THE SYMMETRY AND RENORMALIZATION GROUP FIXED POINTS OF QUARTIC HAMILTONIANS

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Abstract

This paper studies the number and the nature of the fixed points of the renormalization group for the ϕ^4 model, as used for instance in the Landau theory of second order phase transitions. It is shown that when it exists the stable fixed point is unique and a condition on its symmetry is given: it is often larger than the initial symmetry. Finally counter examples, with ν arbitrarily large, are given to the Dzyaloshinskii conjecture that there exist no stable fixed points when the Landau potential depends on more than $\nu = 3$ parameters.

INTRODUCTION

A great interest has risen from the study of the general n-vector model. Following the notations of Brézin's lectures [1] and with summation over repeated indices, the Hamiltonian density of this model is:

$$\mathcal{H}_n(x) = \frac{1}{2}\,\vec{\nabla}\phi_i(x)\cdot\vec{\nabla}\phi_i(x) + \frac{a}{2}\phi_i(x)\,\phi_i(x) + \frac{n^{4-d}}{4!}\,g_{ijkl}\,\phi_i(x)\,\phi_j(x)\,\phi_k(x)\,\phi_l(x)\,, \tag{1}$$

where the scalar field $\phi_i(x)$ has n components (the n values of its index) corresponding to inner degrees of freedom, while d is the space dimension. Brézin, Le Guillou and Zinn Justin [2] have written the renormalization group equations for this model and reached some general conclusion on the properties of stable fixed points, e. g. for $n < 4$ there is only one stable fixed point, that which is $O(n)$ invariant (see Eq. (4.18)).

Such a Hamiltonian density appears in the Landau theory [3] of second order phase transitions: the field values are the n component Landau order parameter. As a mean field theory one obtains good selection rules for the symmetry change in the transition but the wrong critical exponents; so one has to take account of the fluctuations at the critical points and applications to the Landau theory of the renormalization group techniques have been proposed [4-7], and many have been performed since.

The aim of this paper is different. Very few studies have been made up to now of the group covariance properties of the renormalization group techniques [8-10]. Here a more systematic study is made. As a result it is shown that, when it exists and for $n \neq 4$, the stable fixed point is unique; it often has a greater symmetry group than the starting polynomial: some sufficient conditions are given for this phenomenon to occur. In the applications studied in the literature few stable fixed points were found; so there is a conjecture (e.g., Dzyaloshinskii [11]) that there is a topological obstruction to their existence when the quartic part of the Hamiltonian depends on a number ν of parameters larger than three. This is wrong. The stable fixed point of each of a family of Hamiltonians with increasing, and arbitrarily large, values of ν are given explicitly: of course n has to be increasing with ν; in our example $n = 2^{\nu-2}\times 3$. The invariance groups corresponding to these examples form a family of groups which may have some interesting properties.

This paper studies a new example of the equivariant symmetric non-associative algebra obtained from group representations with a third degree invariant (e.g., the "d" algebra introduced by Gell-Mann [12] in his $SU(3)$ paper). Radicati and I [13] have already studied several families of these algebras relevant to physics and shown that directions of spontaneous symmetry breaking are given by idempotents of these algebras. Sattinger [14] has extended this result to bifurcation theory. Here fixed points are also determined by idempotents of the algebra. It seems also that this algebraic method may be more efficient for computing solutions in the case of practical applications.

1. THE RENORMALIZATION GROUP EQUATIONS

In Eq. (1) the g_{ijkl} are coupling constants of the term quartic in Φ. They are completely symmetrical in the four indices so their number is $N = \binom{n+3}{4}$. Their set can be considered as a vector in the N dimensional real vector space $\mathcal{T}_4$ of quartic polynomials with n variables. This vector g generally depends on several parameters. Let λ be the renormalization parameter. The renormalization equations are:

$$\frac{\lambda\, dg_{ijkl}(\lambda)}{d\lambda} = \beta_{ijkl}(g_{i'j'k'l'}) \,. \tag{1.1}$$

To simplify notations we will often use a multi-index α taking N values; then Eq. (1.1) reads:

$$\frac{\lambda\, dg_{\alpha}(\lambda)}{d\lambda} = \beta_{\alpha}(g_{\beta}) \,. \tag{1.2}$$

The fixed points of g^* of these equations satisfy:

$$\beta_{\alpha}(g^*) = 0 \,. \tag{1.3}$$

It can be shown that the matrix $\frac{\partial\beta_\alpha}{\partial g_\beta}$ has real eigenvalues which are related to the value of the critical exponents when the fixed point g^* is stable, that is when the $\frac{\partial\beta_\alpha}{\partial g_\beta}$ satisfy some partial positivity condition that we will make precise below.

The functions β_α are not known exactly, but the first few terms of an expansion in $\epsilon = 4 - d$ have been computed by Brézin et. al. [2]. Imposing the irreducibility condition:

$$g_{iikl} = \gamma(g)\, \delta_{kl} \,, \tag{1.4}$$

they have obtained

$$\begin{aligned}\beta_{ijkl}(\epsilon, g) = {} & -\epsilon\, g_{ijkl} + \frac{1}{2}\left(1 + \frac{\epsilon}{2}\right)\left\{ g_{ijpq}\, g_{pqkl} + g_{ikpq}\, g_{pqjl} + g_{ilpq}\, g_{pqjk} \right\} \\ & - \frac{1}{4}\left\{ g_{ipqr}\, g_{jprs}\, g_{klqs} + 5 \textit{ other terms obtained by permutations of } ijkl \right\} \\ & - \frac{1}{48}\left(1 + \frac{5\epsilon}{4}\right)\left\{ g_{ipqr}\, g_{apqr}\, g_{ajkl} + 3 \textit{ other terms obtained by} \right. \\ & \left. \textit{permutations of } ijkl \right\} .\end{aligned} \tag{1.5}$$

This can be written pictorially in a condensed notation: each g is "tetravalent", internal bonds express saturation of indices and average on the permutation of the indices corresponding to the four external bonds is assumed:

$$\beta(\epsilon,g) = -\,\epsilon(=g=) + \frac{3}{2}\left(1+\frac{\epsilon}{2}\right)(=g=g=) - \frac{3}{2}\left(=g\begin{array}{l}\diagup\, g\,- \\ \;\;\; \| \\ \diagdown\, g\,-\end{array}\right)$$
$$-\left(\frac{1}{12}\right)\left(1+\frac{5\epsilon}{4}\right)\left(-g\equiv g-g\equiv\right). \qquad (1.6)$$

For instance (and we introduce an even more condensed notation for this expression):

$$(=g=g=)_{ijkl} = \frac{1}{3}\left\{ g_{ijpq}\,g_{pqkl} + g_{ikpq}\,g_{pqjl} + g_{ilpq}\,g_{pqjk} \right\} = (g_{\vee}g)_{ijkl}\,. \qquad (1.7)$$

Wallace and Zia [15] have shown that, to this order β_{ijkl} is a gradient. Indeed:

$$\beta_{ijkl}(\epsilon,g) = \frac{d\Phi(\epsilon,g)}{dg_{ijkl}}\,,$$

with:

$$\Phi(\epsilon,g) = -\frac{1}{2}\,\epsilon\, g_{ijkl}g_{ijkl} + \frac{1}{2}\left(1+\frac{\epsilon}{2}\right)g_{ijkl}g_{klpq}g_{pqij} - \frac{3}{8}\,g_{ijkl}g_{ipqr}g_{jpqs}g_{rskl}$$
$$-\frac{1}{48}\left(1+\frac{5\epsilon}{4}\right)g_{ijkl}g_{ipqr}g_{apqr}g_{ajkl}\,. \qquad (1.8)$$

Equivalently:

$$\Phi(\epsilon,g) = -\frac{1}{2}\,\epsilon\left(g\equiv g\right) + \frac{1}{2}\left(1+\frac{\epsilon}{2}\right)\left(g\begin{array}{l}\diagup\!\!\diagup\, g \\ \;\;\;\;\| \\ \diagdown\!\!\diagdown\, g\end{array}\right)$$
$$-\frac{3}{8}\left(\begin{array}{l} g\,\diagdown \\ \| \;\;\; g=g \\ g\,\diagup \end{array}\right) - \frac{1}{48}\left(1+\frac{5\epsilon}{4}\right)\left(\begin{array}{ccc} g & \equiv & g \\ | & & | \\ g & \equiv & g\end{array}\right). \qquad (1.9)$$

The equation for stable fixed points becomes:

$$\frac{d\Phi(g^*)}{dg} = 0\,, \qquad (1.10.a)$$

$$\frac{d^2\Phi(g^*)}{dg^2} \geq 0\,. \qquad (1.10.b)$$

The first one expresses that g^* is an extremum of Φ; it is stable when it satisfies the second one, i.e., when it is minimum. However, the second condition, as it is written in Eq. (1.10.b) is too strong. We will make precise the exact condition in Section 3, after explaining in section 2 some properties of the action of the orthogonal group $O(n)$ on the vector space $\mathcal{T}_4$ of quartic polynomials. In Section 3 we also study when the fixed points can be found by an expansion in ϵ from the solutions in the first order in ϵ. These solutions are extrema of the polynomial:

$$\Phi^{(1)}(g) = -\frac{\epsilon}{2}\,(g,g) + \frac{1}{2}(g_{\vee}g,g)\,. \qquad (1.11)$$

where

$$(g, g) = g_{ijkl}g_{ijkl} , \tag{1.12}$$

and the symbol $g_\vee g$ has been defined in Eq. (1.7). The symbol $\vee$ defines an abelian algebra on $\mathcal{T}^4$ by:

$$g_\vee h = \frac{1}{2}\left[(g + h)_\vee(g + h) - g_\vee g - h_\vee h\right] . \tag{1.13}$$

It has $O(n)$ as the group of automorphisms. This algebra will be studied in Section 4. As said in the introduction, the extrema of $\Phi^{(1)}$ are idempotents of this algebra. Indeed:

$$\frac{d\Phi^{(1)}(g^*)}{dg} = 0 \iff g^*{}_\vee g^* = \frac{2}{3}\,\epsilon\, g^* . \tag{1.14}$$

In Section 5 we will study general properties of the extrema of $\Phi^{(1)}$. The last section, Section 6, will study the family $G_{s_1, \ldots, s_k}$ of irreducible discrete subgroups of $O(n)$, their invariants, and give the counter examples to the Dzyaloshinskii conjecture.

2. GROUP ACTIONS -- THE ACTION OF O(N) ON REAL N-VARIABLE POLYNOMIALS.

We first recall some basic concepts and results on group action. When a group G acts on a set M, the set of transforms of $m \in M$ is denoted by $G{\cdot}m$ and is called the *G-orbit* of m. The elements of g which leave m invariant: $g{\cdot}m = m$, form a subgroup G_m of G which is called the *isotropy* group of m. Note that:

$$G_{g\cdot m} = g\, G_m\, g^{-1} . \tag{2.1}$$

So the set of isotropy groups of an orbit form a conjugation class of subgroups of G. We denote by $[H]$ the conjugation class of $H \subseteq G$. G orbits with the same set $[H]$ of isotropy groups form an equivalence class that we denote by $[G{:}H]$. The set of all points with isotropy groups in a given conjugation class form a *stratum*: this is also the union of all orbits of an equivalence class.

We denote by M^H the subset of M whose elements are invariant under H, i. e.

$$M^H = \{m \in M,\, G_m \supseteq H\} . \tag{2.2}$$

The action of G on the set M defines an action on $\mathcal{T}(M)$, the set of subsets of M. Let $A \subset M$, i. e., $A \in \mathcal{T}(M)$. The *centralizer* $C_G(A)$ of A is the set of elements of G which leave fixed every element of A; it is a subgroup $C_G(A) \subseteq G$:

$$C_G(A) = \bigcap_{m\in A} G_m \ . \tag{2.3}$$

The *normalizer* $N_G(A)$ of A is the set of elements of G which transform A into itself. It is a subgroup of G which contains the centralizer $C_G(A)$. For instance in the action of G on itself by conjugation, $x \rightarrow g\,x\,g^{-1}$, $C_G(A)$ is the subset of elements of G which commute with every element of A; e. g. $C_G(G)$ is the center of G. If $H \subseteq G$ and $N_G(H) = G$, H is called an invariant subgroup of G and we denote it by $H \lhd G$. More generally, $N_G(H)$, the normalizer of H in G, is the largest subgroup of G which has H as an invariant subgroup. For a general G-action, we remark that:

$$C_G(A) \lhd N_G(A) \ . \tag{2.4}$$

(Indeed, let $c \in C_G(A)$, $n \in N_G(A)$; for any $a \in A$, $n(c(n^{-1}\cdot a)) = n\,n^{-1}\,a = a$). So from the definition of the normalizer of $C_G(A)$,

$$N_G(A) \subseteq N_G(C_G(A)) \ . \tag{2.5}$$

We study a situation when the equality holds. We first remark from Eq. (2.1) and Eq. (2.2) that:

$$H \subseteq C_G(M^H) \ , \tag{2.6}$$

but the equality is not necessary. In any case:

$$M^H = M^{C_G(M^H)} \ . \tag{2.7}$$

We now prove that Eq. (2.5) is always an equality when A is of the type M^H:

$$N_G(M^H) = N_G(C_G(M^H)) \ . \tag{2.8}$$

With Eq. (2.5) we need only to prove the inequality $\supseteq$. For every $c \in C_G(M^H)$, $n \in N_G(C_G(M^H))$ and $m \in M^H$, we have $n^{-1}cn\cdot m = m$, i.e., $cn\cdot m = n\cdot m$ so $n\cdot m \in M^{C_G(M^H)} = M^H$ and $n \in N_G(M^H)$.

Finally we recall a non-trivial result. If G is compact, there is a partial order on the set of conjugation classes of closed subgroups: $[H_1] \leq [H_2]$ if a group of $[H_1]$ is a subgroup of a group of $[H_2]$. (This does not define an order relation for general groups). Then Montgomery and Yang [16] have proven for smooth (infinitely differentiable) actions of compact groups that: Among the set of conjugation classes of isotropy groups there is a smallest element and the corresponding stratum is open dense.

We now study the linear action of $O(n)$, the n dimensional orthogonal group on the vector space $\mathcal{T}$ of n variable polynomials. $O(n)$ is the group of $n\times n$ orthogonal matrices, $u^T = u^{-1}$ acting on the real n-dimensional vector space $\mathcal{V}_n$. We denote by ϕ one of its vectors, and by ϕ_i, $1\leq i\leq n$ its coordinates. $O(n)$ also acts on the functions defined on $\mathcal{V}_n$:

$$[u \, . \, f\,](\phi) = f(u^{-1}\phi) \; . \tag{2.9}$$

This action transforms polynomials into polynomials, preserving the degree of homogeneous polynomials. Hence $\mathcal{T}$ is a direct sum of the $O(n)$ invariant vector spaces $\mathcal{T}_k$ containing all homogeneous n-variable polynomials of degree k.

$$\mathcal{T} = \bigoplus_{k=0}^{?} \mathcal{T}_k \quad , \quad \dim \mathcal{T}_k = \binom{n+k-1}{k} \tag{2.10}$$

Note that $\mathcal{T}_0$ is the set of real numbers and that $\mathcal{T}_1 = \mathcal{V}_n$. For $k \geq 2$, the representation of $O(n)$ on the space $\mathcal{T}_k$ is reducible. Indeed, in $\mathcal{T}_2$ there is, up to a factor, an invariant polynomial, the orthogonal product

$$(\phi, \phi) = \phi_i \, \phi_i \; . \tag{2.11}$$

So every quadratic form can be decomposed into two irreducible components.

$$q(\phi) = q_{ij} \, \phi_i \, \phi_j = \left\{ q_{ij} - \frac{q_{kk}\delta_{ij}}{n} \right\} \phi_i \, \phi_j + \frac{q_{kk}(\phi,\phi)}{n} \; . \tag{2.12}$$

Using the Laplacian

$$\triangle = \frac{\partial}{\partial\phi_i} \frac{\partial}{\partial\phi_i} = \nabla_i \, \nabla_i \; , \tag{2.13}$$

we note that

$$\triangle \, q(\phi) = 2 \, q_{ii} = 2 \, \mathrm{Tr} \, q \; . \tag{2.14}$$

More generally all harmonic polynomials, $\triangle \, p(\phi) = 0$, homogeneous of degree k form the space $\mathcal{T}_k^{(k)}$ of an irreducible representation of $O(n)$. In Dynkin labeling this representation is denoted by $(k, 0, 0, 0, \ldots, 0)$ with $\rho - 1$ zeros, where ρ is the rank of $O(n) = E(n/2)$, where $E(n/2)$ is the largest integer $\leq n/2$. We shall simply denote this representation abstractly by (k) and its carrier space by $\mathcal{E}^{(k)}$.

For the readers who still prefer indices, the polynomials $g(\phi) = g_{ijkl\ldots} \; \phi_i \, \phi_j \, \phi_k \, \phi_l \ldots$ of $\mathcal{T}_k^{(k)}$ are tensors of $O(n)$ completely symmetrical in their k indices with partial trace $g_{iiklm\ldots} = 0$.

We can also say that $\triangle$ is a surjective linear map

$$\mathcal{T}_k \xrightarrow{\triangle} \mathcal{T}_{k-2}\ , \tag{2.15}$$

whose kernel is $\mathcal{T}_k^{(k)}$. So, from Eq. (2.8) and Eq. (2.13) we deduce:

$$\dim \mathcal{E}^{(k)} = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \tag{2.16}$$

For instance:

$$\dim \mathcal{E}^{(2)} = (n+2)(n-1)/2 \quad , \dim \mathcal{E}^{(4)} = (n+6)(n+1)n(n-1)/24\ . \tag{2.17}$$

We are especially interested in the case $k = 4$. We denote by $U(u)$ the operator representing the action of $u \in O(n)$ on $\mathcal{T}_4$. The decomposition of U into irreducible representations yields for the carrier space:

$$\mathcal{T}_4 = \mathcal{T}_4^{(4)} \oplus \mathcal{T}_4^{(2)} \oplus \mathcal{T}_4^{(0)} \quad ; \qquad g = g^{(4)} + g^{(2)} + g^{(0)}\ . \tag{2.18}$$

To compute the irreducible components of $g \in \mathcal{T}_4$, we use:

$$\triangle\, a\, b = (\triangle\, a\,)\, b + 2\,(\nabla_i\, a\,)\,(\nabla_i\, b\,) + a \triangle b\ , \tag{2.19}$$

and, for a homogeneous polynomial of degree k,

$$\phi_i\, \nabla_i\, p_k(\phi) = k\, p_k(\phi)\ . \tag{2.20}$$

Moreover, $g^{(0)}$ is proportional to $(\phi,\ \phi)^2$, $g^{(2)}$ contains $(\phi,\ \phi)$ as a factor and $\triangle \left\{(\phi,\phi)^{-1}\, g^{(2)}\right\} = 0$. So we find, computing $\triangle g$ and $\triangle\triangle g$,

$$g^{(0)} = \frac{1}{8n(n+2)}\,(\phi,\ \phi)^2\, \triangle\triangle g \quad , \quad g^{(2)} = \frac{1}{2(n+4)}\,(\phi,\ \phi)\left\{\triangle g - \frac{(\phi,\ \phi)}{2n}\,\triangle\triangle g\right\}. \tag{2.21}$$

We shall set:

$$(\phi,\ \phi)^2 = s = s_{ijkl}\phi_i\, \phi_j\, \phi_k\, \phi_l \tag{2.22.a}$$

with

$$s_{ijkl} = \frac{1}{3}\left\{\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right\}\ . \tag{2.22.b}$$

We have already introduced in Eq. (1.12) the $O(n)$ invariant scalar product

$$(g, h) = g_{ijkl} h_{ijkl} .$$

Then

$$(s, s) = \frac{n(n+2)}{3} , \tag{2.23}$$

and from Eq. (2.19) and $g^{(0)} = s\,(s, g)\,(s, s)^{-1}$ one obtains

$$(s, g) = \frac{1}{24} \triangle \triangle g . \tag{2.24}$$

We will find it convenient to use the shorthand notation

$$\gamma(g) = \frac{1}{n}\,(s, g) \tag{2.25.a}$$

so

$$g^{(0)} = \frac{3\,\gamma(g)}{n+2}\, s . \tag{2.25.b}$$

We will be led to study the polynomials of $\mathcal{T}_4$ whose isotropy groups are irreducible; i.e., their n dimensional representation defined as subgroups of $O(n)$ is irreducible. From the remark that for such a group of the form

$$G = O(n)_g = O(n)_{g^{(4)}} \cap O(n)_{g^{(2)}}$$

and that all isotropy groups of non-vanishing polynomials of $\mathcal{E}^{(2)}$ must be reducible we deduce the

Lemma 2.1: $G = O(n)_g$ irreducible implies $g^{(2)} = 0$, that is $g \in \mathcal{T}_4^{(4)} \oplus \mathcal{T}_4^{(0)}$; With the shorthand introduced in Eq. (2.25.a), and from Eq. (2.21) and Eq. (2.25.a) we get:

$$g^{(2)} = 0 \Longleftrightarrow \frac{1}{12} \triangle(g) = \gamma(g)\,(\phi, \phi) \tag{2.26}$$

or with indices, this is identical to Eq. (1.4).

Let us consider an example (corresponding to the Ising model)

$$c = \sum_{i=1}^{n} \phi_i^4 \tag{2.27.a}$$

so

$$c_{ijkl} = \begin{cases} 1 & for\ \ i=j=k=l \\ 0 & otherwise \end{cases} . \tag{2.27.b}$$

then

$$(c, c) = (s, c) = n \ . \tag{2.28}$$

The isotropy group of c has been labelled B_n by Coxeter (see e. g. [17]). It is the semi-direct product of the abelian group of $(n \times n)$ diagonal matrices with $\pm$ 1 as entries and $(n \times n)$ permutation matrices (i.e., zero everywhere except one element in each row and column which is one). The order of this group (i. e. its number of elements) is

$$|B_n| = 2^n \, n! \ . \tag{2.29}$$

The group B_n is the symmetry group of the hypercube whose vertices have for coordinates $\pm$ 1. In Schönflies notation $B_2 = C_{4v}$, $B_3 = O_h$. For every n, $\mathcal{T}_4^{B_n}$ is generated by s and c:

$$\mathcal{T}_4^{B_n} = \{ \alpha' s + \beta' c \} \ . \tag{2.30}$$

Moreover, for $n = 2$ and $n = 3$, every polynomial $g \in \mathcal{T}_4$ whose isotropy group is irreducible belongs to the stratum of $[B_n]$ or is a multiple of s ; so by an orthogonal transformation it can be brought into a linear combination:

$$\alpha s + \beta c \ \in \mathcal{T}_4^{B_n} \ .$$

It happens that for $n = 2$ the condition (2.26) is also sufficient for the irreducibility of the isotropy group. Indeed:

$$n = 2 \ , \ \mathcal{T}_4^{(4)} = \left\{ \alpha \ \mathrm{Re}(\phi_1 + i \, \phi_2)^4 + \beta \ \mathrm{Im}(\phi_1 + i \, \phi_2)^4 \right\} \tag{2.31}$$

and the isotropy group of any non-zero polynomial of $\mathcal{T}_4^{(4)}$ is conjugated of $C_{4v} = B_2$.

The irreducibility of $G = O(n)_g$ requires that any quadratic form one can form with g and $\triangle$ (equivalently from g_{ijkl} by contraction of indices) is a multiple of $(\phi, \phi) = \delta_{ij} \phi_i \phi_j$; e. g. , using Eq. (2.22.a)

$$G \ irreducible \ \rightarrow g_{ipqr} \, g_{ipqr} = \frac{1}{n} \, (g, g) \delta_{ij} \ . \tag{2.32.a}$$

Note that:

$$g_{ipqr} \, g_{jpqr} \, \phi_i \, \phi_j = \frac{1}{2^9 \, 3^2} \left\{ \triangle^3 g^2 - 6 \, (\triangle g) \, (\triangle^2 g) - 12 \, (\nabla_i \triangle g) \, (\nabla_i \triangle g) \right.$$

$$\left. - 24 \, (\nabla_i \nabla_j g) \, (\nabla_i \nabla_j \triangle g) \right\} . \tag{2.32.b}$$

Both of the necessary conditions Eq. (2.26) and Eq. (2.32.a) for the irreducibility of G are not sufficient and they are inequivalent as shown by the two following examples taken for $n = 3$; see [18] Table 2 for the determination of their isotropy group, which is irreducible:

$$G = D_{3d}\ , \qquad g(\phi) = \left(3\phi_1^{\ 2} - \phi_2^{\ 2}\right)\phi_2\phi_3\ , \qquad \triangle\, g = 0 = g_{iikl}\ , \tag{2.33.a}$$

$$g_{ipqr}g_{jpqr}\ \phi_i\ \phi_j = \frac{1}{8}\ \left(3\phi_1^{\ 2} + 3\phi_2^{\ 2} + 2\phi_3^{\ 2}\ \right)\ . \tag{2.33.b}$$

$$G{=}C_{4h}\ , \qquad g(\phi) = 2\ \left(\phi_1^{\ 2} - \phi_2^{\ 2}\right)\phi_1\phi_2 + \phi_3^{\ 4}, \quad g_{iikl}\phi_k\phi_l{=}\phi_3^{\ 2}{=}\tfrac{1}{12}\triangle g\ , \tag{2.33.c}$$
$$g_{ipqr}\ g_{jpqr}\ \phi_i\ \phi_j = (\phi,\ \phi)\ . \tag{2.33.d}$$

It would be interesting to have a simple set of sufficient conditions on g for the irreducibility of the isotropy group $G = O(n)_g$. As we have seen Eq. (2.26) is sufficient for $n = 2$ and I conjecture that Eq. (2.26) and Eq. (2.32.a) are sufficient for $n = 3$. More generally, for harmonic polynomials $\triangle\ g = 0$, the set of necessary conditions $\triangle^{2k-1}g^k = 0$ for $1 \le k \le n-1$ might be sufficient. For $n = 4$ an exhaustive study has been done in [19]: including that of $O(4)$, there are 15 strata corresponding to irreducible isotropy groups, instead of 2 for $n = 2$ or 3. For $n = 4$, the maximal dimension of a subspace $\mathcal{T}_4^{\ G}$ for G irreducible is 11; all these subspaces are included in $\mathcal{T}_4^{\ (4)} \oplus \mathcal{T}_4^{\ (0)}$ of dimension 26. No similar results are known for higher n.

We end this section with some very important remarks: The isotropy groups of polynomials $g \in \mathcal{T}_4$ are closed subgroups of $O(n)$ and for any mathematical discussion such as those in this paper, one has to consider the isotropy group of $O(n)_g$, i.e., the exact invariance group of g. However this is generally not $\mathcal{G}$, the physical symmetry group of the Hamiltonian defined by g. Indeed $\mathcal{G}$ acts on $\mathcal{V}_n$ (the space of ϕ_i) through an orthogonal representation $\mathcal{G} \xrightarrow{V} O(n)$ of image $V(\mathcal{G})$. For instance, in the Landau theory of second order phase transition $\mathcal{G}$ is the space group (the crystallographic group) of the crystal and either the image $V(\mathcal{G})$ is a finite subgroup of $O(n)$, or it is an infinite discrete subgroup of $O(n)$, so its closure is a compact Lie subgroup of $O(n)$ of positive dimension. In general, because we consider only polynomials of degree 4, any $V(\mathcal{G})$ invariant polynomial of $\mathcal{T}_4$ will have an isotropy group G larger than $V(\mathcal{G})$.

To conclude we emphasize that the symmetry group $\mathcal{G}$ and its image $V(\mathcal{G})$ are given by the physics of the problem. They determine the general quartic term $g(\phi)$ of the Hamiltonian. In general $g(\phi)$ is a function of several parameters and $g(\phi) \in \mathcal{T}_4^{\ G}$ where, in general $G \supseteq V(\mathcal{G})$.

3. THE DEFINITION OF STABLE FIXED POINTS

As we have seen, the isotropy group $O(n)_g = G$ of the quartic term $g(\phi)$ of the Hamiltonian density (1) is the symmetry group of the Hamiltonian. It is a closed subgroup of $O(n)$, hence it is a Lie group of dimension $m < \binom{n}{2}$; in the particular case $m = 0$, G is a finite group.

Physically we need only to consider the case of irreducible subgroups $G \subset O(n)$. Indeed, when G is reducible, by choosing suitable linear combinations ϕ'_c of the components of the fields $\phi_k(x)$, one can split the Hamiltonian into a sum of non-interacting but similar Hamiltonians, each one with a field of $n^{(\alpha)}$ components, (with $\sum_\alpha n^{(\alpha)} = n$) and with isotropy subgroups G_α, irreducible subgroups of $O(n^{(\alpha)})$. So one has only to study the Hamiltonians with an irreducible symmetry group.

As explained in Section 1, the deduction of the renormalization equations (1.5) requires only the weaker hypothesis Eq. (1.4), equivalent to $g \in \mathcal{T}_4^{(0)} \otimes \mathcal{T}_4^{(4)}$. However, the renormalization equations (1.1) are equivariant for the whole action of $O(n)$ on $\mathcal{T}_4$.

$$\forall\, u \in O(n) \quad , \quad U(u)_{\alpha\beta}\beta_\beta(g_{\alpha'}) = \beta_\alpha\,(U_{\alpha'\beta'}g_{\beta'}) \; . \tag{3.1}$$

The quartic term $g(\phi)$ is the value of $g(\lambda)$ for a fixed value λ_0 of λ (e. g. $\lambda_0 = 1$). If G is its isotropy group, Eq. (3.1) implies that for every value of λ, $g(\lambda)$ is invariant under G. We can also say that the trajectory of $g(\lambda)$ stays in the space $\mathcal{T}_4^G$. More precisely, the isotropy group of $g(\lambda)$ has to be independent of λ in a neighborhood of λ_0 when $\frac{dg}{d\lambda} \neq 0$ and it may become larger at the fixed points g^*. The physical requirement of stability is the positivity of the restriction of the matrix $\frac{\partial \beta_\alpha}{\partial g_\beta}$ to the subspace $\mathcal{T}_4^G$.

This is expressed by the equation for g^* , the stable fixed point of g of isotropy group G:

$$\beta_\alpha(g^*) = 0 \; , \tag{3.2.a}$$

$$\frac{\partial \beta_\alpha}{\partial g_\beta}\Big|_{\mathcal{T}_4^G} \geq 0 \; . \tag{3.2.b}$$

Since G is the isotropy group of $g(\lambda_0) \in \mathcal{T}_4^G$, G is also the centralizer $C_{O(n)}\,(\mathcal{T}_4^G)$ (equality in Eq. (2.6)); so from Eq. (2.8), the normalizer of $\mathcal{T}_4^G$ is the normalizer of G in $O(n)$

$$N_{O(n)}(\mathcal{T}_4^G) = N_{O(n)}(G) \; . \tag{3.3}$$

So $N_{O(n)}(G)$ acts on $\mathcal{T}_4^G$ through the quotient group

$$Q(G) = N_{O(n)}(G)/G \ , \tag{3.4}$$

which acts effectively (i.e. no element of $Q(G)$ different from the identity leaves fixed every point of $\mathcal{T}_4^{\,G}$). When $Q(G)$ is not trivial, from a solution g^* of Eq. (3.2.a), by the action of $Q(G)$ on $\mathcal{T}_4^{\,G}$ one builds in general an orbit of solutions. This was already noted in [8] and [10]. If $Q(G)$ is a Lie group of positive dimension n' , that is the dimension of the orbit $Q(G)(g^*)$ of g^* and the tangent plane at g^* to this orbit is in the kernel of $\frac{\partial \beta_\alpha}{\partial g_\beta}\Big|_{\mathcal{T}_4^G}$. We will show later that the stable fixed point, when it exists, is unique.

We have now to take into account the fact that $\beta(g)$ is known only through an ϵ expansion $\beta(\epsilon,g)$, so the solutions of $\beta(\epsilon,g^*) = 0$ define g^* as a function of ϵ. Only the solutions $g^*(\epsilon) \to 0$ when $\epsilon \to 0$ are physically relevant. It is difficult to study the convergence of the ϵ expansion for $\epsilon = 4 - d = 1$ (generally it is not convergent; it is an asymptotic expansion). We completely ignore this problem here.

Assume the expansion

$$g^*(\epsilon) = \epsilon \sum_{k=0}^{\infty} \epsilon^k \tilde{g}_k \ .$$

The first term $\tilde{g}_0$ is defined by a non-linear equation.

$$0 = \tilde{\beta}(\epsilon,\tilde{g}_0) \longleftrightarrow -\tilde{g}_0 + \frac{3}{2}\,\tilde{g}_{0\vee}\tilde{g}_0 = 0 \ . \tag{3.5}$$

The other terms are defined by a system of linear equations. For instance, that for $\tilde{g}_1$ reads:

$$\epsilon^{-2}\,\frac{d\beta}{dg}(\epsilon,\tilde{g}_0)\,\tilde{g}_1 = \frac{1}{2}\,\tilde{g}_0 \ . \tag{3.6}$$

This solution is unique if

$$\frac{d\tilde{\beta}}{dg}(\epsilon,\tilde{g}_0)\Big|_{\mathcal{T}_4^G}$$

is invertible. When this is not the case, one says that there is a bifurcation: indeed, in general, new solutions appear. We shall see that this is the case for $n = 4$.

In this paper we consider the cases without bifurcation. Then the solutions found to the first order of ϵ can be computed to the next orders. For some range of $\epsilon \geq 0$, $g^*(\epsilon)$ keeps the same isotropy group and does not change its stability character. We are only interested in these properties of the solutions and not in their precise location. We need only to study the extrema of $\Phi^{(1)}(g)$ defined in Eq. (1.14).

To simplify notations we will drop the index $\tilde{g}_0$ and use the shorthand notation

$$H(g^*) = \frac{d^2\Phi^{(1)}}{dg^2}(g^*) \tag{3.7}$$

for the Hessian of a fixed point. We can now reformulate the simplified mathematical problem we have to solve for finding stable points (from first order in the ϵ expansion) in the renormalization of Landau theory of second order phase transitions.

One is given a positive quartic polynomial on $\mathcal{V}_n$:

$$0 \neq \phi \in \mathcal{V}_n \ , \quad g(\phi) > 0 \quad , \quad g(\lambda\phi) = \lambda^4\, g(\phi) \tag{3.8}$$

with an irreducible isotropy group $G = O(n)_g$. Find the extrema g^* of $\Phi^{(1)}(g)$ (defined in Eq. (1.11)) on $\mathcal{T}_4{}^G$. They are defined by:

$$g^* = \epsilon\, \tilde{g} \ , \quad \tilde{g} \vee \tilde{g} = \frac{2}{3}\tilde{g} \ , \quad \tilde{g} \in \mathcal{T}_4{}^G \ . \tag{3.9}$$

Such an extremum is a stable fixed point if and only if the Hessian at g^* is strictly positive.

$$\epsilon > 0 \qquad H(g^*)\big|_{\mathcal{T}_4{}^G} > 0 \ . \tag{3.10}$$

Moreover this stable fixed point is physically acceptable for the given polynomial g if it is in the attraction basin of g^*, i.e., $\phi^{(1)}(g)$ never increases from g to g^* on the integral line of the gradient field. Finally one has also to verify that $g^*(\phi) > 0$ for $\epsilon \neq 0$.

4. THE $O(N)$ COVARIANT SYMMETRIC ALGEBRA $\mathcal{T}_4$ AND ITS IDEMPOTENTS.

Equation (3.9) means that the fixed points g^* are idempotents of the algebra defined by the symbol $\vee$. In this section we study some properties of this algebra similar to those studied in [12]. For the linear representation $O(n)$, $u \mapsto U(u)$ on $\mathcal{T}_4$, the expression:

$$\Theta(g) = g_{ijkl}\, g_{klpq}\, g_{pqij} = \Theta(U(u)g) \tag{4.1}$$

is a third degree polynomial invariant; by polarization we define a trilinear form

$$\tilde{\Theta}(u,v,w) = \frac{1}{6}\left[\Theta(u+v+w) - \Theta(u+v) - \Theta(v+w) - \Theta(w+u) + \Theta(u) + \Theta(v) + \Theta(w)\right] \tag{4.2.a}$$

$$= \left(u_{ijkl}\, v_{klpq}\, w_{pqij} + u_{ijkl}\, v_{klpq}\, w_{pqij}\right) , \tag{4.2.b}$$

which is invariant under any permutation of its three arguments and which is $O(n)$ invariant. If we fix u and v, $\tilde{\Theta}(u,v,w)$ is a linear form in w; so with the $O(n)$ invariant scalar product (g,g) defined in Eq. (1.12), it can be written as the scalar product of w by a fixed vector that we denote by $u_\vee v$:

$$\tilde{\Theta}(u,v,w) = (u_\vee v,\ w) \ . \tag{4.3}$$

The correspondence from the pair u, v to $u_\vee v$ is a linear map:

$$\mathcal{T}_4 \otimes \mathcal{T}_4 \xrightarrow{\vee} \mathcal{T}_4 \ ,$$

which defines an algebra on $\mathcal{T}_4$:

$$(u_\vee v)_{ijkl} = \frac{1}{6} \left\{ u_{ijpq}\, v_{pqkl} + v_{ijpq}\, u_{pqkl} + u_{ikpq}\, v_{pqjl} + v_{ikpq}\, u_{pqjl} + u_{ilpq}\, v_{pqjk} + v_{ilpq}\, u_{pqjk} \right\} . \tag{4.4}$$

The symmetry of $\tilde{\Theta}$ in its argument implies:

$$(u_\vee v,\ w) = (v,\ u_\vee w) = (u,\ v_\vee w) = etc \ . \ . \ . \tag{4.5}$$

For each $g \in \mathcal{T}_4$ we can define a linear operator $D_g \in \mathcal{L}(\mathcal{T}_4)$, on $\mathcal{T}_4$ by:

$$D_g\, w = g_\vee w \ .$$

The first equality of Eq. (4.5) shows that D_g is a symmetric operator

$$D_g^T = D_g \ , \tag{4.6}$$

and from the $O(n)$ invarinace of $\Theta(g)$, one proves that the linear map

$$\mathcal{T}_4 \xrightarrow{D} \mathcal{L}(\mathcal{T}_4) \tag{4.7}$$

is $O(n)$ covariant:

$$D_{U(u)g} = U(u)\, D_g\, U(u)^{-1} \ . \tag{4.8}$$

The Hessian $H(g)$ is simply:

$$\frac{d^2\Phi(g)}{dg^2} = H(g) = -\epsilon I + 3\, D_g \ . \tag{4.9}$$

To compute easily with this algebra on fourth degree homogeneous polynomials $g(\phi)$ one can introduce the $n \times n$ matrix

$$T_g(\phi)_{ij} = \frac{1}{12} \frac{\partial^2 g}{\partial \phi_i \, \partial \phi_j} , \tag{4.10}$$

which is quadratic in ϕ. Note that:

$$g(\phi) = T_g(\phi)_{ij} \, \phi_i \, \phi_j , \tag{4.11}$$

$$T_g(\phi)_{ii} = \mathrm{Tr} \, T_g(\phi) = \frac{1}{12} \triangle \, g(\phi) . \tag{4.12}$$

The algebra product of g and h is simply:

$$g \vee h = \mathrm{Tr} \, T_g T_h . \tag{4.13}$$

As an example we easily compute for $s(\phi) = (\phi, \phi)^2$ (see Eq. (2.22.a)

$$T_s(\phi)_{ij} = \frac{1}{3} \left(\delta_{ij}(\phi, \phi) + 2 \, \phi_i \, \phi_j \right) \tag{4.14}$$

so

$$s \vee s = \frac{n+8}{9} \, s \tag{4.15}$$

and more generally with

$$g = g^{(0)} + g^{(2)} + g^{(4)} , \tag{4.16}$$

(see Eq. (2.18) for this $O(n)$ covariant decomposition)

$$D_s \, g = s \vee g = \frac{n+8}{9} \, g^{(0)} + \frac{n+16}{18} \, g^{(2)} + \frac{2}{3} \, g^{(4)} . \tag{4.17}$$

Eq. (4.15) implies that $\Phi^{(1)}(g)$ has an extremum for

$$s^* = \epsilon \, \bar{s} \qquad , \qquad \bar{s} = \frac{6}{n+8} \, s . \tag{4.18}$$

and from Eq. (4.17) we obtain its Hessian

$$H(s^*) = \epsilon \left(P^{(0)} + \frac{6}{n+8} \, P^{(2)} + \frac{4-n}{n+8} \, P^{(4)} \right) \tag{4.19}$$

where $P^{(x)}$ is the orthogonal projector on $\mathcal{T}_4^{(x)}$, $x = 0, 2, 4$. Since $\mathcal{T}_4^{O(n)} = \mathcal{T}_4^{(0)}$, the fixed point s^* *is a stable fixed point for any* n, a well known result. For $n < 4$, s^*

is even a minimum on the whole space $\mathcal{T}_4$. We can prove the (purely mathematical) result, independent of the value of n:

Theorem 4.1: $\Phi^{(1)}(g)$ has no minimum $\neq s^*$ on $\mathcal{T}_4$ or on $\mathcal{T}_4^{(4)} + \mathcal{T}_4^{(0)}$.
We first establish some general relations. From Eq. (4.5) and Eq. (4.17):

$$\gamma(g_{\surd} g) n = (s, g_{\surd} g) = (s_{\surd} g, g) = $$
$$= \frac{n+8}{9}\left(g^{(0)}, g^{(0)}\right) + \frac{n+16}{18}\left(g^{(2)}, g^{(2)}\right) + \frac{2}{3}\left(g^{(4)}, g^{(4)}\right), \tag{4.20.a}$$

$$\gamma(g_{\surd} g) n = \frac{2}{3}(g, g) + \frac{n+2}{9}\left(g^{(0)}, g^{(0)}\right) + \frac{n+4}{18}\left(g^{(2)}, g^{(2)}\right). \tag{4.20.b}$$

For an extremum $g^* = \epsilon \bar{g}$ not collinear to s, (i.e. $\left(g^{(2)}, g^{(2)}\right) + \left(g^{(4)}, g^{(4)}\right) > 0$) with the use of Eq. (3.5) and Eq. (2.25.b) we obtain $\gamma(\bar{g})\left\{\gamma(\bar{s}) - \gamma(\bar{g})\right\} > 0$, i.e.,

$$0 < \gamma(\bar{g}) < \gamma(\bar{s}) = 2\,\frac{n+2}{n+8}. \tag{4.21}$$

The Hessian at g^* is, from Eq. (4.9):

$$H(g^*) = \epsilon\left\{3 D_{\bar{g}}(0) - I + 3 D_{\bar{g}}(2) + 3 D_{\bar{g}}(4)\right\}$$
$$= \epsilon\,\frac{\gamma(\bar{g})}{\gamma(\bar{s})}\,H(\bar{s}) - \epsilon\left\{\left(1 - \frac{\gamma(\bar{g})}{\gamma(\bar{s})}\right) I - 3 D_{\bar{g}}(2) - 3 D_{\bar{g}}(4)\right\}. \tag{4.22}$$

The trace of the product of 2 positive operators is > 0; for $n \geq 4$ we will show that $\mathrm{Tr}\, H(\epsilon\bar{g})P^{(4)} < 0$ so that $H(\epsilon\bar{g})P^{(4)}$ is not a positive matrix and this will prove the theorem for $n \geq 4$.

We just remark that $\mathrm{Tr}\, D_g$ and $\mathrm{Tr}\,\{D_g P^{(4)}\}$ are linear forms on $\mathcal{T}_4$; moreover Eq. (4.8) and the fact that $\mathcal{T}_4^{(4)}$ is an invariant space for $O(n)$ imply that these linear forms are $O(n)$ invariant. So they must be proportional to (s, g). This requires $\mathrm{Tr}\,\{D_{\bar{g}^{(2)}} P^{(4)}\} = 0$ and $\mathrm{Tr}\,\{D_{\bar{g}^{(4)}} P^{(4)}\} = 0$; so Eq. (4.22) with Eq. (4.19) and Eq. (4.21) imply that $\mathrm{Tr}\,\{H(g^*)P^{(4)}\} < 0$ for $n \geq 4$. We did not need to compute the proportionality factors K' and K'' in $\mathrm{Tr}\, D_g = K'(s,g)$, $\mathrm{Tr}\,\{D_g P^{(4)}\} = K''(s,g)$. They can be computed from $g = s$ and the use of Eq. (4.19) and Eq. (2.17). One finds

$$\mathrm{Tr}\, D_g = \frac{(n+2)(n+3)}{12}(s, g), \tag{4.23.a}$$

$$\mathrm{Tr}\, D_g P^{(4)} = -\frac{(n+6)(n+1)(n-1)(n-4)}{8(n+2)(n+4)}(s, g), \tag{4.23.b}$$

and similarly

$$\frac{(s, g_{\surd} s)}{(s, s)} = \mathrm{Tr}\, D_g P^{(0)} = \frac{n+8}{3n(n+2)}. \tag{4.23.c}$$

The proof of the theorem when $n < 4$ will be a simple consequence of Theorem 5.2 in the next section.

5. EXTREMA OF $\Phi^{(1)}$

Let $g^* = \epsilon\tilde{g}$ be an extremum of $\Phi^{(1)} = -\frac{\epsilon}{2}(g, g) + \frac{1}{2}(g_\vee g, g)$. It satisfies $\tilde{g}_\vee\tilde{g} = \frac{2}{3}\tilde{g}$ so:

$$\Phi^{(1)}(g^*) = -\frac{\epsilon^3}{6}(\tilde{g}, \tilde{g}) . \tag{5.1}$$

From Eq. (4.20.b) we deduce:

$$(\tilde{g}, \tilde{g}) = \frac{n}{2}\gamma(\tilde{g})(2 - \gamma(\tilde{g})) - \frac{n+4}{12}(g^{(2)}, g^{(2)}) . \tag{5.2}$$

Assume we have a second extremum $h^* = \epsilon\tilde{h}$ and consider the restriction of $\Phi^{(1)}$ to the straight line containing g^* and h^*. It is a third degree polynomial in λ:

$$\psi(\lambda) = \Phi^{(1)}((1-\lambda)g^* + \lambda h) = \frac{\epsilon^3}{6}\left\{ \left[(\tilde{h}, \tilde{h}) - (\tilde{g}, \tilde{g}) \right] \lambda^2 (2\lambda - 3) - (\tilde{g}, \tilde{g}) \right\} . \tag{5.3}$$

Since $\lambda = 0$ and $\lambda = 1$ correspond respectively to the extrema g^* and h^* of $\Phi^{(1)}$ these values must be extrema of $\psi(h)$. Indeed:

$$\frac{d\psi}{d\lambda} = \epsilon^3 \left[(\tilde{h}, \tilde{h}) - (\tilde{g}, \tilde{g}) \right] \lambda (1 - \lambda) . \tag{5.4}$$

We know that a third degree polynomial has no other extrema. So when $(\tilde{h}, \tilde{h}) = (\tilde{g}, \tilde{g})$ we verify from Eq. (5.3) that ψ is constant. Then we can prove that the direction $\tilde{g} - \tilde{h}$ does not correspond to a zero eigenvalue of the Hessian $H(g^*)$ or $H(h^*)$. Indeed, assume that:

$$0 = H(g^*)(\tilde{g} - \tilde{h}) = \epsilon(3D_{\tilde{g}} - I)(\tilde{g} - \tilde{h}) = \epsilon(\tilde{g} + \tilde{h} - 3\tilde{g}_\vee\tilde{h}) = 0 . \tag{5.5}$$

By scalar multiplication with $\tilde{g}$ and $\tilde{h}$ one obtains $(\tilde{g}, \tilde{g}) = (\tilde{h}, \tilde{h}) = (\tilde{g}, \tilde{h})$, i.e., $\tilde{g} = \tilde{h}$ which is absurd. So $\tilde{g} - \tilde{h}$ is not an eigenvector with zero eigenvalue of $H(g^*)$ or $H(h^*)$ although the expectation value of these operators on $\tilde{g} - \tilde{h}$ vanishes when $(\tilde{g}, \tilde{g}) = (\tilde{h}, \tilde{h})$:

$$(\tilde{g} - \tilde{h}, H(g^*)(\tilde{g} - \tilde{h})) = \epsilon\left[(\tilde{g}, \tilde{g}) - (\tilde{h}, \tilde{h}) \right] = -(\tilde{g} - \tilde{h}, H(h^*)(\tilde{g} - \tilde{h})) . \tag{5.6}$$

We can therefore conclude, if $G = O(n)_{\tilde{g}} \cap O(n)_{\tilde{h}}$

Lemma 5.1: If two extrema of $\Phi^{(1)}$ on $\mathcal{J}_4^{\,G}$ have the same length, they are not minima of $\Phi^{(1)}|_{\mathcal{J}_4^{\,G}}$.

Of course neither are they minima of the whole polynomial $\Phi^{(1)}$. When $(\tilde{g}, \tilde{g}) \neq (\tilde{h}, \tilde{h})$, since $\psi(\lambda)$ has no other extrema than 0 and 1, the extremum with the shortest length is in the attraction basin of that with the biggest length (see Eq. (5.1)). This is true for any pair of extrema. This discussion establishes the following theorem:

> **Theorem 5.2:** For any subspace $\mathcal{E} \in \mathcal{T}_4$, if $\Phi^{(1)}$ has a minimum on $\mathcal{E}$, this minimum is unique and any other extremum of $\Phi^{(1)}$ on $\mathcal{E}$ is on the boundary of the attractor basin of this minimum.

This completes the proof of Theorem 4.1 for $n < 4$ since in that case $\tilde{s}$ is a minimum. We also remark that for $n \leq 4$, $\gamma(\tilde{s}) \leq 1$ (see Eq. (4.21)) and from $\gamma(\tilde{g}) < \gamma(\tilde{s})$ and Eq. (4.21) we deduce $(\tilde{g}, \tilde{g}) < (\tilde{s}, \tilde{s})$ for any extremum $\epsilon\, \tilde{g}$ when $n \leq 4$.

The discussion in Section 2 on the action of the normalizer $N_{O(n)}(\mathcal{T}_4^{\ G})$ on $\mathcal{T}_4^{\ (G)}$ and the equalities (2.6) and (2.8) gives the following addition to Theorem 2.

> **Corollary 5.3:** If g is an isotropy group on $\mathcal{T}_4^{\ G}$ and if $\Phi^{(1)}|_{\mathcal{T}_4^{\ G}}$ has a minimum, this minimum is unique and its invariant group is $\supseteq N_{O(n)}(\mathrm{G})$, the normalizer of G in $O(n)$.

The interesting question would be to decide for which conjugate classes $[G]$ of subgroups of $O(n)$, and more specifically for which conjugate classes of irreducible strict subgroups of $O(n)$, $\Phi^{(1)}|_{\mathcal{T}_4^{\ G}}$ has a minimum. Indeed this minimum yields a stable fixed point $\tilde{g}(\epsilon)$ of the renormalization problem when its Hessian is not degenerate; (when $detH(g^*) = 0$, a study of the bifurcation has to be done). For $n = 2$ or 3 we have seen that the class of irreducible strict subgroups is unique, it is $[B_n]$ and , as is well known, the isotropic fixed point is the only stable one. When $n = 4$, the list of the classes of irreducible subgroups which are isotropy groups on $\mathcal{T}_4$ is known [19]; we have shown here that $\tilde{s}$ is still the only minimum of $\Phi^{(1)}$, but it is degenerate. We are studying the case $n = 4$ in collaboration with J. C. and P. Toledano. When $n > 4$, we have seen that $\tilde{s}$ is never a minimum of $\Phi^{(1)}|_{\mathcal{T}_4^{\ G}}$ for G irreducible. In the next section we will construct a family of irreducible G such that $\Phi^{(1)}|_{\mathcal{T}_4^{\ G}}$ has a minimum when $n > 4$ and is not prime. For all $n > 4$ and $G = B_n$, there is a well known cubic invariant minimum.

To end this section we consider the case of a pair of extrema $\epsilon\, \tilde{u}$, $\epsilon\, \tilde{v}$ such that $\tilde{u} \vee \tilde{v}$ is a linear combination of $\tilde{u}$ and $\tilde{v}$:

$$\tilde{u} \vee \tilde{u} = \frac{2}{3}\, \tilde{u}\ , \qquad \tilde{v} \vee \tilde{v} = \frac{2}{3}\, \tilde{v}\ , \qquad \tilde{u} \vee \tilde{v} = \frac{1}{3}\left(\alpha\, \tilde{u} + \beta\, \tilde{v} \right). \quad (5.7)$$

The scalar product with $\tilde{u}$ and $\tilde{v}$ yields

$$\alpha(\bar{u}, \bar{u}) = (2 - \beta)(\bar{u}, \bar{v}) . \quad (5.8.a)$$

$$\beta(\bar{v}, \bar{v}) = (2 - \alpha)(\bar{u}, \bar{v}) . \quad (5.8.b)$$

We note $\bar{u} = \bar{v} \Rightarrow 2 - \alpha - \beta = 0$. The exact converse is not true, but one easily verifies that:

$$\alpha \beta \neq 0 \quad and \quad 2 - \alpha - \beta = 0 \quad \Rightarrow \bar{u} = \bar{v} . \quad (5.9)$$

The Schwartz inequality yields from Eq. (5.9):

$$\alpha \beta \neq 0 \quad and \quad \bar{u} \neq \bar{v} \quad \Rightarrow \alpha \beta (2 - \alpha - \beta) = 0 . \quad (5.10)$$

Moreover, when $\alpha \beta \neq 1$ there is a third extremum $\bar{w}^* = \epsilon \bar{w}$ in the 2-plane $\bar{u}$, $\bar{v}$:

$$\bar{w} = \zeta \bar{u} + \eta \bar{v} . \qquad \zeta = \frac{1 - \alpha}{1 - \alpha \beta} , \qquad \eta = \frac{1 - \beta}{1 - \alpha \beta} . \quad (5.11)$$

This equation is also valid for $\alpha = \beta = 0$. Indeed, the three cases $\alpha = \beta = 0$; $\alpha = 0$, $\beta = 2$; and $\alpha = 2$, $\beta = 0$ correspond to the same 2-plane:

$$\bar{u} \underset{\surd}{} \bar{v} = 0 = (\bar{u}, \bar{v}) \quad , \quad \bar{w} = \bar{u} + \bar{v} . \quad (5.12)$$

For any extremum g^* of $\Phi^{(1)}(g)$ we have

$$H(g^*)g^* = \epsilon g^* . \quad (5.13)$$

It is easy to compute the other eigenvalue and eigenvector in the 2-plane spanned by u^*, v^*, and w^* of the respective Hessians:

$$H(u^*)(\alpha \bar{u} + (\beta - 2)\bar{v}) = \epsilon (\beta - 1)(\alpha \bar{u} + (\beta - 2)\bar{v}) , \quad (5.14.a)$$

$$H(v^*)((\alpha - 2)\bar{u} + \beta \bar{v}) = \epsilon (\alpha - 1)((\alpha - 2)\bar{u} + \beta \bar{v}) , \quad (5.14.b)$$

$$H(w^*)(\alpha \bar{u} - \beta \bar{v}) = \epsilon \frac{(1 - \alpha)(1 - \beta)}{1 - \alpha \beta}(\alpha \bar{u} - \beta \bar{v}) . \quad (5.14.c)$$

We will see several examples of such 2-planes in the next section. Here we consider a 2-plane which contains $\bar{s}$ and we use the notation x^*, y^*, s^* instead of u^*, v^* and w^* in order to distinguish this particular case. We denote by $\mathcal{E}_x$ this 2-plane and we require moreover

$$\mathscr{E}_x \subset \mathscr{E} = \mathscr{T}_4^{(0)} + \mathscr{T}_4^{(4)} \tag{5.15}$$

since we are interested in extrema invariant under an irreducible subgroup $G \subset O(n)$. We will also use the direct sum of orthogonal subspaces

$$\mathscr{E} = \mathscr{E}_x \oplus \mathscr{E}_x^{\perp} . \tag{5.16}$$

Then we get from Eq. (4.17) and Eq. (4.18)

$$g \in \mathscr{E} \qquad \bar{s} \vee g = \frac{1}{3}\left(\gamma(g)\bar{s} + \frac{12}{n+8} g\right) . \tag{5.17}$$

If we apply $D_{\bar{x}}$ and $D_{\bar{y}}$ to $\bar{s} = \zeta\bar{x} + \eta\bar{y}$ and use $\bar{x} \vee \bar{y} = \frac{1}{3}\left(\alpha\bar{x} + \beta\bar{y}\right)$ we obtain $\alpha = \gamma(\bar{y})$, $\beta = \gamma(\bar{x})$, i.e.,

$$\bar{x} \vee \bar{y} = \frac{1}{3}\left(\gamma(\bar{y})\, \bar{x} + \gamma(\bar{x})\, \bar{y}\right) \tag{5.18}$$

and

$$\zeta + \eta = \frac{12}{n+8} = \frac{2 - \gamma(\bar{x}) - \gamma(\bar{y})}{1 - \gamma(\bar{x})\, \gamma(\bar{y})} . \tag{5.19}$$

From the inequalities (4.21) and (5.10), and from the preceding equation, we obtain the inequalities:

$$0 < \gamma(\bar{x}) + \gamma(\bar{y}) < \mathrm{Inf}\left(2, \frac{4\,(n+2)}{n+8}\right) , \tag{5.20.a}$$

$$0 < \gamma(\bar{x})\, \gamma(\bar{y}) < 1 . \tag{5.20.b}$$

Eq. (5.19) can also be written equivalently:

$$(\gamma(\bar{x}) - 1)\,(\gamma(\bar{y}) - 1) = -\frac{n-4}{n+8}\left(1 - \gamma(\bar{x})\, \gamma(\bar{y})\right) . \tag{5.21}$$

When $n = 4$ either $\gamma(\bar{x}) = 1$ or $\gamma(\bar{y}) = 1$ which implies that $\eta = 0$ or $\zeta = 0$, hence

Lemma 5.4: For $n = 4$, any 2-plane containing $\bar{s}$ can contain at most one other extremum.

When $n \neq 4$, any extremum $\epsilon\bar{x}$ defines a 2-plane $\mathscr{E}_x$ containing two other extrema $\epsilon\bar{s}$ and $\epsilon\bar{y}$. When $n < 4$ we verify again that $\epsilon\bar{s}$ is the minimum of $\Phi^{(1)}|_{\mathscr{E}_x}$. When $n > 4$, Eq. (5.20.b) implies that the right hand side of Eq. (5.21) is negative so either $\gamma(\bar{x}) - 1$ or $\gamma(\bar{y}) - 1$ is positive and equations (5.14.a), (5.14.b) and (5.14.c) show that

Lemma 5.5: When $n > 4$, either $\epsilon\tilde{x}$ or $\epsilon\tilde{y}$ is the minimum of $\Phi^{(1)}|_{\mathscr{E}_x}$.

We can have more knowledge of the Hessians of $\epsilon\tilde{x}$ and $\epsilon\tilde{y}$. Indeed, from $\tilde{s} = \zeta\,\tilde{x} + \eta\,\tilde{y}$ we have a linear relation among the Hessians

$$\zeta\, H(\epsilon\tilde{x}) + \eta\, H(\epsilon\tilde{y}) = H(\epsilon\tilde{s}) + \epsilon I(1 - \zeta - \eta) \,. \tag{5.22}$$

From Eq. (4.19) and Eq. (5.19), and the values of ζ and η given by Eq. (5.11) and Eq. (5.18) we obtain by projection on $\mathscr{E}_x^{\perp}$:

$$(\gamma(\tilde{y}) - 1)\, H(\epsilon\tilde{x})\big|_{\mathscr{E}_x^{\perp}} + (\gamma(\tilde{x}) - 1)\, H(\epsilon\tilde{y})\big|_{\mathscr{E}_x^{\perp}} = 0 \tag{5.23}$$

The two Hessians are proportional on $\mathscr{E}_x^{\perp}$ and the proportionality factor is positive when $n > 4$.

Let us apply these results to the 2-plane $\mathscr{T}_4^{B_n}$ spanned by $c = \sum_i \phi_i^4$ and $s = (\phi,\phi)^2$. Note that:

$$(c,\, c) = (s,\, c) = n \quad , \quad \gamma(c) = 1 \quad , \quad c \vee c = c \quad , \quad s \vee c = \frac{1}{3}(s + 2c) \,. \tag{5.24}$$

Hence $\tilde{c} = \dfrac{2}{3}\, c$

$$\tilde{c}' = \frac{2}{n}\, s + \frac{n - 4}{3}\, c \quad \Longleftrightarrow \quad \tilde{s} = \frac{3}{n + 8}\left\{ (4 - n)\, \tilde{c} + n\, \tilde{c}' \right\} . \tag{5.25}$$

When $n > 4$, $\epsilon c'$ is the minimum of $\Phi^{(1)}$ in $\mathscr{E}_c = \mathscr{T}_4^{B_n}$; indeed

$$H(c'^{*})\, (s - 2c) = \epsilon\, \frac{n - 4}{3n}\, (s - 2c) \,. \tag{5.26}$$

We can also write the whole spectrum of $H(c^{*})$ and $H(c'^{*})$ on $\mathscr{T}_4$. Indeed, let us denote by $\mathscr{E}_{(4)}$, $\mathscr{E}_{(3,1)}$, $\mathscr{E}_{(2,2)}$, $\mathscr{E}_{(2,1,1)}$, $\mathscr{E}_{(1,1,1,1)}$ the subspaces of $\mathscr{T}_4$ defined by the following properties of the values of indices of g_{ijkl} corresponding to non-vanishing values of this tensor: the four indices are equal, only three are equal, two different pairs of equal indices, only two equal indices, the four indices are different. We give below the dimension of these spaces and the eigenvalues of $H(c^{*})$:

space:	$\mathscr{E}_{(4)}$	$\mathscr{E}_{(3,1)}$	$\mathscr{E}_{(2,2)}$	$\mathscr{E}_{(2,1,1,1)}$	$\mathscr{E}_{(1,1,1,1)}$
dimension:	n	$n(n-1)$	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$	$\binom{n}{4}$
eigenvalues of $H(c^{*})$:	ϵ	0	$-\frac{\epsilon}{3}$	$-\frac{2\epsilon}{3}$	$-\epsilon$

$H(c'^*)$ has the same eigenspaces in $\mathcal{E}_c$ and the eigenvalues are multiplied by $(n-4)/n$. Since B_n is finite, the dimension of the orbit of these extrema is $n(n-1)/2$. This is half the dimension of Ker $H(c^*)$ = Ker $H(c'^*)$; so these extrema also have an accidental degeneracy of dimension $n(n-1)/2$.

6. THE ISOTROPY GROUPS $G_{r_1,r_2,\ldots,r_k}$, $N = \prod_{i=1}^{k} r_i$, AND THEIR INVARIANT POLYNOMIALS.

Consider the polynomials $x_{pq} \in \mathcal{E} = \mathcal{T}_4^{(0)} + \mathcal{T}_4^{(4)}$ with

$$p\,q = n \qquad x_{p,q} = \sum_{j=1}^{q} \Big\{ \sum_{i=1}^{p} \phi_{ij}^2 \Big\}^2 . \tag{6.1}$$

For each value of j, $\Big\{ \sum_{i=1}^{p} \phi_{ij}^2 \Big\}^2$ has $O(p)$ as isotropy group; with the summation over j the isotropy group is the semidirect product

$$\Gamma_{p,q} = O(p)^q \quad \Pi_q \tag{6.2}$$

where $O(p)^q$ is the direct product of q factors isomorphic to $O(p)$ and Π_q is the permutation group of q objects. Particular cases are:

$$\Gamma_{n,1} = O(n) \quad , \quad \Gamma_{1,n} = B_n . \tag{6.3}$$

The group $\Gamma_{p,q}$ is realized as a subgroup of $O(n)$. The $n \times n$ orthogonal matrices are made of q^2 blocks of $p \times p$ submatrices; $O(p)^q$ has all its blocks zero except the diagonal ones and each of these diagonal blocks is a $p \times p$ orthogonal matrix $\in O(p)$. The permutation group Π_q is represented by matrices which have only q blocks different from zero, one per row and per column and each of these non-zero blocks is equal to the matrix I_p, the $p \times p$ unit matrix. We will prove later that the groups $\Gamma_{p,q}$ are irreducible subgroups of $O(n)$. We prove now that

$$\dim \mathcal{T}_4^{\Gamma_{pq}} = 2 . \tag{6.4}$$

To be invariant under $O(p)^q$, the quartic polynomial $g(\phi)$ has to be a quadratic polynomial in the quadratic forms

$$Q_j = \sum_{i=1}^{p} \phi_{ij}^2 \quad .$$

The quadratic invariants of the group Π_q of permutations of the Q_j are known, they are

$$s = \left\{ \sum_{j=1}^{q} Q_j \right\}^2 \quad \text{and} \quad x_{p,q} = \sum_{j=1}^{q} Q_j^2 .$$

It is convenient for our purposes to consider the n dimensional space $\mathcal{V}_n$ as a tensor product, $\mathcal{V}_n = \mathcal{V}_p \otimes \mathcal{V}_q$ so the coordinates ϕ_{ij} can be written as $\phi_{ij} = \rho_i \otimes \sigma_j$. Then:

$$x_{p,q} = \left\{ \sum_i \rho_i^2 \right\}^2 \otimes \sum_j \sigma_j^4 = s_p \otimes c_q \quad , \quad x_{n,1} = s \quad , \quad x_{1,n} = c \; ; \tag{6.5}$$

similarly one finds

$$s = s_p \otimes s_q \quad , \quad c = c_p \otimes c_q \quad . \tag{6.6}$$

Then it is easy to compute:

$$(x_{p,q}, x_{p,q}) = \frac{p\,q(p+2)}{3} = (s,\, x_{p,q}) \quad , \qquad (c, x_{p,q}) = p\; q \; , \tag{6.7}$$

$$x_{p,q} \vee x_{p,q} = \frac{p+8}{9}\, x_{p,q} \; , \tag{6.8.a}$$

$$s \vee x_{p,q} = \frac{p+2}{9}\, s \; + \frac{2}{3}\, x_{p,q} \quad , \quad c \vee x_{p,q} = \frac{1}{3}\, x_{p,q} + \frac{2}{3}\, c \; . \tag{6.8.b}$$

In the particular cases $p = 1$ or $q = 1$ we find Eq. (5.24). We can also generalize these equations for different pairs p,q; e.g.

$$(x_{p,qr} \, , \, x_{pq,r}) = p\; q\; r\, \frac{p+2}{3} \quad , \qquad x_{p,qr} \vee x_{pq,r} = \frac{2}{3}\, x_{p,qr} + \frac{p+2}{9}\, x_{pq,r} \, . \tag{6.9}$$

Now we can consider a family of groups $\Gamma_{p,q}$, $pq = n$ and their intersections. Let

$$r_1\, r_2 \, . \, . \, . \, r_k = n = \prod_{i=1}^{k} \; r_i \tag{6.10}$$

be a decomposition of n in k factors, not necessarily prime, listed in a fixed order. With $r_0 = 1$, define

$$p_l = \prod_{i=1}^{l} \; r_i \quad , \quad q_l = n/p_l \quad , \quad \textit{so that} \quad p_0 = q_k = 1 \; . \tag{6.11}$$

The group $G_{r_1, \ldots, r_k}$ is defined by:

$$G_{r_1, \ldots, r_k} = \bigcap_{0 \le i \le k} \; \Gamma_{p_i, q_i} \, . \tag{6.12}$$

We could have defined it by recursion:

$$G_{r_1} = B_{r_1} \quad , \quad G_{r_1, r_2, \ldots\ldots, r_k} = (G_{r_1, r_2, \ldots\ldots, r_{k-1}})^{r_k} \quad \Pi_{r_k} \, . \tag{6.13}$$

The order of this subgroup of B_n is:

$$|G_{r_1,r_2,\ldots,r_k}| = 2^n \prod_{i=1}^{k} (r_i\,!)^{q_i}\,. \tag{6.14}$$

It is interesting to note that for $r_1 = r_2 = \ldots = r_k = 2$,

$$G_{2,2,\ldots,2} = \mathrm{Syl}_2\,(B_2k)\,, \tag{6.15}$$

i.e., $G_{2,2,\ldots,2}$ is isomorphic to the Sylow-2 groups[1] of the Coxeter group B_2k.

We verify that $G_{r_1,r_2,\ldots,r_k}$ is an irreducible subgroup of $O(n)$. Indeed it contains the Abelian subgroup Δ_n of diagonal matrices with elements $\pm$ 1. The matrices which commute with Δ_n are the diagonal matrices. We see that the permutation matrices of $G_{r_1,r_2,\ldots,r_k}$ act transitively on the basis $\{\phi_i\}$ of the n dimensional space $\mathcal{V}_n$; indeed the ordered set of n basis vectors can be decomposed into nested sets of $p_1, p_2, \ldots, p_{k-1}$ elements for the different levels of nesting. The permutations of Π_{r_k} permute the p_{k-1}-element sets[2]; inside one such set $\Pi_{r_{k-1}}$ is the group of permutations of the p_{k-2} element sets and so on. So ϕ_1 can be sent to the place of any ϕ_a, $1 \leq a \leq$ n. By conjugation the permutation matrices permute the diagonal elements of diagonal matrices; so the only matrices which commute with all elements of $G_{r_1,\ldots,r_k}$ are multiples of the identity, i.e., $G_{r_1,\ldots,r_k}$ is irreducible. This is also true of all $O(n)$ subgroups which contain it. So the subgroups $\Gamma_{p,q}$ are also irreducible.

7. THE STABLE FIXED POINTS OF THE POLYNOMIAL $\sum_{i=0}^{k} \lambda_i\, x_i(\phi)$.

We use a still more condensed notation

$$0 \leq i \leq k\,, \quad x_i = x_{p_i,q_i}(\phi)\,, \quad x_0 = c\,, \quad \lambda_k = s\,. \tag{7.1}$$

where p_i and q_i are defined in Eq. (6.11) and $x_{p,q}(\phi)$ in Eq. (6.1). The polynomials x_i span the space $\mathcal{T}_4^{G_{r_1,\ldots,r_k}}$ whose dimension is

$$\dim \mathcal{T}_4^{G_{r_1,\ldots,r_k}} = k + 1\,. \tag{7.2}$$

Eq. (6.9) *can be written in the present notation:*

[1]Given a finite group whose order (that is, the number of elements) is $|G|$, let $|G| = 2^{l_2} 3^{l_3} \ldots p^{l_p}$ be the decomposition of $|G|$ into prime factors. Then all subgroups of G of order p^{l_p} are conjugated and are called Sylow p subgroups of G

[2]That is they transform any basis vector into any other.

$$(x_i, x_j) = n\frac{p_i+2}{3}, \quad i \le j, \tag{7.3.a}$$

$$x_i \vee x_j = \frac{2}{3}x_i + \frac{p_i+2}{9}x_j, \quad i \le j. \tag{7.3.b}$$

For $i = j$ this equation shows that the x_i are idempotents; they yield the fixed points:

$$x_i^* = \epsilon\,\tilde{x}_i = \epsilon\frac{6}{p_i+8}x_i \quad , \quad \tilde{x}_i \vee \tilde{x}_i = \frac{2}{3}\tilde{x}_i. \tag{7.4}$$

From equations (7.3.a) and (7.3.b) we see that any pair of these fixed points satisfies an equation of the form of Eq. (5.7):

$$\tilde{x}_i \vee \tilde{x}_j = \frac{1}{3}\left(\frac{12}{p_j+8}\tilde{x}_i + 2\frac{p_i+2}{p_i+8}\tilde{x}_j\right), \quad i \le j. \tag{7.5}$$

From Eq. (7.3.a) we obtain the orthogonality relations:

$$(x_i, x_j - x_k) = 0, \quad i \le j \le k \tag{7.6}$$

and similarly:

$$\tilde{x}_i \vee (x_j - x_k) = \frac{2}{3}\frac{p_i+2}{p_i+8}(x_j - x_k), \quad i \le j \le k. \tag{7.7}$$

We deduce immediately for the Hessian of x_i^* (see Eq. (4.9))

$$H(x_i^*)(x_j - x_k) = \epsilon\frac{p_i-4}{p_i+8}(x_j - x_k), \quad i \le j \le k. \tag{7.8}$$

For $i = 0$, $x_i^* = c^*$, we simply find that, in the hyperplane of $\mathcal{T}_4^{G_{r_1,\dots,r_k}}$ orthogonal to c, $H(c^*)$ is $-(\epsilon/3)\,I$ (proportional to the identity); indeed this hyperplane is in $\mathcal{E}_{(2,2)}$. It is more interesting to consider the case $i = 1$. Then for $p_1 = r_1 > 4$ the Hessian $H(x_1^*)$ is a positive multiple of the unit operator on the $k-1$ dimensional subspace of $\mathcal{T}_4^{G_{r_1,\dots,r_k}}$ orthogonal to $c = x_0$ and x_1. The restriction to the 2-plane spanned by c, x of the Hessians $H(c^*)$, $H(x_i^*)$ are not positive; however we know, from Eq. (5.11), how to form a third fixed point, that we call here z^* whose restriction of the Hessian is positive in the 2-plane; in Eq. (7.5) with

$$i = 0, \quad j = 1, \quad \alpha = \frac{12}{p_1+8}, \quad \beta = \frac{2}{3}$$

and

$$z = \frac{r_1-4}{r_1}\tilde{c} + \frac{r_1+8}{3r_1}\tilde{x}_1 \tag{7.9}$$

(we have used $p_1 = r_1$). The eigenvectors and eigenvalues of $H(z^*)$ in the 2-plane $\bar{c}$, $\bar{x}_1$ are (we use Eq. (5.14.c))

$$H(z^*)\,\bar{z} = \epsilon\,\bar{z} \quad . \tag{7.10}$$

$$H(z^*)\left\{\frac{12}{r_1+8}\,\bar{c} - \frac{2}{3}\,\bar{x}_1\right\} = \frac{r_1-4}{3r_1}\left\{\frac{12}{r_1+8}\,\bar{c} - \frac{2}{3}\,\bar{x}_1\right\}.$$

The linear relation (7.9) implies for the Hessian:

$$H(z^*) = \frac{r_1-4}{r_1}\,H(c^*) + \frac{r_1+8}{3\,r_1}H(x_1^*) + \epsilon\frac{r_1-4}{3r_1}I \tag{7.11}$$

and from Eq. (7.8)

$$H(z^*)\,(x_1 - x_j) = \epsilon\,\frac{r_1-4}{3r_1}\,I \quad , \qquad 1 \le j \le k\,.$$

To summarize: when n is divisible by $r_1>4$ and n/r_1 can be written as the product of at least $k-1$ factors, one can consider the Hamiltonian Eq.(1) with a quartic polynomial

$$\sum_{i=0}^{k} \alpha_i\,x_i$$

depending on $k+1$ parameters α_i; it has a stable fixed point $z^*(\phi)$, given by Eq. (7.9), which is physically relevant since it is a positive polynomial (the coefficients in Eq. (7.9) are positive and the x_i are positive). Mathematically, $k+1$ can be arbitrarily large. This example simply destroys the "conviction" in reference [11] that as "a general consequence of some kind of topological properties of the renormalization group" no stable fixed points can exist for $k > 2$. We will end this section by giving another counter example when $r_1 = 2$ or 3 (of course the case $r_1 = 4$ is excluded). If $r_1 = 2$, we require $r_2 > 2$ so that $p_2 = r_1 r_2 > 4$. Using [Eqs. (7.3.a) to (7.8)], we leave it to the reader to check that $b^* = \epsilon\,\tilde{b}$ where :

$$\tilde{b} = \frac{1}{r_1(r_1r_2+8(r_2-2)+16)}\left\{(r_1r_2-4)(r_1+8)\,\bar{x}_1 - (r_1-4)(r_1r_2+8)\,\bar{x}_2\right\} \tag{7.12}$$

is a stable fixed point when $r_1 < 4 < r_1 r_2$. Indeed:

$$\text{Spectrum } H(b^*)\big|_{\mathcal{B}_4^{G}} = \left\{\epsilon,\ k\ times\ \ \epsilon\,\frac{(4-r_1)(r_1r_2-4)}{r_1(r_1r_2+8(r_2-2)+16)}\right\}. \tag{7.13}$$

The corresponding minimum value of n is:

Table 7-1: Fixed points g^* in the 3-plane $\mathscr{E}_{hij}$ spanned by $\bar{x}_h$, $\bar{x}_i$ and $\bar{x}_j$, $0 \le h < i < j \le k$.

g^*	Component on $6\epsilon x_h$	$6\epsilon x_i$	$6\epsilon x_j$	Spectrum of $\epsilon^{-1} H(g^*)_{\mathscr{E}_{hij}}$		
0	0	0	0	-1	-1	-1
$\bar{x}_h$	$\frac{1}{p_h+8}$	0	0	1	$\frac{p_h-4}{p_h+8}$	$\frac{p_h-4}{p_h+8}$
$\bar{x}_i$	0	$\frac{1}{p_i+8}$	0	1	$-\frac{p_i-4}{p_i+8}$	$\frac{p_i-4}{p_i+8}$
$\bar{x}_j$	0	0	$\frac{1}{p_j+8}$	1	$-\frac{p_j-4}{p_j+8}$	$-\frac{p_j-4}{p_j+8}$
$\bar{y}_{ij}$	0	$(p_j-4)\zeta_{ij}^{-1}$	$-(p_i-4)\zeta_{ij}^{-1}$	1	$-\zeta_{ij}(p_i-4)(p_j-4)$	$\zeta_{ij}(p_i-4)(p_j-4)$
$\bar{y}_{hj}$	$(p_j-4)\zeta_{hj}^{-1}$	0	$-(p_h-4)\zeta_{hj}^{-1}$	1	$-\zeta_{hj}(p_h-4)(p_j-4)$	$\zeta_{hj}(p_h-4)(p_j-4)$
$\bar{y}_{hi}$	$(p_i-4)\zeta_{hi}^{-1}$	$(p_h-4)\zeta_{hi}^{-1}$	0	1	$-\zeta_{hi}(p_h-4)(p_i-4)$	$-\zeta_{hi}(p_h-4)(p_i-4)$
$\bar{z}_{hij}$	$(p_i-4)(p_j-4)\zeta^{-1}$	$-(p_h-4)(p_j-4)\zeta^{-1}$	$(p_h-4)(p_i-4)\zeta^{-1}$	1	$-\frac{1}{6}(p_h-4)(p_i-4)(p_j-4)\zeta^{-1}$	$\frac{1}{6}(p_h-4)(p_i-4)(p_j-4)\zeta^{-1}$

Notation: $\zeta_{ij}=p_ip_j-16p_i+8p_j+16>0$; similar expressions for ζ_{hi}, ζ_{hj}.

$$\zeta=\frac{1}{6}(p_hp_ip_j-8p_ip_j-16p_hp_j+8p_hp_i-16p_h-80p_i+16p_j+128)$$

Note that $\bar{z}_{hij}$ is at the intersection of the three 2-planes spanned respectively by $\bar{x}_h$, $\bar{y}_{ij}$; $\bar{x}_i$, $\bar{y}_{hj}$; $\bar{x}_j$, $\bar{y}_{hi}$. This table is valid only if $(p_h-4)(p_i-4)(p_j-4)\neq 0$. The minimum is $\bar{y}_{ij}$ when $p_h<p_i<4$, $\bar{y}_{hj}$ when $p_h<4<p_i$, and $\bar{x}_h$ when $4<p_h$.

$$n \geq 2^{k-1} \times 3 \ . \tag{7.14}$$

Why both $H(z^*)$ and $H(b^*)$ have the eigenvalues $\neq \epsilon$ of multiplicity k is not clear to me. It is to be noted that the isotropy group of the fixed points is much larger than the normalizer of the isotropy group $G_{r_1, \ldots, r_k}$. Indeed

$$O(n)_{z^*} = \Gamma_{r_1, q_1} \quad , \quad O(n)_{b^*} = \Gamma_{p_1, q_1} \cap \Gamma_{p_2, q_2} \ . \tag{7.15}$$

The proof is easy in the second case since dim $O(n)_{b^*} = n(r_1 - 1)/2$ while the normalizer of $G_{r_1, \ldots, r_k}$ is finite since $G_{r_1, \ldots, r_k}$ is finite and irreducible and $\subset O(n)$. The polynomials x_i appear often in actual studies of Landau transitions.

Finally, I hope that the powerful method developed in this paper for computing fixed points will also be useful for practical applications.

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APPENDIX

In Section 5 we studied the fixed points of some 2-planes. We give here similar results for 3-planes spanned by x_i, x_j and x_h with $0 \leq h < i < j \leq k$; this implies that $1 \leq p_h < p_i < p_j \leq$ n; we add the condition that every p is different from 4. An interesting particular case is $h = 0$, $j = n$, so $p_0 = 1$, $p_k = n$, $\tilde{x}_h = c$ and $\tilde{x}_j = s$. The component of the fixed points and the spectrum of the restriction of their Hessians to the 3-plane are given in Table 7.1.

We find that such a 3-plane has 8 fixed points. They can be considered as the vertices of a polyhedron which a has the same faces and edges as a cube. Similarly, we conjecture that in the $k + 1$ dimensional space $\mathcal{T}_4^{G_{r_1, \ldots, r_k}}$ there are 2^{k+1} fixed points forming a cube-like figure.

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