COVARIANT VECTOR FIELDS AND RENORMALIZATION GROUP

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Abstract : This review paper studies explicitly the important example of Landau theory of second order phase transitions. The consequence of the $O(n)$ covariance of the renormalization equation is explicitly analyzed. Some purely group theoretical criteria for the existence, non existence of stable fixed points and their symmetry are presented.

The "renormalization group" equations are a system of first order differential equations of the form

$$\frac{du(\lambda)}{d \log \lambda} = \beta(u(\lambda)) \tag{1}$$

where λ is the scale parameter and u is a variable defined in a vector space P (generally a set of coupling constants) and $\beta(u)$ is a vector field on P. We will be interested by the case where the vector field is covariant for a compact group acting on P. The fixed points of the flow defined by (1) are solutions of the non linear equations

$$\beta(u^*) = 0 . \tag{1'}$$

The stability of the fixed point u^* is given by some conditions that we will precise later, on the spectrum of the linear operator on P :

$$H(u^*) = \frac{d\beta}{du}(u^*). \tag{2}$$

In a quantum field theory the renormalization may introduce a spontaneous symmetry breaking, Lee [1], Lee and Gervais [2], Symanzik [3]. In the proposed theories which unify the fundamental interactions (so-called GUT's for grand unified theory) the spontaneous symmetry breaking is introduced by Higgs fields or technicolor condensates; these theories are renormalizable and the renormalization restores the symmetry at very high energy ($\sim 10^{16}$ GeV) . In order to deal with a concrete example, I will restrict this lecture to a rich family of applications : the study of second order phase transitions with symmetry changes.

The theory proposed by Landau [4] and several times reviewed in these colloquia considers a finite dimensional orthogonal representation $g \mapsto D(g)$ of the physical symmetry group Γ. Often Γ is a crystallographic space group or

the symmetry group of a mesomorphic phase [5] (e.g. liquid crystals...) . Let E be the real vector space carrying the representation $\mathcal{D}(g)$ which is assumed irreducible over the reals;we denote by $\vec{\phi}\cdot\vec{\phi}$, $\vec{\phi} \in E$ a Γ invariant scalar product. The equilibrium state is described by the lowest value of a Γ-invariant Landau potential :

$$V = \frac{a}{2} \vec{\phi}\cdot\vec{\phi} + p(\vec{\phi}) \tag{3}$$

with $\frac{1}{\lambda^4} p(\lambda\vec{\phi}) = p(\vec{\phi}) = p(\mathcal{D}(g)\vec{\phi}) > 0$ when $\vec{\phi} \neq 0$. $\tag{3'}$

The coefficient a is an increasing function of the temperature which vanishes for $T = T_c$. When $T > T_c$ the minimum is at the origin $\vec{\phi} = 0$ and Γ is the symmetry group of the system. When $T < T_c$, the potential V has, at least, one Γ orbit of minima : the system is in another phase whose symmetry group is described by the isotropy group of a point of the orbit of the minima (when V takes its minimum value on several Γ-orbits there is an exceptional degeneracy). If V had a degree 3 term, when a decreases but is still positive, V would develop another minimum which would become negative and be the lowest minimum for a value $a_c > 0$ of a ; then the equilibrium state would jump from $\vec{\phi} = 0$ to a $\vec{\phi}$ of this minimum orbit; this would describe a first order phase transition. In order to avoid it, Landau assumed that there are no Γ-invariants of degree 3 on E .

The Landau theory gives in general good predictions for the symmetry changes in second order phase transition, but it fails completely for predicting the correct critical exponents. Learning from Kadanoff and Wilson, one understands that fluctuations have to be taken in account by renormalization. One considers $\vec{\phi}$ as the value of an n-component field $\vec{\phi}(x)$ and the potential V becomes a part of the Hamiltonian density :

$$H(x) = V(\phi(x)) + \sum_{\alpha=1}^{3} \frac{\partial}{\partial x^\alpha} \vec{\phi} \cdot \frac{\partial}{\partial x^\alpha} \vec{\phi} \tag{4}$$

with the historical normalisation of the quartic polynomial [6]

$$p(\vec{\phi}) = \frac{n^{4-d}}{4!} u(\phi) \ , \ u(\phi) = \sum_{ijk\ell} u_{ijk\ell} \phi_i \phi_j \phi_k \phi_\ell. \tag{5}$$

The renormalization equation (1) is applied to u, vector of the $\binom{n+3}{4}$ dimensional vector space P_4 of quartic polynomials in n variables. This is also the space of "coupling constants" $u_{ijk\ell}$ which form a rank-4 completely symmetrical tensor on E. (These two points of view correspond to a mathematical isomorphism). The renormalization equation (1) may yield a stable fixed point u^* ; if it is positive for all $\vec{\phi}(x) \neq 0$, it yields in (3) and (4) respectively the effective Landau potential and corresponding Hamiltonian.

The special quartic polynomials for u :

$$s = (\vec{\phi} \cdot \vec{\phi})^2 \ , \quad c = \sum_{i=1}^{n} \phi_i^4 \tag{6a,b}$$

were respectively studied by Wilson [7] and Aharony [8]. The general case of an arbitrary quartic polynomial was studied in 1974 by Brezin, Le Guillou, Zinn-Justin [6].

The renormalized version of Landau theory has had good success in its predictions although it seems still imperfect for describing some experiments : crystals are not perfect and their defects may plan an important role in the phase transition. I refer to more competent reviewers for giving a balanced account of the present situation.

The aim of this lecture, as is fit for this colloquium, is to give a review on the group theoretical properties of this theory. As you will see, from this point of view one can say much concerning the existence of fixed points,

their stability, their symmetry and the value of the critical exponents. Four years ago when I began to study these questions I found that few papers were devoted to them : two general papers by Wegner [9] and Khorzhenevskii [10] and most of the others dealing with particular points. I will quote those I know and ask the authors I may not know to excuse me and send me reprints of their work. I will have to rely essentially on three papers [11], [12] , [13], the middle one published two months ago, the two others unpublished yet[(*)].

The explicit form of the quartic polynomials depends on the orthonormal basis chosen on the space E . Such a change of basis is an orthogonal transformation (element of $O(n)$) . Physics must be independent from this choice of basis. Hence : <u>In the renormalization equation, β is an $O(n)$ covariant vector field on</u> P_4 <u>, the space of quartic polynomials. Similarly the critical exponents are $O(n)$ covariants.</u>

We are led to study the orthogonal representation of $O(n)$ on P_4 . It is an orthogonal representation : it leaves invariant the scalar product :

$$(u,u) = \sum_{ijk\ell} u_{ijk\ell} u_{ijk\ell}. \tag{7}$$

(It can be expressed directly from $u(\phi)$ with the n-dimensional gradient and Laplacian, see [11] equ. (15)).

$$P_4 = P_4^{(o)} \oplus P_4^{(2)} \oplus P_4^{(4)}, \tag{8}$$

dimension $\quad \binom{n+3}{4} = 1 + \frac{(n+2)(n-1)}{2} + \frac{(n+6)(n+1)n(n-1)}{24}$, $\tag{8'}$

Dynkin notation of irreps : $(0,0,\dots) + (2,0,\dots) + (4,0,\dots.)$. $\tag{8''}$

$P_4^{(o)}$ is spanned by the $O(n)$ invariant polynomial s ,

$P_4^{(4)}$ is the space of harmonic polynomials :

(*) Although [11] was written before March 1982, it seems to have been blocked in a Yale computer. Unhappily I have not had the possibility to correct the too many misprints it contains. I found Khozhenevskii's paper very inspiring.

$$u^{(4)} \in P_4^{(4)} \leftrightarrow \Delta u^{(4)} = 0 \quad \text{where} \quad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial \phi_i}. \tag{9}$$

$P_4^{(2)}$ is the space of polynomials $\vec{\phi} \cdot \vec{\phi} \, q(\phi)$, where $q(\phi)$ is quadratic and $\Delta q = 0$.

To simplify, let us denote simply by g. the action of $g \in O(n)$ on P_4. The covariance of β is simply

$$\forall g \in O(n) \quad , \quad \forall u \in P_4 \, , \, g \cdot \beta(u) = \beta(g \cdot u). \tag{9'}$$

We denote by P_4^g the set of quartic polynomials invariant by $g: u(\phi) = g \cdot u(\phi) = u(\mathcal{D}(y)\phi)$.

We denote by H the image of the irreducible representation of Γ on E (i.e. H is the set of matrices $\mathcal{D}(\gamma)$, $\gamma \in \Gamma$). The physical problem forgets the kernel of $\mathcal{D}$ and "feels" only the image H, which is an irreducible subgroup of $O(n)$. No polynomials of $P_4^{(2)}$ can be invariant by H. So the space of H-invariant quartic polynomials

$$P_4^H = \bigcap_{g \in H} P_4^g \subset P = P_4^{(o)} \oplus P_4^{(4)} \tag{10}$$

is in P just defined. Our global approach will concentrate on the study of the polynomials in P.

The $O(n)$-covariance of β (equ. (9') implies that the trajectory $u(\lambda)$ of u by equation (1) is tangent to the stratum of u in the $O(n)$ action on P. (We recall that the stratum is the union of all orbits of the same type, i.e., with the same conjugation class of isotropy groups). Moreover, if $O(n)_u$ is the isotropy group of a non fixed u, it leaves invariant every point of the trajectory $u(\lambda)$. It is the isotropy group of all points of a dense subset of the trajectory and we call it the isotropy group of the trajectory. So the isotropy group of a fixed point contains the isotropy group of all trajectories passing through this fixed point : this renormalization process can only increase the symmetry. We can now precise the stability condition : Given

the physical symmetry $H = \mathrm{Im}\, \mathcal{D} = \mathcal{D}(\Gamma)$, the quartic term in the Landau potential (3) and the Hamiltonian density (4) is defined by a vector $u \in P_4^H$; its trajectory stays in this subspace and the stability condition for u^*, one of the fixed points, depends only on the restriction to P_4^H of the operator $\frac{d\beta}{du}(u^*)$. Precisely

$$\mathrm{Re}(\mathrm{Spectrum}\ \frac{d\beta}{du}(u^*)\Big|_{P_4^H}) > 0 . \tag{11}$$

The stable fixed point leads to a second order phase transition only if

$$\forall \phi \neq 0 \quad , \quad u^*(\phi) > 0 . \tag{11'}$$

In [12] formulae giving $\dim P_4^H$ for a given H are recalled.

To go deeper in the study of the covariance, we must use the following group theoretical concept. The centralizer $C(Q)$ (respectively the stabilizer $S(Q)$) in $O(n)$ of an arbitrary subspace $Q \subset P$ is the largest $O(n)$ subgroup which leaves fixed every point of Q (resp. transforms Q in itself). We denote by $C(G)$ and $N(G)$ respectively the centralizer and normalizer of $G < O(n)$; they are the largest $O(n)$ subgroups which centralizes (resp. stabilizes) G by conjugation in $O(n)$ (equivalently : $N(G)$ is the largest $O(n)$ subgroup which contains G as invariant subgroup). Finally, we use the general notation $<$ for subgroup, $\lhd$ for invariant subgroup, and when $< X$, $B < X$, $A.B = \{ab, \forall a \in A , \forall b \in B\}$; $A.B$ is a subgroup only when $A.B = B.A$; this is the case for instance if $B \lhd X$. Then the general results are well known and easy to prove :

$$C(Q) \lhd S(Q) \ , \ (C(G).G) \lhd N(G) \ , \ N(H) < S(P_4^H) \tag{12a,b,c}$$

and for $u \in P_4^H$,

$$H < C(P_4^H) < O(n)_u < S(P_4^H) = N(C(P_4^H)) . \tag{12'}$$

It may happen that $C(P_4^H)$ is a strict subgroup of all isotropy groups $O(n)_u$.

Equation (12') leads us to two remarks : i) When $H = \mathcal{D}(\Gamma)$ is a strict subgroup of $C(P_4^H)$ the relevant symmetry group of our particular problem is $C(P_4^H)$, larger than the physical symmetry (this was for instance emphasized in [14]) and $O(n)_u$ is the relevant symmetry group for a chosen quartic polynomial. ii) The stabilizer $S(P_4^H)$ acts on P_4^H effectively through the quotient

$$Q_H = S(P_4^H)/C(P_4^H) \tag{13}$$

so, as was noted in [10], [15] , if u^* is a fixed point in P_4^H , all points of the $S(P_4^H)$ (or Q_H) orbit of u^* are also fixed points and the corresponding operators $\frac{d\beta}{du}(u^*)$ have the same spectrum :

$$g \in S(P_4^H) , \frac{d\beta}{du}(gu^*) = g \frac{d\beta}{du}(u^*)g^{-1}. \tag{14}$$

As we shall show in the appendix, Q_H is either finite or a compact Lie group of dimension 1 (when n is even and $\mathcal{D}(\Gamma)$ is reducible in the complex) or 3 (when n is divisible by 4 and $\mathcal{V}(\Gamma)$ is a quaternionic representation). Then note that

$$\text{Ker } \frac{d\beta}{du}(u^*) \supset T_{u^*}(Q_H(u^*)), \tag{15}$$

the tangent plane at u^* to the Q_H orbit of u^* . So when $\dim Q_H > 0$ the stability is "marginal". More generally what is the meaning of this multiplicity of equivalent stable fixed points ? As it seems suggested in [10] it is tempting to make the assumption : <u>When it exists the stable fixed point is unique.</u> This has been proven in [11], [12], [13] for the β computed in [6] up to two-loop expansion in $\varepsilon = 4-d$.

Indeed β can only be computed approximatively. We will not discuss

here the validity of the ε-expansion used for computing it. We remark that the $O(n)$ covariance limits strongly the number of possible terms occurring in an expansion in powers. Let us consider first $O(n)$ invariant homogeneous polynomials[*] of the degree k in u . They are obtained by saturating the indices of k copies of $u_{ijk\ell}$ and an arbitrary number of the invariant s whose tensor is $s_{ijk\ell} = \frac{1}{3}(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$. To write these invariants we use a short hand "chemical" notation where s and u are tetravalent. Up to degree 4 in u , a basis of algebraically independent invariants is found to be (this list will be partly justified below : see equations (22))

$$\text{degree } 1 : \quad s \equiv u \leftrightarrow (s,u) \quad ; \quad 2 : u \equiv u \leftrightarrow (u,u) \quad ; \quad 3 : u \overset{u}{\underset{\ }{=}} u \qquad (17)$$

$$4 : \quad \begin{matrix} u = u \\ \| \quad \| \\ u = u \end{matrix} \quad , \quad \begin{matrix} u = u \\ |\!\times\!| \\ u = u \end{matrix} \quad , \quad \begin{matrix} u \equiv u \\ | \quad | \\ u \equiv u \end{matrix} \, . \qquad (17')$$

By the same method one can find the $O(n)$ covariant vector field as chemical radical with 4 free valences : average over the permutations of the 4 free indices has to be performed. We obtain the gradient (∇) of these six invariants

e.g.,

$$u = \frac{1}{2}\nabla(u,u) \quad , \quad u_{\vee}u \leftrightarrow (= u = u =) = \frac{1}{3}\nabla \begin{matrix} u \\ /\!/\ \backslash\!\backslash \\ u = u \end{matrix} \qquad (18)$$

and the constant vector field $s = \frac{1}{2}\nabla(s,u)$. The chemical notation is nothing else than the short hand notation for Feynman diagrams :

e.g.,

$$= u\!\!\Big\langle \begin{matrix} u- \\ \| \\ u- \end{matrix} = \frac{1}{4}\nabla \cdot \begin{matrix} u = u \\ |\!\times\!| \\ u = u \end{matrix} \longleftrightarrow \rightarrow\!\!\bowtie\!\!\oslash \quad . \qquad (18')$$

The symbol $\vee$ can also be defined by

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[*] They are polynomials on the space P of polynomials in ϕ !

$$(u_v v)(\phi) = \mathrm{tr}\, T_u T_v \quad \text{with} \quad (T_u)_{ij} = \frac{1}{12}\, \frac{\partial^2}{\partial \psi_i \partial \phi_j}\, u \qquad (19)$$

so
$$u = (u, T_u u). \qquad (19')$$

Note that as trace of the square of a symmetric matrix,

$$\phi \neq 0 \;\Rightarrow\; u_v u > 0 \,. \qquad (20)$$

More precisely, using also (19'), the Schwarz inequality yields [6] :

$$u_v u(\phi)\; s(\phi) \geq u(\phi)^2 \,. \qquad (21)$$

The product $_v$ on P_4 defines a $O(n)$ covariant commutative but not associa-
tive algebra. In the IVth colloquium of this series I explained how to construct
algebras of this type from a third degree invariant and I gave some of their
properties [16] (see also [11] for more details)[*]. Here are some typical
relations satisfied by $s, u, v \in P$:

$$s_v u = \frac{1}{3n}\,(s, u)\,s + \frac{2}{3}\,u \quad , \quad s_v s = \frac{n+8}{9}\,s \qquad (22)$$

$$\overset{s}{\underset{u\,=\,v}{/\!\!/\,\backslash\!\!\backslash}} \;\leftrightarrow\; (s, u_v v) = (s_v u, v) = \frac{1}{3n}\,(s, u)\,(s, v) + \frac{2}{3}\,(u, v) \qquad (22')$$

$$= s\!\!\overset{u-}{\underset{v-}{\Big\langle\!\!\Big\|}} \;\leftrightarrow\; \frac{1}{3n}\,(u, v)\,s + \frac{2}{3}\,u_v v \;,\; = u\!\!\overset{s-}{\underset{v-}{\Big\langle\!\!\Big\|}} \;\leftrightarrow\; \frac{1}{3n}\,(s, v)\,u + 2u_v v \;. \qquad (22'')$$

We must also point out that the irreducibility of $O(n)_u$ implies

$$\begin{matrix} u & \equiv & u \\ | & & | \\ u & \equiv & u \end{matrix} \;\leftrightarrow\; \frac{1}{n}\,(u, u)^2 \,. \qquad (23)$$

(*) These algebras have often been used in physics, e.g. Gell-Mann d-algebra
for SU(3) (see more references in [5] works with Radicati) and in mathematics :
the "monster" or "friendly giant" can be defined as the automorphism of such an
algebra built from a smaller sporadic finite simple group.

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If we had listed the degree 5 invariants in u alone we would have given five linearly independent ones; similarly there are ten degree 4 linearly independent vector fields; so only half of them are gradients. The situation is quite different for the vector fields of degree 3 or less. They are all gradients [12]. However, this is no longer true when we multiply them with coefficients containing s : the $O(n)$ covariant fields of degree less than 3 in u (up to linear independence) which are not gradients are :

$$(s,u)u, \quad (s,u)^2 u, \quad (s,u)\, u_\vee u \, . \tag{24}$$

Nevertheless, as remarked in [17], the β-vector field computed up to two-loop expansion in [6] is a gradient :

$$\beta(u) = \frac{d}{du}\, \phi(u) \tag{25}$$

$$\phi(u) = -\frac{\epsilon}{2}\,(u,u) + \frac{1}{2}\,(1 + \frac{\epsilon}{2})(u_\vee u,u) - \frac{3}{8}\, \Big|\!\!\underset{u\,=\,u}{\overset{u\,=\,u}{\bowtie}}\!\!\Big| + \frac{1}{48n}\,(1 + \frac{5}{4}\,\epsilon)(u,u)^2 . \tag{25'}$$

To the same order, the corresponding critical exponents are [*]

$$\eta = \frac{1}{24n}\,[\,(1 + \frac{5}{4}\epsilon)(u,u) - \frac{3}{4}\,(u_\vee u,u)\,], \tag{26}$$

$$\frac{1}{\nu} = 2 - \frac{1}{2n}\,(1 + \frac{\epsilon}{2})(s,u) + \frac{5}{24}\,(1-\epsilon)(u,u)_. \tag{26'}$$

In this approximation the stable fixed points in P_4^H are the minima of the restriction of $\phi(u)$ on this subspace. Assuming an ϵ expansion for the fixed points :

$$u^* = \epsilon\tilde{u} + \sum_{k=2}^{\infty} \epsilon^k \tilde{u}_k \tag{27}$$

the first term $\epsilon\tilde{u}$ is an extremum of the polynomial :

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[*] Equations (25'), (26) correct some misprints in [12].

$$\phi^{(1)}(u^*) = \frac{1}{2}\left(-\epsilon(u,u)+(u_\vee u,u)\right) , \tag{28}$$

and it is a solution of the equation

$$\beta^{(1)}(u^*) = \frac{d\phi^{(1)}}{du}(u^*) = 0 \Leftrightarrow \widetilde{u}_\vee \widetilde{u} = \frac{2}{3}\widetilde{u}. \tag{29}$$

At this first order on ϵ ,

$$\frac{d^2\phi^{(1)}}{du^2}(u^*) = \frac{d\beta}{du}(u^*) = \epsilon(3D_{\widetilde{u}}-I) \tag{30}$$

where the linear operator D_u , is defined by

$$\forall v \in P_4 , \quad D_u v = u_\vee v . \tag{31}$$

If for a u^* solution of (28)

$$H(u^*) = \epsilon H(\widetilde{u}) = \frac{d\beta}{du}(u^*)\Big|_{P_4^H} \tag{32}$$

is invertible, the higher terms in the expansion (27) can be computed unambiguously; they are of the form $\widetilde{u}_k = H(\widetilde{u})^{-1}p(\widetilde{u}_i)$, where p is a polynomial in the $\widetilde{u}_i$'s , $i < k$. ($\widetilde{u}$ is a short for $\widetilde{u}_1$) .

As in many other physical problems, equation (29) shows that we are led to study the idempotent of a $\vee$ algebra [5] [16] . In any 2-dimensional subalgebra, this leads to a degree-3 polynomial equation on the real field : it yields either one or 3 idempotents. For 2 dimensional subalgebra containing the $O(n)$ invariant vector s , I do not know of examples with one idempotent; however, for $n = 4$, s becomes a double solution of the equation (and even a triple solution in one case [13]). Then $H(u^*)$ is not invertible.

When this is the case, there is a "bifurcation". This is the case for

instance for all fixed points for $n = 4$. This special value of n has been thoroughly studied in [13]. To resolve the bifurcation one expands in the neighbourhood of the u^* solution of (28). If a quadratic equation is sufficient for resolving the bifurcation, one has exactly the same type of problem to solve. For instance for $n = 4$ the stable fixed points are near $\epsilon \breve{s} = \epsilon \frac{s}{2}$, i.e

$$u^* = \epsilon \left(\frac{s}{2} + a\right) \quad , \quad \epsilon \to 0 \Rightarrow a \to 0. \tag{33}$$

Then

$$\epsilon^{-5} \beta(a) = \left(\frac{1}{2n}(s,a) - \frac{1}{48}\right) s + \epsilon \left(-\frac{1}{3n}(s,a) s - \frac{5}{24} a + \frac{3}{2} a \vee a\right) + O(\epsilon). \tag{34}$$

In all cases the functions $u^*(\epsilon)$ is defined by an ϵ expansion, and the lowest significant order in ϵ is a solution of a quadratic equation in the $\vee$-algebra.

Here we will deal only with the simplest case (28) but, as shown in [12], the same results can be generalized (for instance for the bifurcation at $n = 4$). If $u^* = \epsilon \tilde{u}$ is a solution of (29), from (20) we deduce that $u^* = \epsilon \tilde{u} > 0$ for $\phi \neq 0$ so it is a physically acceptable solution (see (11')). With such a solution the polynomial $\phi^{(1)}(u^*)$ in (28) has the value

$$\phi^{(1)}(u^*) = -\frac{\epsilon^3}{6} (\tilde{u}, \tilde{u}) , \tag{35}$$

so the stable points have the greatest length. More generally, for the idempotents, all algebraic invariants [17] become related. Indeed, with (29) equation (22') reads [8]:

$$\tilde{u} \vee \tilde{u} = \frac{2}{3} \tilde{u} \Rightarrow 2 \frac{(\tilde{u}, \tilde{u})}{n} = \frac{(s, \tilde{u})}{n} \left(2 - \frac{(s, \tilde{u})}{n}\right). \tag{36}$$

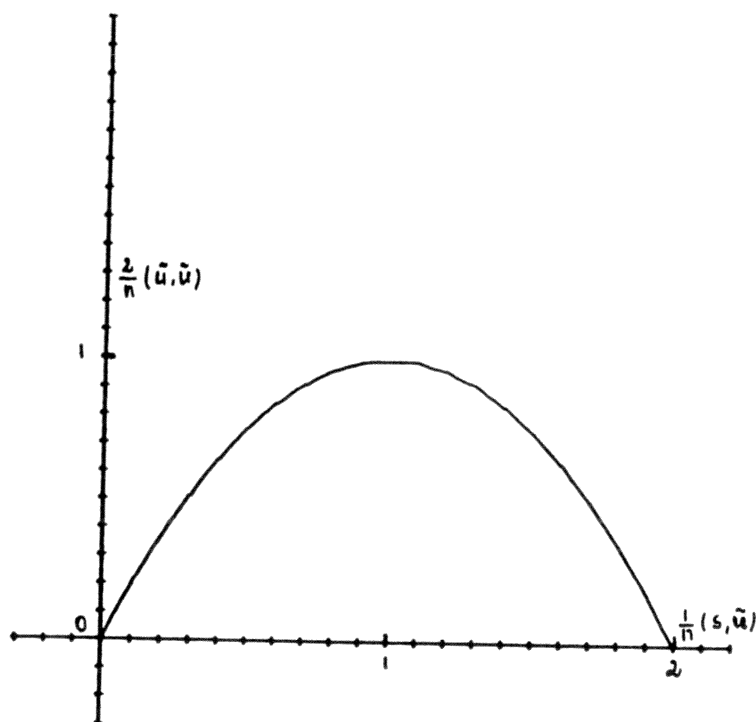This function is plotted (parabola) in Fig. 1. The conditions $(\tilde{u}^{(4)}, \tilde{u}^{(4)}) \geq 0$ yields [6]:

Figure 1

$$0 < \frac{1}{n} (s,\widetilde{u}) < \frac{1}{n}(s,\widetilde{s}) = 2\, \frac{n+2}{n+8} \,. \tag{37}$$

With equation (36) this shows that for $n < 4$ the isotropic fixed point

$$s^* = \varepsilon \widetilde{s} = \varepsilon\, \frac{6}{n+8}\, s \tag{38}$$

is the longest one and therefore it is the only stable fixed point (This result was found in [6]). For $n > 4$, $\varepsilon \widetilde{s}$ is stable only for the one dimensional subspace $P_4^{0(n)}$, i.e. for the $0(n)$ invariant problem.

The algebraic relation (36) between the invariants of an idempotent leads to the simpler expression for the lowest ε-order of the initial exponents :

$$\eta = \frac{\varepsilon^2}{24} (\widetilde{u},\widetilde{u}) \quad , \quad \frac{1}{\nu} = 2 - \frac{\varepsilon}{2n} (s,\widetilde{u}). \tag{39}$$

Since the stable ones are the longest among the fixed points, equation (39) proves the "intriguing conjecture" at the end of [6] that the stable fixed point has the largest critical exponent η : see [12] and in [3] it is shown that at two loop order all fixed points have the same critical exponents. Equation (36) yields a quadratic relation between η and ν at a fixed point u^*, given in [6] :

$$12\eta(u^*) = (2-\nu(u^*)^{-1})(\varepsilon-2+\nu(u^*)^{-1}). \qquad (40)$$

Finally, the best result known on the stable fixed point is the

Theorem : If it exists the fixed point is unique.

The proof in the lowest order of ε was given in [11] (see also [12]); since it is essentially due to the idempotent property it extends to two loop approximation (the extension is done in [13]). It would be very interesting to know if this is valid independently from this approximation as suggested in [13].

We give here the sketch of the proof per absurdum. Assume that there exist in P_4^H two stable distinct fixed points $\varepsilon\widetilde{u}$, $\varepsilon\widetilde{v}$; from (35) we know that $(\widetilde{u},\widetilde{u}) = (\widetilde{v},\widetilde{v})$ and on the line $\lambda\widetilde{u}+(1-\lambda)\widetilde{u}$ the degree 3 polynomial $\psi^{(1)}$ has two extrema with the same value, so it is constant. This implies the vanishing of the Hessian expectation values:

$$(\widetilde{u}-\widetilde{v},H(\varepsilon\widetilde{u})(\widetilde{u}-\widetilde{v}) = 0 = (\widetilde{u}-\widetilde{v},H(\varepsilon\widetilde{v})(\widetilde{u}-\widetilde{v})H \qquad (41)$$

However, $\widetilde{u}-\widetilde{v}$ is not an eigenvector with zero eigenvalues : indeed from (30), (32) and (29) (this extends to (33), (34)),

$$\varepsilon^{-1}H(\varepsilon\widetilde{u})(\widetilde{u}-\widetilde{v}) = \widetilde{u}+\widetilde{u}-3\widetilde{u}\vee\widetilde{v} = \varepsilon^{-1}H(\varepsilon\widetilde{v})(\widetilde{v}-\widetilde{u}) = 0. \qquad (42)$$

Taking the scalar product with u and v and using again (29) (or (33), (34)

for the next order) this implies $\tilde{u} = \tilde{v}$ which is absurd. However, $H(\epsilon\tilde{u})$ can have a zero expectation value for a non eigenvector only if it has negative eigenvalues: this contradicts the assumption that $\epsilon\tilde{u}$ is a minimum.

Incidentally this proof shows that the other fixed points are on the boundary of the attracting basin of the fixed point (see [12] for more details).

Several obvious corollaries can be deduced from this theorem, e.g., the isotropy group of the stable fixed point (if it exists) of P_4^H is not smaller than the stabilizer $S(P_4^H)$. The neatest corollaries were deduced by J.C. Toledano [19] after common discussions and examination of the complete results of [13]. For a given n :

I. The isotropy groups G_i of the stable fixed points of the different sub-spaces P_4^H satisfies $G_i = N_{O(n)}(G_i)$ (consequence of (12')).

II. If no G_i contains $S(P_4^H)$, this subspace has no stable fixed point.

We can now ask relevant questions on the stable fixed points, based purely on group theory considerations.

Question 1. For a given n , what are the strict subgroups G_i of $O(n)$ such that $G_i = N(G_i)$, candidates as symmetry groups of stable fixed points ? Of course we must also have $\dim P_4^{G_i} > 1$.

In the appendix it is proven that a finite $G < O(n)$ must be completely irreducible (i.e. irreducible also in the complex). In [11] I gave a set of $\sigma_0(n)$ (i.e. the number of divisors of n) of such groups that I denoted Γ_{pq} , where the integers p,q satisfy $pq = n$. The group Γ_{pq} is the wreath product $O(p) \textcircled{w} S_q$ where S_q is the permutation group of q objects, i.e. Γ_{pq} is the semi-direct product

$$\Gamma_{pq} = 0(p)^q \rtimes S_q \quad , \qquad \dim \Gamma_{pq} = \frac{1}{2} n(p-1), \tag{43}$$

where S_q acts by permutation of the q factors of $0(p)^q$. Note that $\Gamma_{n1} = 0(n)$; so from now on we assume

$$p \quad \text{divides} \quad n \quad , \quad 1 \le p < n. \tag{44}$$

For $p = 1$, Γ_{1n} is the symmetry group of the hypercube. It is a group generated by reflexions and is denoted B_n by Coxeter (e.g. [20])

$$\Gamma_{1n} \sim B_n \sim (Z_2)^n \rtimes S_n \quad , \qquad |B_n| = 2^n . n! \tag{45}$$

The groups Γ_{pq} have the properties :

$$p < n = pq, \Gamma_{pq} \text{ is irreducible on the complex },$$
$$\dim P \, _{4}^{\Gamma_{pq}} = 2. \tag{46}$$

If we split the indices i of ϕ as a pair α, β , the polynomial

$$x_{pq} = \sum_{\beta=1}^{q} (\sum_{\alpha=1}^{p} \phi_{\alpha\beta}^2)^2 \tag{47}$$

is invariant by Γ_{pq}. When $p \neq 4 \neq n$, the 2 dimensional sub algebra $P \, _{4}^{\Gamma_{pq}}$ has 3 distinct idempotents

$$\tilde{s} = \frac{6}{n+8} s \, , \, \tilde{x}_{pq} = \frac{6}{p+8} x_{pq} \, , \, \tilde{y}_{pq} = \frac{6}{n(p+8)-16(p-1)} [(4-p)s+(n-4)x_{pq}]. \tag{48}$$

When $n > 4$ I have shown [11], that

$$p < 4 \, , \tilde{y}_{pq} \text{ is stable,} \quad p > 4, \tilde{x}_{pq} \text{ is stable.} \tag{49}$$

(For $p = 1$, $x_{1n} = c$, this result was proven by Aharony [8]). Incidentally this proves (with extension to two loop order) that for $n \ge 4$, Γ_{pq} is its own normalizer in $0(n)$!

To these Γ_{pq}'s one can add another group for all $n \geq 4$. Indeed the invariants of the Coxeter groups (i.e. the groups generated by reflections are all known [20]. The group $A_n \sim S_{n+1}$ has a third degree invariant, so it is excluded by Landau theory, but $Z_2(I,-I) \times A_n$ of $2.(n+1)!$ elements is acceptable and for $n \geq 3$ its space of quartic invariants is 2-dimensional. Moreover, it is its own normalizer in $O(n)$ [(*)]. Hence $Z_2 \times A_n$ is a good candidate as symmetry group of a stable fixed point in n dimensions. The other Coxeter groups are not [(**)].

For $n < 4$, we have seen that $\varepsilon \tilde{s}$ is the only stable fixed point.

For $n = 4$, there are 17 irreducible isotropy groups (and 22 irreducible centralizers of P subspaces) [13], which correct some errors of [22]. Only $O(4)$ and three subgroups are equal to their normalizers : these are $\Gamma_{22} = O(2)^2 \rtimes S_2$, $\Gamma_{14} = B_4$, $Z_2 \times A_4$. They are indeed the symmetry groups of the only 4 stable fixed points for subspaces of maximal dimension equal respectively to $1,4,2,2$.

When n is odd, $n = 2\ell+1$, the orthogonal representation of spin ℓ of $SO(3)$ is completely irreducible. For $n \geq 7$, dim $P_4^{SO(3)} > 1$ (e.g. [24] which computes it as 2 for $n = 7$, and for $n = 9,...$). Note that $C(SO(3)) = Z_2 \times SO(3) = O(3)$. Since Aut $SO(3) = SO(3)$, equation A(1) of

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(*) The normalizer transforms reflections into reflections, so it transforms A_n into itself in the product $A_n \times Z_2$. Of all permutation groups S_{n+1}, only $A_5 = S_6$ has an outer automorphisms, but those do not preserve reflections (which are the transportions of two objects). To complete the proof, see Appendix and the fact that A_n is irreducible on the complex.

(**) $G_2, F_4, E_6, E_7, E_8, H_3, H_4$ have no quartic invariants $\neq s$. D_n is defined for $n \geq 4$. It is a subgroup of index 2 of B_n. For $n > 4$ it has the same quartic invariants as B_n and therefore it is not an isotropy group. For $n = 4$, dim $P_4^{D_4} = 3$. However $N_{O(u)}(D_4) = F_4$ so this 3 dimensional space has no stable fixed points.

the appendix shows that $N(SO(3)) = O(3)$ (for $\ell > 3$ these $O(3)$ are even maximal subgroups of $O(2\ell+1)$ so they are also good candidates as isotropy groups of stable fixed points).

With the table [24], it is possible to find all compact semi-simple Lie groups K with an orthogonal representation of dimension p or w·h an essentially complex representation of dimension q. Then choose $n = 1$ or $n = 2q$, or consider the wreath product $K \textcircled{w} S_k$ with $n = pk$ or $2qk$. One has then to check if they have quartic invariants different from s. Finally all finite irreducible groups with complex representation of dimension ≤ 7 are classified : see [25], §8.5 and references there.

Question 2. Give families of irreducible subgroups $H < O(n)$ such that P_4^H has no stable fixed points.

Dzyaloshinskii [26] had a "strong conviction" that this was the case if $\dim P_4^H > 3$. As we have seen, there is already a counter-example for $n = 4$. Infinite families of counter examples were given before in [27] and [11].

J.C. Toledano's criterion II (quoted above) is the more precise. It shows that for half of the 22 irreducible centralizers for $n = 4$, there are no stable fixed points, e.g. the finite groups with quaternionic representations.

A systematic method to find some families of subspace answering question 2 would be to look for irreducible subgroup $G < O(n)$ with $\dim P_4^G = 1$, reduce their representation on P and determine which of the invariant spaces have irreducible centralizers.

Whatever the interest of the pure mathematical problems (easier to solve for n prime!) one must not forget that we are particularly interested by the three dimensional crystallographic space groups Γ. So we must use

what is known about the images of their irreducible representations (see e.g. [28]) . Their complex dimension divides 48. They are finite when induced from a momentum wave vector with rational coordinates (taken modulo 1). When at least one coordinate is irrational, the closure of the real image is $< \Gamma_{2n}$, n divides 48.

Appendix. Structure of the normalizers of real irreducible subgroups of $O(n)$.

We denote by $C(G)$ the center of a group G ; its group of inner automorphisms is the quotient In Aut $G = G/C(G)$; it is an invariant subgroup of Aut G , the groups of automorphisms of G . We define Out G = Aut G/In Aut G. When G is a compact Lie group, Out G is discrete [29]. Moreover let G be a closed subgroup of $O(n)$: it is compact, or, as a particular case finite. There is a natural injective homomorphism θ :

$$1 \to \frac{N(G)}{C(G).G} \overset{\theta}{\to} \text{Out } G \quad \text{and} \quad \text{Im } \theta \text{ is finite.} \tag{A1}$$

We denoted by Z_2 the center of $O(n)$ (it contains I and $-I$). We assume $\mathbf{Z_2} < G$. If n odd, the irreducibility of G on the real implies its irreducibility on the complex. Moreover $C(G) = C(G) = Z_2$ and one can prove that if G is the centralizer of a subspace P^G , it is also an isotropy group of a polynomial in P^G .

If n even, the real irreducible $G < O(n)$ might be reducible on the complex; if $\frac{n}{2}$ is odd, then $C(G) \sim O(2)$: it is the diagonal subgroup of $O(2)^{n/2}$. When $\frac{n}{2}$ is even, the complex irreducible representation might be reducible on the quaternions. Then $C(G) \sim SU(2)$ diagonal subgroup of $O(4)^{n/4}$.

If G is finite and reducible in the complex, then $N(G) \neq G$ as used in the text.

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