{A}^G$ is the set of Ω such that:

$$\Omega = \Omega^* = \Omega^{-1}$$

or

$$\Omega^2 = 1$$

that is

$$\mathscr{A}^G = {}_2\mathscr{A} = {}_2\mathscr{A}^G.$$

We have studied this case in III (48′):

$$H^2(G, \mathscr{A}) = {}_2\mathscr{A}.$$

The element of ${}_2\mathscr{A}$ which characterizes the extension of G by $\mathscr{A}$, hence of $\mathscr{P}'$ by $\mathscr{A}$ is $\omega(T, T)$; the square of time reversal.

3. The Invariance *CP*

Since the discovery of non-conservation of parity, as predicted by Lee and Yang, we know that neither P nor C are an automorphism of $\mathscr{N}$, the algebra generated by the observables. However, it seems that physics in invariant under the product $PC = CP$. This transformation is a linear homomorphism of $\mathscr{N}$. What is its effect on $\mathscr{N}'$, the center of $\mathscr{N}$?

If we admit axiom c, the elements of $\mathscr{N}'$ are functions of the charges. As a shorthand we write $\mathbf{Q}$ for the set of operators which represent the charges. The unitary elements of $\mathscr{N}'$ can be written

$$\Omega = \exp[i\pi f(\mathbf{Q})] \tag{18}$$

where f is a real analytic function. By the automorphism PC

$$\Omega \xrightarrow{PC} \Omega_{pc} = \exp[i\pi f(-\mathbf{Q})]. \tag{19}$$

We call here G the two-element group generated by CP. We have given in III.46 the value of $H^2(G, \mathscr{A})$:

$$H^2(G, \mathscr{A}) = \operatorname{Ker} D/\operatorname{Im} N \tag{20}$$

where

$$N\Omega = \exp[i\pi f(\mathbf{Q})] + \exp[inf(-\mathbf{Q})] \tag{21}$$

$$D\Omega = \exp[i\pi f(\mathbf{Q})] - \exp[i\pi f(-\mathbf{Q})] \tag{21′}$$

We do not discuss CP invariance independently. We will discuss now the full Poincaré group.

4. The Full Poincaré Group of Quantum Mechanics

This is the title of a paragraph of Professor Wigner's lecture. The Poincaré group defined in V.1b, the geometrical Poincaré group, is not the group of invariance of quantum mechanics. Following Wigner, we call quantum mechanical Poincaré group $\mathscr{P}$ the semi-direct product:

$$\mathscr{P} = \mathscr{P}_0 \times G \tag{22}$$

where G is the four-element group:

$$G\ ;\ [1, PC = CP, T, PCT]. \tag{22'}$$

The group $\mathscr{P}$ is a subgroup of the group of automorphisms of $\mathscr{N}$. We know how G, and $\mathscr{P}$, act on $\mathscr{A}$ the group of the unitary operators of $\mathscr{N}'$. The computation of $H^2(G, \mathscr{A})$ in the general case is simplified by the property of $\mathscr{A}$ to be divisible. Indeed, let $\Omega \in \mathscr{A}$. Its spectral decomposition is:

$$\Omega = \sum_{\mathbf{q}} \exp[if(\mathbf{q})]P_{\mathbf{q}} \tag{23}$$

where $\mathbf{q}$ is the set of values of the charges $\mathbf{q}$: q, b, l, . . . and $P_{\mathbf{q}}$ is the projector on the corresponding coherent Hilbert space of state vectors. Then:

$$\Omega_n = \sum_{\mathbf{q}} \exp[(i\pi/n)f(\mathbf{p})]P_{\mathbf{q}} \tag{24}$$

is a n-th root of Ω, that is $\Omega_n{}^n = \Omega$.

The computation of ${}_2\mathscr{A}^G$ is easy: it is the set of Ω such that:

for every set $\mathbf{q}$, $f(\mathbf{q})$ and $f(-\mathbf{q})$ are integers of the same parity. (25)

5. Physical Interpretation

We leave to the reader, if he is interested, a more general discussion. In order to indicate briefly the possible physical interpretation, we restrict the discussion to the simplest possible $\mathscr{A}$, that group given in Eq. (III.9).

$$\mathscr{A} = \{\exp[i(\alpha_q Q + \alpha_b B + \alpha_l L)]\} \equiv \exp(i\boldsymbol{\alpha} \,.\, \mathbf{Q}). \tag{26}$$

In this simple case:

$$\exp(i\alpha \,.\, \mathbf{Q}) \xrightarrow{T} \exp(-i\boldsymbol{\alpha} \,.\, \mathbf{Q}),\ \exp(i\boldsymbol{\alpha} \,.\, \mathbf{Q}) \xrightarrow{PC} \exp(-i\boldsymbol{\alpha} \,.\, \mathbf{Q}),$$
$$\exp(i\boldsymbol{\alpha} \,.\, \mathbf{Q}) \xrightarrow{PCT} \exp(i\boldsymbol{\alpha} \,.\, \mathbf{Q}). \tag{27}$$

The *PCT* transformation leaves invariant every element of $\mathscr{A}$ and from our point of view is more trivial than *P*, *C* and *T*. Then

$$_2\mathscr{A}^G = {}_2\mathscr{A} = (-1)^{\epsilon_q Q + \epsilon b B + \epsilon l L} = \{(-1)^{\boldsymbol{\epsilon}\cdot \mathbf{Q}}\} \qquad (28)$$

where each $\epsilon = 1$ or 0. The group $_2\mathscr{A}$ is the group of the eight square roots of the unit element of $\mathscr{A}$.

According to theorem 1 and to Eq. (III.49″)

$$H^2(\mathscr{P}, \mathscr{A}) = {}_2\mathscr{A} \oplus {}_2\mathscr{A} \oplus {}_2\mathscr{A}. \qquad (29)$$

There are $8^3 = 512$ inequivalent extensions. Each is characterized by an element of $H^2(\mathscr{P}, \mathscr{A})$. Every element of this group is characterized by the triplet:

$$(-1)^{\boldsymbol{\epsilon}\cdot \mathbf{Q}},\ (-1)^{\boldsymbol{\epsilon}'\cdot \mathbf{Q}},\ (-1)^{\boldsymbol{\epsilon}''\cdot \mathbf{Q}}. \qquad (30)$$

This represents more ϵ whose value 1 or 0 has to be determined:

$(-1)^{\boldsymbol{\epsilon}\cdot \mathbf{Q}}$ is $f(-1)$ the image into $_2\mathscr{A}^G$ of the "rotation of 2π" (31)

$(-1)^{\boldsymbol{\epsilon}'\cdot \mathbf{Q}}$ is $\omega(T, T)$ the square of T (31′)

$(-1)^{\boldsymbol{\epsilon}''\cdot \mathbf{Q}}$ is $\omega(PC, PC)$ the square of PC. (31″)

The square of *PCT* is immaterial. It can be normalized to $1 \in \mathscr{A}$.

The first question to be answered for a physical interpretation is: which extension $\mathscr{E}(\epsilon, \epsilon', \epsilon'')$ is realized in nature?

To be able to choose among the 512 $\mathscr{E}$ we have to know their irreducible unitary representations and "corepresentations" (see Prof. Wigner's lectures) for the operators $T\mathscr{P}_0$ and $PCT\mathscr{P}_0$. These are representations and corepresentations up to a factor of the quantum mechanical Poincaré group. They have been determined in Prof. Wigner's lectures. Here we give to the arbitrary phase a new physical interpretation, as representation of $\mathscr{A}$ the invariant subgroup of $\mathscr{E}$ the extension of $\mathscr{P}$. A complete discussion from this new point of view of the theory of "types" will not be carried here. It is left to the enjoyment of the eventually interested reader.

We recall that in Chapter III we have found:

$$\boldsymbol{\epsilon} = (\epsilon_q = 0,\ \epsilon_{\mathbf{q}} = \epsilon_l = 1). \qquad (32)$$

This implies for the representation of $\mathscr{E}_0$ the relation

$$(-1)^{2j} = (-1)^{b+l} \qquad (33)$$

between spin and charge.

To end this chapter, we will prove that for a one-particle state, the eigen value of

$$\omega(CP, CP) \equiv (-1)^{\epsilon''\cdot Q}$$

is what is called "particle–antiparticle relative parity".

6. The Particle–antiparticle Relative Parity

Except of K^0, $\bar{K}^0$ (and their resonances), the states of one particle and the states of one antiparticle have at least one charge with opposite value and therefore are separated by a superselection rule. So the expression "particle–antiparticle relative parity" used in the folklore of elementary particle physics is an image for something which is not a relative parity.

Let $|+, u_+\rangle$, a state vector representing a one-particle state. We choose $k(CP)$ as representative of CP in the extension $\mathscr{E}$ which corresponds to the description of nature. We define the state vector $|-, u_-\rangle$ by:

$$k(CP)|+, u_+\rangle = |-, u_-\rangle. \tag{34}$$

The vector $|-, u_-\rangle$ represents the physical state obtained by CP transformation. As we have seen, because of the normalization $k(1) = 1$,

$$[k(CP)]^2 = \omega(CP, CP) = (-1)^{\epsilon''\cdot Q} \tag{35}$$

is an element of ${}_2\mathscr{A}$ and is independent of the choice of the representative $k(CP)$. For short, we will denote by ϵ'' the number which represents $(-1)^{\epsilon''\cdot Q}$ in the group representation describing one-particle states: that is

$$[k(CP)]^2|\pm, u_\pm\rangle = \epsilon''|\pm, u_\pm\rangle. \tag{36}$$

Multiplying both sides of (34) by $k(CP)$ we obtain:

$$k(CP)|-, u_-\rangle = \epsilon''|+, u_+\rangle. \tag{37}$$

Let us consider the physical state of one particle and one antiparticle which is described by

$$|+, u_+\rangle \otimes_S |-, u_-\rangle \tag{38}$$

where $\otimes_S$ is a tensor product which has been symmetrized or antisymmetrized according to the statistics. This state is a proper state of $k(CP)$. Indeed from (34) and (37):

$$\begin{aligned} & k(CP)|\,+,u_+\rangle \otimes_S |\,-,u_-\rangle \\ & = \epsilon''(-1)^{2j}|\,+,u_+\rangle \otimes_S |\,-,u_-\rangle \end{aligned} \tag{39}$$

where j is the particle spin.

We leave to the reader physicist to compare (39) to his usual treatment of invariance of a state of one particle and one antiparticle under CP and to check that ϵ'' is what is abusively called the particle–antiparticle relative parity (Hint: CP does not change the spin state, so (39) is symmetrical for the spins; let 1 be the relative orbital momentum, then the proper value of C is $(-1)^{2j+l}$, so the proper value of P is $\epsilon''(-1)^l$).

The Dirac equation yields $(CP)^2 = -1$, so it is generally assumed $\epsilon'' = (-1)^{2j}$. From our general scheme developed in III, ϵ'' is a function of the charges. This function should be experimentally measured in order to check if it is equal or not equal to $(-1)^{2j} = (-1)^{b+l}$.

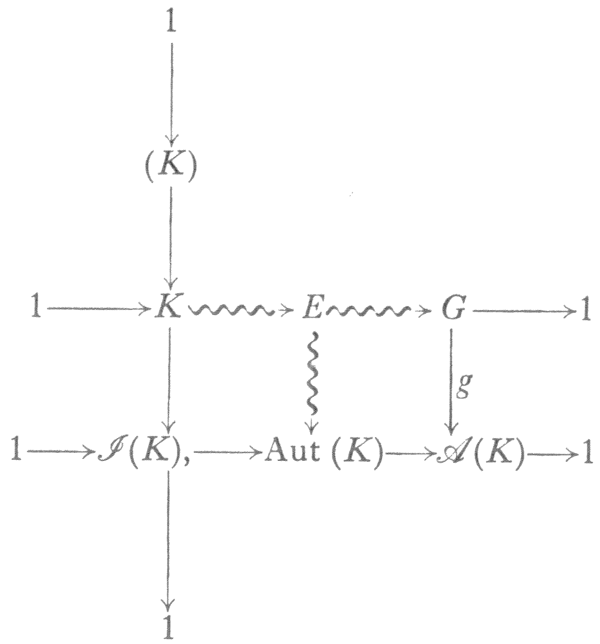
7. Group Extensions by Non-Abelian Groups. Miscellaneous Applications to Particle Physics

1. Results of the General Theory

We have already defined the general problem of the extensions of a group G by a non-abelian group K. We must first determine all homomorphisms:

$$G \xrightarrow{g} \mathscr{A}(K).$$

Then for each homomorphism g, we have to find the set $\text{Ext}_g(G, K)$ of solutions E of the diagram IV.9 reproduced here:



We just give here some results of the paper "Cohomology theory in abstract groups II. Group extension with a non-Abelian kernel" by S. Eilenberg and S. MacLane, 1947, *Ann. Math.*, **48**, 326. We refer the reader to this paper for more details and for the proofs. We shall quote this paper as "E.M.".

1.a. The three-cocycle $\zeta(a, b, c) \in Z^3(G, \mathscr{C}(K))$. Let k be a mapping of $\mathscr{A}(K)$ into Aut(K) such that (see Diagram VII.1):

$$p_0 k = I, \text{ the identity transformation on } \mathscr{A}(K). \tag{1}$$

Similarly, we consider a mapping k' such that:

$$p'_0 k' = I, \text{ the identity on } \mathscr{I}(K). \tag{1'}$$

Since Aut(K) is an extension of $\mathscr{A}(K)$ by $\mathscr{I}(K)$, the choice of representatives in Aut(K) given by the mapping k defines a factor system $\tilde{\omega}$ whose values are in $\mathscr{I}(K)$. If we consider the arguments of $\tilde{\omega}$ which belongs to Img, we have a "cocycle"

$$\forall a, b \in G, \quad \omega(a, b) \in Z^2[G, \mathscr{I}(K)] \tag{2}$$

defined by:

$$\omega(a, b) \,.\, (k_0 g)(ab) = (k_0 g)(a) \,.\, (k_0 g)(b) \tag{3}$$

we wrote "cocycle $\tilde{\omega}$" between quotation marks because the range of $\tilde{\omega}$ is in a non-abelian group $\mathscr{I}(K)$. So we cannot extend to this

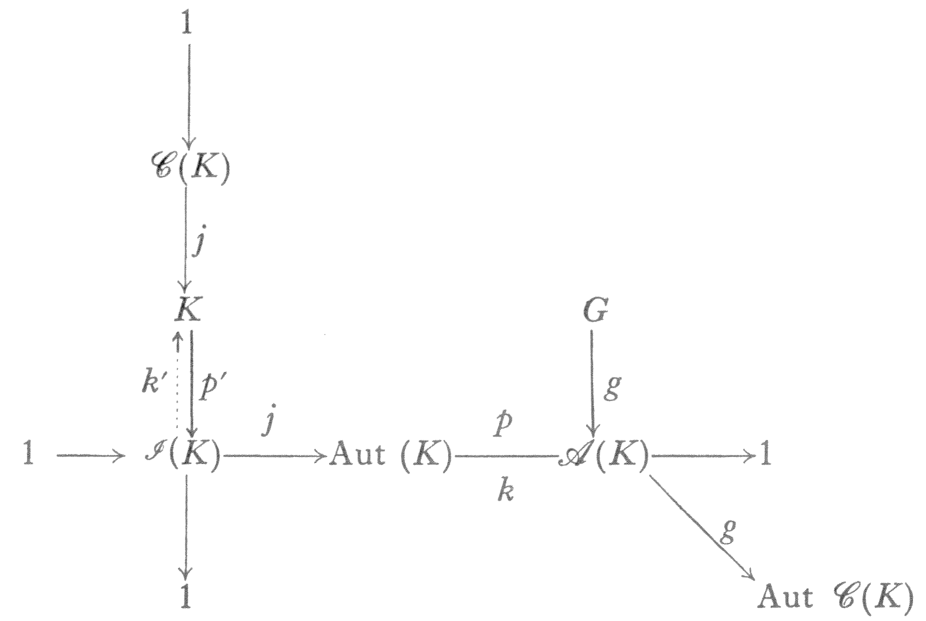


Diagram VII.1

case all results of Chapter IV. However the associativity of the group law of Aut(K) yields indeed:

$$\tilde{\omega}(a, b)\tilde{\omega}(ab, c) = a[\tilde{\omega}(b, c)]\tilde{\omega}(a, bc) \tag{4}$$

where

$$a[\tilde{\omega}] = k_0 g(a)\omega[k_0 g(a)]^{-1} \tag{4'}$$

The relation (4) among elements of $\mathscr{I}(K)$ can be mapped by k' into a relation among elements of K up to an element of $\mathscr{C}(K)$, the center of K. We call $\zeta(a, b, c)$ this element. It is defined by:

$$\zeta(a, b, c)k'[\tilde{\omega}(a, b)]k'[\tilde{\omega}(ab, c)] = k'(a[\tilde{\omega}(b, c)])k'[\tilde{\omega}(a, bc)]. \tag{5}$$

This function $\zeta(a, b, c)$ is a three-cochain

$$\zeta(a, b, c) \in C^3[G, \mathscr{C}(K)]. \tag{6}$$

The way G acts on $\mathscr{C}(K)$ is unambiguous. Indeed to each element $\hat{x}$ of $\mathscr{A}(K)$ corresponds a class of automorphisms of K defined modulo an inner automorphism. Since inner automorphisms of K leave fixed every element of the center $\mathscr{C}(K)$, there is a homomophism $\mathscr{A}(K) \xrightarrow{g'}$ Aut $\mathscr{C}(K)$. Hence G acts on $\mathscr{C}(K)$ by the homomorphism:

$$G \xrightarrow{g'0g} \text{Aut}\, \mathscr{C}(K). \tag{6'}$$

If we calculate $a[b[\tilde{\omega}(c, d)]\omega(b, cd)]\omega(a, bcd)$ in two ways, we check (E. M., lemma 7.1, p. 331) that $\zeta(a, b, c)$ is cocycle:

$$\zeta(a, b, c) \in Z^3[G, \mathscr{C}(K)] \tag{7}$$

One also checks that: by changing the mappings k or k', the cocycle ζ is changed by a coboundary; moreover, by a suitable change of k, the cocycle ζ can be changed to any cohomologuous cocycle (E. M., lemma 7.2 and 7.3.). Hence, from Diagram VII.1, that is from G, K and $G \xrightarrow{g} \mathscr{A}(K)$ we have determined an element ζ of $H^3[G, \mathscr{C}(K)]$ by the 3-cocycle ζ, defined up to a coboundary.

We can now state the theorem:

1.b. *Theorem* 1. There are extensions E of G by K corresponding to the homomorphism $G \xrightarrow{g} \mathscr{A}(K)$, if, and only if, the corresponding ζ is the zero element of $H^3[G, \mathscr{C}(K)]$; in other words, the cocycle ζ defined by (5) is a coboundary. Proof of the necessary condition: Let E be an extension and h a mapping such that $s_0h = I$, the identity automorphism of G (see Diagram VII.2).

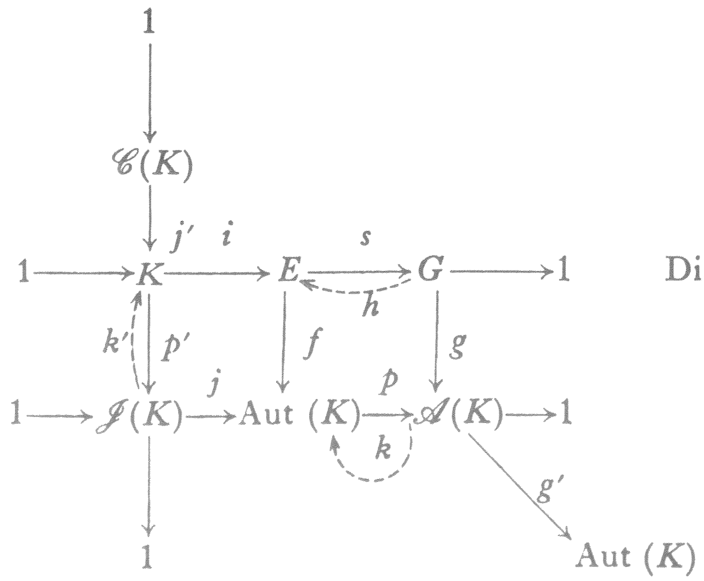


Diagram VII.2

We define the mapping k such that

$$k_0 g = f_0 h.$$

The factor system $\omega(a, b)$ of the extension E, corresponding to the mapping h:

$$\omega(a, b)h(ab) = h(a)h(a) \tag{8}$$

yields for the $\tilde{\omega}$ defined in (3):

$$\tilde{\omega}(a, b) = f[\omega(a, b)]. \tag{9}$$

From (9) we define k' by

$$\omega(a, b) = k'[\tilde{\omega}(a, b)].$$

The ω satisfy (4), hence $\zeta(a, b, c) = 0$. It is the null coboundary. Proof of the sufficient condition: If $\zeta(a, b, c)$ is a coboundary, by E.M., lemma 7.2, already quoted, we can choose k and k' such that $\zeta = 0$. We denote $k'(\tilde{\omega})$ by ω. Then, on the set product of the sets K and G, whose elements are pairs (α, a), we define the composition law:

$$(\alpha, a)(\beta, b) = (\alpha \cdot a[\beta] \cdot)\omega(a, b), ab) \tag{10}$$

where $a[\beta]$ is the transformed of β by the automorphism $k_0 g(a)$. With the relation (4) satisfied by the ω's, Eq. (10) defines a group law. One checks that the corresponding group is an extension E, solution of Diagram IV.9. (Hint for the computation: $\omega(a, b)ab[\gamma] = a[b[\gamma]]\omega(a, b)$.)

1.c. The next question to answer is: When G, K, $G \xrightarrow{g} \mathscr{A}(K)$ are such that ζ is a coboundary, how many extensions E are there? Or, how many different mappings k yield inequivalent extensions?

The general definition of equivalence for extensions has been given in IV.31 where Aut(K) has to be replaced by $\mathscr{A}(K)$. The answer is:

Theorem 2: For a given homomorphism $G \xrightarrow{g} \mathscr{A}(K)$, there is a one-to-one correspondence between the elements of the set $\text{Ext}_g(G, K)$ and the elements of $H^2{}_{g'_0 g}[G, \mathscr{C}(K)]$, that is the set of extensions of G by the center of K corresponding to the homomorphism $G \xrightarrow{g'_0 g}$ Aut $\mathscr{C}(K)$.

For the proof, we refer to M.E. Without proof, we shall give here an explicit construction of this one-to-one correspondence. For this we have to introduce few definitions and new kinds of group product.

Definition of G-kernels. Given a group G, an abelian group A and a homomorphism $G \xrightarrow{f} \text{Aut}(A)$, we call G-kernel any pair of a group K and of a homomorphism g of $G \longrightarrow \mathscr{A}(K)$ such that: $A = \mathscr{C}(K)$, the center of K and $f = g'_0 g$ where g' is the canonical homomorphism $\mathscr{A}(K) \xrightarrow{g'} \text{Aut}(A)$.

Among G-kernels, there is the pair A, f since $\mathscr{A}(A) = \text{Aut}(A)$.

Equivalence of G-kernels. K_1, g_1 and K_2, g_2 are equivalent if there exists an isomorphism $K_1 \xrightarrow{i} K_2$ which leaves invariant every element of their common center A and such that, if j_1 is an automorphism of K_1 which belongs to the class $g_1(a)$, then $i_0 j_1 0 i^{-1} \in g_2(a)$ for all $a \in G$.

Product of G-kernels. K_1, g_1 and K_2, g_2. The direct product $K_1 \otimes K_2$ has for elements the pairs (α_1, α_2). Let $\tilde{A}$ be the subgroup of pairs (α, α^{-1}) where $\alpha \in A$ the common center of K_1 and K_2. We define

$$K = K_1 \wedge K_2 = (K_1 \otimes K_2)/\tilde{A}. \qquad (11)$$

The corresponding homomorphism g is deduced naturally from the homomorphisms g_1 and g_2.

We can now define the product of two extensions E_1 and E_2 or G by the G-kernels K_1 and K_2 respectively. (Compare with the product $\vee$ defined at the end of Chapter IV). Consider the subgroup F of $E_1 \otimes E_2$ whose elements are pairs (a_1, a_2) such that $s_1(a_1) = s_2(a_2) = a$ (see Diagram VII.2 for the notations). The subgroup $\tilde{A}$ we have defined above (its elements are the pairs (α, α^{-1}) where $\alpha \in A$) is invariant subgroup of F. We define:

$$E = E_1 \wedge E_2 = F/\tilde{A}. \qquad (14)$$

One can check that it is an extension of G by the G-kernel $K = K_1 \wedge K_2$. In the particular case $K_1 = K_2 = A$, the product $\wedge$ is just the product $\vee$ defined at the end of Chapter IV for the extensions of G by the abelian group A.

In IV we have indicated the general methods to solve the extension problem in the particular case where the G-kernel A, f is abelian; let us denote by D_i the corresponding extensions, elements of $H_f^2(G, A)$. M.E. prove that, given E_0, an extension of G by the extendible G-kernel K, g all the non-equivalent extensions of G by the same G-kernel are given by

$$E_i = E_0 \wedge D_i \qquad (15)$$

where $D_i \in H_f^2(G, A)$. Please note that the set $\mathrm{Ext}_g(G, K)$ of the E_i does not form a group.

2. Some Particular Cases

2.a. *Extensions by a group without center.* If $A = \mathscr{C}(K) = 1$, then $H^3(G, A) = 0$ and K, a group without center, is an extendible

G-kernel for all groups G. Furthermore $H^2(G, A) = 0$, hence there is only one extension of G by K for every homomorphism $g \in \mathrm{Hom}(G, \mathscr{A}(K))$. (This was proven by R. Baer, 1934, *Math. Zeit.*, **38**, 375.)

2.b. *The homomorphism g is trivial*. In this case we know there is at least one solution: the direct product. (As a particular case of the above theory, all $\tilde{\omega}$ defined in (3) are equal to 1, hence ζ is the null coboundary). The other solutions are obtained by Eq. (15).

$$E_i = (K \otimes G)\Lambda D_i \tag{15'}$$

where the D_i are the central extensions of G by $\mathscr{C}(K)$. The characterization of this particular case is: E is an extension of G by K such that the inner automorphisms of E induce on the invariant subgroup K inner automorphisms of K.

Remark that g must be trivial when $\mathscr{A}(K) = 1$, that is when K has no outer automorphisms; every automorphism of K is an inner automorphism.

2.c. *Extension by a "complete" group*. A group K is called complete if it has no center and no outer automorphisms: $\mathscr{C}(K) = 1, \mathscr{A}(K) = 1$ (Kurosh, **13**, 92). In other words: K complete $\Leftrightarrow K \approx \mathrm{Aut}(K)$.

Then K belongs to both particular cases 2.a and 2.b. Hence: the direct product $K \otimes G$ is the only extension of the arbitrary group G by the complete group K.

Examples of complete groups are:

ζ_n, the permutation group of n objects, for $n \geq 3$ and $n \neq 6$ (Hölder, 1895, *Math. Ann.* **46**, 321).

Aut G, when G is non-abelian and simple (no invariant subgroup) (N. Bourbaki, *Algèbre*, I, **7**, *no.* 7, exerc. 3 and 4).

SO_3, the three-dimensional rotation group and other examples among classical Lie groups, given in Prof. Speiser's lectures.

3. Relativistic Invariance under $\mathscr{P}_0$ (the Connected Poincaré Group) in the Strong Coupling Approximation

We should study the strong coupling approximation in particle physics in the spirit of Chapter III. Here, we will only, as an example, look for the extensions of $\mathscr{P}_0$ by $U_1 \otimes U_2$, the smallest strong coupling invariance group defined in II. These extensions can only correspond to the trivial homomorphism since $\mathrm{Hom}[\mathscr{P}_0, \mathscr{A}(U_1 \otimes U_2)] = 1$. On the other hand $\mathscr{C}(U_1 \otimes U_2) = U_1 \otimes U_1$, hence $H^2(\mathscr{P}_0, U_1 \otimes U_2) = {}_2(U_1 \otimes U_1) = Z_2 \otimes Z_2 =$ the group of square roots of 1 in the center $U_1 \otimes U_1$ of $U_1 \otimes U_2$. These four square roots are $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$; they will be denoted by ϵ_i. The four inequivalent extensions of $\mathscr{P}_0$ by $U_1 \otimes U_2$ are

$$E_i = [(U_1 \otimes U_2) \otimes \bar{\mathscr{P}}_0]/Z_2 \tag{16}$$

where $\bar{\mathscr{P}}_0$ is the universal covering group of $\mathscr{P}_0$ and the two element group Z_2 is generated by (ϵ_i, ω) where ω is the "rotation of 2" (the non-trivial element of the center of $\bar{\mathscr{P}}_0$).

The irreducible unitary representation of E_i are characterized by the baryonic charge b (for U_1), the hypercharge y and the isospin t (for U_2) and by the mass m and the spin j (for $\mathscr{P}_0$). By the method used in II, we find for each E_i some relation among these quantum numbers. For one, and one extension only, the corresponding relation is verified by Nature. This extension corresponds to the square root $(-1, -1)$ (which can also be written $(-1)^{B \pm Y}$ or $(-1)^S$ where S is the strangeness operator); it implies the relation

$$(-1)^{b+y+2t+2j} = 1 \tag{17}$$

where any $+$ sign can be replaced by a $-$ sign.

The inclusion of the discrete operations P, C, T would yield many inequivalent extensions. We shall discuss only the case of the charge conjugation, since this is an historical example.

4. The Isotopic Parity

In 1952 I asked for myself the question "How to relate charge independence and charge conjugation invariance?" I thought at that time that the strong interaction invariance group was SO_3. How is it enlarged when one adds charge conjugation to it? Since SO_3 is a complete group, the only extension of $Z_2 = \{1, C\}$ by SO_3 is the direct product $SO_3 \otimes Z_2$ (SU_2 is an extension of SO_3 by Z_2 which is not interesting for our physical problem). The group $SO_3 \otimes Z_2$ is isomorphic to O_3, the three-dimensional orthogonal group (rotations and symmetries). Hence the quantum numbers for the invariance under both, charge independence and charge conjugation, are the isotopic spin t and the isotopic parity η.

For example: what is the isotopic parity of the π-meson? The π^+, π^0, π^- states are the three components of an isovector, and they are invariant under the rotations around the third axis in isospace. The state π^0 is a proper state of C, the states π^+ and π^- are exchanged by C. Hence C is a symmetry through a plane which contains the third axis of isospace. The proper value c of $C\pi^0 = c\pi^0$ indicates the nature, vector or pseudo-vector, of the π-meson for isotopic parity. Since $c = 1$, the π-meson is an isovector (and not an isopseudo-vector) and its isoparity $\eta = -1$. Hence the isoparity of a system of n π-mesons is $(-1)^n$. For a state t, $t_3 = 0$, one has the relation $\eta = c(-1)^t$. For these relations, for applications to nucleon–antinucleon system, and for the representation of the inversion through the origin of isospace by the operator $U = i\tau_2 C$, see L. Michel, 1953, *N. Cim.*, **10**, 319; footnotes 1 to 8 of this paper are references to anterior works.*

However in these notes I tried to convince you that the strong coupling invariance group was not SO_3 but contains $U_1 \otimes U_2$ as subgroup. So, as a simple exercise we will look for the extensions of $Z_2\{1, C\}$ by $U_1 \otimes U_2$. We first determine the homomorphism g $Z_2 \xrightarrow{g} \mathscr{A}(U_1 \otimes U_2)$ of physical interest. We know that B and Y are changed into $-B$ and $-Y$ by charge conjugation, and we have seen above the relations between C and isospin transformations. Hence, if $(\alpha, \sigma) \in U_1 \otimes U_2$ (α is a phase, σ is a 2 by 2 unitary matrix),

$$g(C)(\alpha, \sigma) = (\bar{\alpha}, \bar{\sigma}) \tag{18}$$

where – means complex conjugate. The fixed elements $(\mathscr{C}(U_1 \otimes U_2))Z_2$ of the center $\mathscr{C}(U_1 \otimes U_2) = U_1 \otimes U_1$ are the square roots of the unit, as we have seen in Section 3, they form the group $Z_2 \otimes Z_2$. The Extensions are in a one-to-one correspondence with the elements of $H^2(Z_2, \mathscr{C}(V_1 \otimes V_2))$ and from Eq. IV.(48′) this group is

$$Z_2 \otimes Z_2 = {}_2(U_1 \otimes U_1), \tag{19}$$

* I am very grateful to T. D. Lee and C. N. Yang for the great advertisement they gave to this new quantum number: *N. Cim*, **13,** 749 (1956), their footnote (3); however I disagree with them for having changed the name "isotopic parity" into the unexpressive G-parity.

This explicit correspondence is:

$$\omega(U, U) = \epsilon \in Z_2 \otimes Z_2 \tag{19'}$$

where $U = i\tau_2 C$. That is, independently of the choice of the representative $k(U)$, the square of $k(U)$ is ϵ: we can say that it is the square of the inversion operator in isospace. We met a similar situation (for the square of CP) in VI of Section 6. The connection between the symmetry and the isoparity of a state composed of a particle and of its antiparticle depends on the value ϵ' of ϵ in the group representation describing one-particle states. Let us show it explicitly (without proof). The irreducible unitary representations of the group $K = U_1 \otimes U_2$ can be labelled ${}_{b,\,y}D_t$ (with $y + 2t$ even). According to the Frobenius theorem (quoted in Prof. Wigner's notes) we deduce easily the irreducible representations of any of our four extensions:

	representation of E	its restriction to K reduces to:
when $b \neq 0$ or $y \neq 0$	${}_{b,\,y}D_t$	${}_{b,\,y}D_t \otimes {}_{-b,\,-y}D_t$
when $b = 0 = y$	${}_{00}D_t{}^{\eta}$ where η = isoparity	${}_{00}D_t$

Let us consider a state of one particle of baryonic charge b, hypercharge y and isospin t, and of one of its anti-particle $(-b, -y)$. This state belongs to the representation $({}_{b,\,y}D_t)^2$. One easily computes the decomposition of this representation into a direct sum of irreducible representations of E. This yields

$({}_{b,\,y}D_t)^2 \supset$ for the part $y = b = 0$

symmetrical

$$\bigoplus_{\lambda=0}^{2t} {}_{00}D_{2t-\lambda}{}^{\lambda\epsilon'}$$

antisymmetrical

$$\bigoplus_{\lambda=0}^{2t} {}_{00}D_{2t-\lambda}{}^{-\lambda\epsilon'}$$

where ϵ' is the value of the matrix which represents $\omega(U, U) = \epsilon \in \mathscr{C}(E)$ in the representation ${}_{b,\,y}D_t$.

In the physical literature one always assumes $\epsilon' = 1$. However ϵ' could be ± 1 and it is a function of b and y (with $\epsilon' = 1$ if $b = y = 0$). This function should be measured experimentally.