Argonne Summer Study "High Energy Physics with Polarized Beams"

Talk given by Louis MICHEL

at the Workshop "Acceleration and Storage of Polarized Beams"

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My first duty is to answer your question : "How to derive equation (1), the starting equation of the fundamental Froissart and Stora paper. "Depolarisation d'un faisceau de protons polarisés dans un synchrotron" (Nucl. Inst. and methods 7 (1960) p. 297-305). This equation reads (their notations[1]) for the precession of the polarization $\vec{s}$ in the magnetic field $\vec{B}$

$$\frac{d\vec{s}}{dt} = \frac{e}{m\gamma} \vec{s} \times [\vec{B} + \frac{g-2}{2}(B_{/\!/} + \gamma B_{\perp}] \qquad (1)$$

It can also be written in the form :

$$\frac{d\vec{s}}{dt} = \frac{e}{m\gamma} \vec{s} \times [\frac{g}{2} \vec{B}_{/\!/} + (1 + \gamma \frac{g-2}{2})\vec{B}_{\perp}] \qquad (1')$$

Froissart and Stora give the following reference for this equation

1) Case et Mendlowitz, Phys Rev. 97 (1955) 33; Case, Phys. Rev. 406 (1957) 175

L. Michel communication privée

V. Bargmann, L. Michel et V.L. Telegdi, Phys. Rev. lett. 2 (1959 435.

(quoted hereafter BMT)

Equation (1) shows that the polarization precession is obtained as the sum of three rotations :

$$\frac{d\vec{s}}{dt} = \vec{s} \times \vec{\Omega} \quad \text{with} \quad \vec{\Omega} = \vec{\Omega}_L + \vec{\Omega}_{/\!/} + \vec{\Omega}_{\perp} . \qquad (2)$$

1) $\vec{B}_{/\!/}$ and $\vec{B}_{\perp}$ are the components of $\vec{B}$ parallel and perpendicular to the velocity $\vec{v}$ of the particle

Ω_L is the Larmor precession : in the rest frame of the particle, with time τ (= proper time) $\vec{\Omega}_L = \frac{e}{m}\vec{B}$ ($\hbar = c = 1$) . In the laboratory $t = \gamma\tau$ with $\gamma = 1/\sqrt{1-v^2}$, so $\frac{d}{dt} = \frac{1}{\gamma}\frac{d}{d\tau}$ and

$$\vec{\Omega}_L = \frac{e}{m\gamma}\vec{B}_\perp = \omega\,\vec{B}_\perp \qquad (2')$$

$\vec{\Omega}_{/\!/} + \vec{\Omega}_\perp$ is the Thomas precession. In the BMT paper $\vec{\Omega}_{/\!/}$ and $\vec{\Omega}_\perp$ are deduced from the covariant equation respectively in case (A) and (B) ;

$$\vec{\Omega}_{/\!/} = \frac{ge}{2m\gamma}\vec{B}_{/\!/} = \frac{g}{2}\,\omega\,\vec{B}_{/\!/} \qquad (2'')$$

does not affect the longitudinal polarization and rotate the transverse polarization while

$$\vec{\Omega}_\perp = \frac{g-2}{2}\,\frac{e}{m}\,\vec{B}_\perp = \frac{g-2}{2}\,\gamma\omega\,\vec{B}_\perp \qquad (2''')$$

transforms longitudinal polarization into transverse at a rate proportional to $(g-2)$.

Professor Ernest Courant will give you a direct derivation of equation (1)(which was first written by Thomas in 1926, see Appendix 1 for some points of history) and, as you will see, we all agree !

I will derive (1) again from first principles, but this will be more incidental in my talk. I wish to use my allotted time for some teaching : distinguish spin and polarization, and for pleading for covariance.

We all agree that a spin $\frac{1}{2}$ particle at rest has an intrinsic angular momentum $\frac{1}{2} < \vec{\sigma} > \hbar$, where $< \vec{\sigma} >$ is the expectation value of the three Pauli matrices. For spin j particles, the three generators of the rotation group, $\frac{1}{2}\sigma_x, \frac{1}{2}\sigma_y, \frac{1}{2}\sigma_z$ have to be replaced by the $2j+1$ by $2j+1$ corresponding matrices. The particle magnetic moment is

$$\vec{\mu} = g \frac{e\hbar}{2mc} < \vec{\sigma} > \qquad (3)$$

(this define the gyromagnetic ratio g); in a magnetic field, the particle stays at rest and the spin precesses according to the equation

$$\frac{d<\vec{\sigma}>}{d\tau} = \vec{\mu} \times \vec{B} = \frac{g}{2}\frac{e}{m} < \vec{\sigma} > \times \vec{B} \qquad (4)$$

To find the relativistic generalization of (4) one can always transform the electromagnetic field $\vec{E}$, $\vec{B}$ in the rest frame of the particle. (This was actually done by Thomas in 1926). The method I prefer is to write a covariant equation : it is a simpler.(1)

Let $\underline{x}(t,\vec{x})$ be the four coordinates of the particle (2)

$$\frac{d}{d\tau}\,\underline{x} \equiv \dot{\underline{x}} = \underline{p}/m = \underline{u} = (\gamma, \gamma\vec{v}) \qquad (5)$$

where $\underline{u}$ is the four velocity and $\underline{p} = (E,\vec{p})$ is the energy momentum.

$$\underline{p}^2 = m^2 \sim \underline{u}^2 = 1 \qquad (6)$$

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(1) You have the right to disagree if you prefer the form $(E-\frac{1}{2}\,\vec{\alpha}\cdot\vec{p}+\beta m)\psi = 0$ of the Dirac equation $(\gamma^{\mu})\partial_{\mu} + m)\psi = 0$ and if you never consider momentum $\vec{p}$ and energy E as components a four vector $\underline{p}$, but you prefer to obtain the new velocity $\vec{v}'$, from the old one $\vec{v} = \vec{p}/E$, by the Lorentz transformation of velocity $\vec{w}$, according to the law :

$$\vec{v}' = (\vec{w} + \frac{1}{\gamma_0}\vec{v} - \frac{\gamma_0}{1+\gamma_0}(\vec{v},\vec{w})\vec{w})(1+\vec{v},\vec{w})^{-1}$$

with $\gamma_0 = (1-w^2)^{-1/2}$

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(2) Massless particles are not localizable except however for spin 0 , and four-component spin $\frac{1}{2}$ where a position operator can be defined.

The electric field $\vec{E}$ and the magnetic field $\vec{B}$ form a skew symmetric tensor[3] $\underline{\underline{F}}(-\vec{E}, -\vec{B})$.

The motion of the particle is given by the Lorentz equation

$$\dot{\underline{u}} = \frac{e}{m} \underline{\underline{F}} \cdot \underline{u} \tag{7}$$

(Remark that $\dot{u} \cdot u = \frac{e}{m} \underline{u} \cdot \underline{\underline{F}} \cdot \underline{u} = 0$ since $\underline{\underline{F}}$ is antisymmetric; so (6) is compatible with (7)).

In a frame with time $t = \gamma\tau$, fields $\vec{E}(t,x)$, $\vec{B}(t,x)$, equation (7) is equivalent to

$$\frac{d}{dt} \gamma = \frac{e}{m} \vec{E} \cdot \vec{v} \ , \ \frac{d}{dt} \vec{v} = \frac{e}{m\gamma} (\vec{v} \times \vec{B} + \vec{E} - (\vec{E} \cdot \vec{v})\vec{v}) \tag{8}$$

For example, a particle at rest, $\vec{v} = 0$, has an acceleration $\frac{d\vec{v}}{dt} = \frac{e}{m\gamma} \vec{E}$. For a pure magnetic field $(\vec{E} = 0)$, equation (8) becomes

$$\frac{d}{dt} \vec{v} = \frac{e}{m\gamma} \vec{v} \times \vec{B} = \vec{v} \times \vec{\Omega}_L \tag{9}$$

(see 2' and 2)

The relativistic generalization of angular momentum $\vec{J}$ is a skew symmetric tensor $\underline{\underline{M}}(\vec{K}, \vec{J})$ which is an operator in quantum mechanics, sum of the orbital momentum and of the spin (= intrinsic) angular momentum $\underline{\underline{M}}_s$:

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(3) With the explicit use of components indices (see e.g. BMT); greek letter 0,1,2,3, latin letters 1,2;3; $g_{oo} = g^{oo} = 1$, $g^{ii} = -1 = g_{ii}$. Vector $\underline{a} = (a^o, a^i)$, skew symmetric tensor $\underline{\underline{T}}(\vec{T}^t, \vec{T}^s)$ with $(\vec{T}^t)^i = T^{oi}$, $(\vec{T}^s)^i = T^{jk}$, ijk circular permutation of 1,2,3. The polar tensor of $\underline{\underline{T}}$ is $\underline{\underline{T}}^D = (\vec{T}^s, -\vec{T}^t)$, i.e. $T^D_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} T^{\rho\sigma}$. Then $\underline{a} \cdot \underline{b} = a^\mu g_{\mu\nu} b^\nu = a^o b^o - \vec{a} \cdot \vec{b}$, $(T \cdot u)^\mu = T^{\mu\nu} g_{\nu\rho} a^\rho$. With two vectors $\underline{a}, \underline{b}$ we form a tensor $(\underline{a} \otimes \underline{b})^{\mu\nu} = a^\mu b^\nu$ and an antisymmetric tensor $\underline{a} \wedge \underline{b} = \underline{a} \otimes \underline{b} - \underline{b} \otimes \underline{a}$.

$$\underline{\underline{M}} = \underline{x} \wedge \underline{p} + \underline{\underline{M}}_s \qquad (10)$$

The four components of $\underline{P}$, energy momentum operator ($\underline{p} = < \underline{P} >$) and the six of $\underline{\underline{M}}$ form the ten generators of the Poincaré group and are the kinematical observables.

As early as 1926 Frenkel wrote the classical equation satisfied by $< \underline{\underline{M}} >$ for a particle in a constant electromagnetic field, but it is not easy to extract from it the motion of $< \underline{\underline{M}}_s >$. There is another relativistic generalization of the spin operator $\vec{\sigma}$, the polarization operator $\underline{W}$, which is an axial vector :

$$\underline{W} = \underline{\underline{M}}^D \cdot \underline{P} \qquad (11)$$

(in the rest frame $\underline{\underline{M}} = (\vec{0}, \ j\sigma)$, $\underline{W} = (0, j\sigma)$) .

The polarization $\underline{W}$ has two properties not shared by the angular momentum $\underline{\underline{M}}$: it does not depend on the orbital momentum; the components of $\underline{P}$ and those of $\underline{W}$ commute :

$$[P^{\lambda}, W^{\rho}] = 0 \qquad (12)$$

and this is not true for P^{λ} and $M^{\mu\nu}$. To describe completely particle states we need a complete set of commuting operators. The P^{λ} commute and give energy and momentum; by definition, particle polarization is what completes the description. It is obtained from $\underline{W}$ which commutes with $\underline{P}$ (while $\underline{\underline{M}}$ does not). Kinematically the particle is characterized by the two invariants of the Poincaré group

$$\underline{P}^2 \ , \ \underline{W}^2 \qquad (13)$$

whose eigenvalues label its unitary irreducible representations :

$$\underline{P}^2 = m^2 \qquad \underline{W}^2 = -j(j+1)m^2 \tag{13'}$$

Furthermore $\quad \underline{P}\cdot\underline{W} = 0$ (14)

(from (11) and the antisymmetry of $\underline{\underline{M}}^D$) .

For $m = 0$, $\underline{P}^2 = \underline{W}^2 = 0 = \underline{P}\cdot\underline{W}$ so

$$m = 0, \ \underline{W} = \lambda\underline{P} \quad , 2\lambda \ \text{integer} \tag{15}$$

where the pseudo scalar λ is the helicity.

For $m \neq 0$ the components W^ρ do not commute with each either and a complete set of observables is obtained with the four P^λ and the component $-\underline{W}\cdot\underline{n}$ of W along the quantization axis $(\underline{n}^2 = -1 \ , \ \underline{n}\cdot\underline{p} = 0)$.

A complete description of a $m \neq 0$, spin j particle requires the knowledge of $2j$ multipoles : dipole, quadrupole, etc.... which are the expectation values of the completely symmetric tensors[1] $\frac{1}{m^k}\underbrace{\underline{W}\otimes\underline{W}\otimes\cdots\otimes\underline{W}}_{k \text{ factors}}$ $(1 \leq k \leq 2j)$. The magnetic moment is carried by the dipole.

$$\underline{s} = \langle\frac{\underline{\underline{W}}}{m}\rangle \tag{16}$$

which satisfies

$$\underline{s}^2 = -1 \quad , \ \underline{s}\cdot\underline{u} = 0 = \underline{s}\cdot\underline{p} \qquad (17a) \quad (17b)$$

Equation (4), in the rest frame is linear in the spin and in the electromagnetic field[2]. In the same linear approximation, the most general invariant equation between $\underline{s}$, $\underline{\underline{F}}$, $\underline{u}$ is : (remember that $\underline{\underline{F}}$ is antisymmetric and $\underline{u}\cdot\underline{s} = 0$, $\underline{u}^2 = 1$)

$$\dot{\underline{s}} = \alpha_1\underline{\underline{F}}\cdot\underline{s} + \alpha_2\underline{u}(\underline{u}\cdot\underline{\underline{F}}\cdot\underline{s}) \quad .$$

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(1) For more details, see L. Michel, N. Cim. Suppl. 14 (1959) 95.

(2) Hence new effects due to field gradients, polarisibility of the magnetic moment (as a function of B), etc.... are neglected.

This is compatible with (17a) since $\dot{\underline{s}}\cdot\underline{s} = 0$; the compatibility with (17b) requires $0 = \underline{u}\dot{\underline{s}} + \dot{\underline{u}}\underline{s} = (\alpha_1 + \alpha_2 - \frac{e}{m})\underline{u}\cdot\underline{\underline{F}}\cdot\underline{s}$. In the rest frame $\underline{u}(1,\vec{0})$, $\underline{s} = (0, <\vec{\sigma}>)$, the comparison with (4) yields $\alpha_1 = \frac{g}{2}\frac{e}{m}$ so the final equation is (BMT)

$$\dot{\underline{s}} = \frac{e}{2m}\left(g\,\underline{\underline{F}}\cdot\underline{s} - (g-2)\,\underline{u}(\underline{u}\cdot\underline{\underline{F}}\cdot\underline{s})\right) \tag{18}$$

Since parity and time reversal are violated by weak and CP violating interactions, we can also consider particles carrying also an electric dipole (this was also done in BMT), cf (3)

$$\vec{\delta} = \frac{g'e}{2m} < \vec{\sigma} > \tag{19}$$

The complete equation, to be compatible with (19), can be written in the form

$$\dot{\underline{s}} = \frac{e}{m}\,\underline{\underline{G}}\cdot\underline{s} \tag{20}$$

where the antisymmetric tensor $\underline{\underline{G}}$ is

$$\underline{\underline{G}} = \underline{\underline{F}} + \tfrac{1}{2}\,\underline{\underline{Q}}\cdot((g-2)\underline{\underline{F}} + g'\underline{\underline{F}}^D)\cdot\underline{\underline{Q}} \tag{20'}$$

$$\underline{\underline{Q}} = \underline{\underline{I}} - \underline{u}\otimes\underline{u} \qquad (\text{i.e. } Q^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}) \tag{20''}$$

<u>Remarks</u> 1) If $g = 2$, $g' = v$, $\underline{s}$ satisfies the same equation than $\underline{u}$, i.e. (7).

2) For electrically neutral particle $e = 0$, but $\frac{eg}{2m}$, $\frac{eg'}{2m}$ should be replaced by μ , δ , the magnetic and electric dipole values.

The space part of $\underline{s} = (s^0, \vec{s})$ should not be confused with the polarization denoted by $\vec{s}$ in Froissart's and Stora's equation (1). Indeed, the length of space part $\vec{s}$ of a constant length four vector $\underline{s}^2 = -1$ varies. At each time t it is convenient to introduce for the particle a "tetrad" $\underline{n}^{(\alpha)}$ of four vectors such that

$$\underline{n}^{(o)} = \underline{u} \quad , \quad \underline{n}^{(\alpha)} \cdot n^{(\beta)} = g^{\alpha\beta} \quad , \quad \det(n^{(\alpha)\mu}) = 1 \tag{21}$$

We introduce the 3 components :

$$\zeta^i = -\underline{s} \cdot n^{(i)} \quad \text{i.e.} \quad \underline{s} = \Sigma\, \zeta^i n^{(i)} \tag{22}$$

and $\underline{s}^2 = -1$ requires

$$\sum_i (\zeta^i)^2 \equiv \vec{\zeta}^2 = 1 \tag{23}$$

The notation "$\vec{\zeta}$" is symbolic, except in the rest frame where $\vec{\zeta} = \langle\vec{\sigma}\rangle$, the polarization. So we call $\vec{\zeta}$,the rest frame polarization (and it is the $\vec{s}$ of Froissart and Stora). And following BMT (see also L. Michel, axiomatic Field Theory 1965, Brandeis University Summer School., ed. Chrétien Deser p. 355, Gordon and Breach 1966) we decompose the precession polarization into the sum of two notions, that of the tetrad, which follows the particle (Larmor precession), and that of $\vec{\zeta}$ relatively to the tetrad. Indeed, we assume that the four $n^{(\alpha)}$'s satisfy equation (7)

$$\dot{\underline{n}}^{(\alpha)} = \frac{e}{m}\, \underline{\underline{F}} \cdot \underline{n}^{(\alpha)} \tag{24}$$

From (22) and (20-20') we obtain

$$\dot{s} = \sum_k \dot{\zeta}^k n^{(k)} + \zeta^k \dot{n}^{(k)}) = \frac{e}{m} \sum_\gamma \zeta^j (\underline{\underline{F}} + \tfrac{1}{2}\, \underline{\underline{Q}} \cdot ((g-2)\underline{\underline{F}} + g'F^D) \cdot \underline{\underline{Q}}) \cdot n^{(j)} \quad .$$

We take the scalar product of both members with $\underline{n}^{(i)}$,we use (21) and remark that $\underline{\underline{Q}} \cdot \underline{n}^{(i)} = \underline{n}^{(i)} \cdot \underline{\underline{Q}} = \underline{n}^{(i)}$. We obtain

$$\dot{\zeta}^i = -\frac{e}{m} \sum_j \left(\frac{g-2}{2}\, \underline{n}^{(i)} \cdot \underline{\underline{F}} \cdot n^{(j)} + g' n^{(i)} \cdot \underline{\underline{F}}^D \cdot \underline{n}^{(j)}\right) \zeta^j \tag{25}$$

From now on ijk is an permutation of 123 . We introduce the notations

$$\omega_T^k = \omega_T^{ij} = \underline{n}^{(i)} \cdot \underline{\underline{F}} \cdot \underline{n}^{(j)} \quad , \quad \omega_E^k = \omega_E^{ij} = \underline{n}^{(i)} \cdot \underline{\underline{F}}^D \cdot n^{(j)} \tag{26}$$

Equation (25) can be symbolically written :

$$\dot{\vec{\zeta}} = \frac{e}{m} \vec{\zeta} \times \left(\frac{g-2}{2} \vec{\omega}_T + g' \vec{\omega}_E\right) = \vec{\zeta} \times (\vec{\Omega}_T + \vec{\Omega}_E) \tag{27}$$

For $g' = 0$ this is exactly the term proportional to $\frac{g-2}{2}$ in equation (1). The first term in (1), proportional to $\vec{B}$, is the Larmor precession (here the tetrad motion). Equation (27) represents the Thomas precession $\Omega_T = \Omega_{/\!/} + \Omega_{\perp}$ (defined in equation 2).

I leave as an exercise how to write (27) in term of $\vec{B}$ and $\vec{E}$ in the lab frame. I give below the solution for the simple case $g' = 0$, $\vec{E} = 0$. I conclude with the following remark (see e.g. L. Michel, N. Cim. Supp. 14 (1959)95) which resumes the relation between spin and polarization. Consider the Hilbert space $\mathcal{H}$ of the one-particle states for a particle of mass $m \neq 0$ and spin j . Diagonalize the operator $\underline{P}$ and for each $\underline{p}$ introduce the tetrad (21), and define the operators (compare with (22))

$$S(\underline{p})^{(i)} = -\frac{1}{m} \underline{W} \cdot \underline{n}^{(i)} \tag{28}$$

For each value of $\underline{p}$ the $S(\underline{p})^{(i)}$ are the quantum mechanical operators[(1)] representing the observables ζ^i. From the commutations relation of the $\underline{W}$'s (which are obtained from the commutation relations of the Poincaré group generators $\underline{P}$ and $\underline{\underline{M}}$) one obtains those of the $S^{(i)}$, which are

$$[S(\underline{p})^{(i)},\ S(\underline{p})^{(j)}] = i\ S(\underline{p})^{(k)} \ (ijk,\ \text{circular permutation of } 1,2,3) \tag{29}$$

and from (13'), 21), (28) one also obtain

$$\sum_i (S(\underline{p})^{(i)})^2 = j(j+1) \tag{30}$$

I do hope that everything is perfectly clear to all of us.

Solution of the exercise : Obtain (1) from (27) .

If $\underline{a} = (a^o, a)$, $\underline{b} = (b^o, b)$, one finds

$$\underline{a}\cdot\underline{\underline{F}}\cdot\underline{b} = \vec{E}(ab^o - ba^o) - B\cdot\vec{a} \times \vec{b} \tag{31}$$

Let us choose the tetrad for which $\underline{n}^{(1)}$ and $\underline{n}^{(2)}$ correspond to transverse polarization and $\underline{n}^{(3)}$ to longitudinal polarization. Indeed, since we consider only the case $g' = 0$, $\vec{E} = 0$, in the tetrad motion given by (24) the longitudinal polarization stays longitudinal. Explicitly

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(1) To be mathematically rigorous, one says that $\mathcal{H}$ is the direct integral of Hilbert space $\mathcal{H} = \int^{\oplus}_{p^o>0} H(p)\delta(\underline{p}^2 - \underline{m}^2)d^4\underline{p} = \int^{\oplus} H(p)d\mu(p)$ where the integrand $H(\underline{p})$ (which is not a subspace of $\mathcal{H}$) is the $2j+1$ dimensional space of polarization states for the value $\underline{p}$ of the energy momentum. Then every operator on $\mathcal{H}$ has a spectral decomposition such as $\underline{P} = \int \underline{p}\, I(p)d\mu(p)$ where $I(\underline{p})$ is the identity on $H(\underline{p})$, and $\underline{W} = \sum_i \int S(\underline{p})^{(i)} \underline{n}^{(i)} d\mu_p$. Of course (30) should have been written $\sum_i (S(\underline{p})^{(i)})^2 = j(j+1)I(p)$.

$$\underline{n}^{(1)} = (0,\vec{n}^{(1)}) \ , \ \underline{n}^{(2)} = (0,\vec{n}^{(2)}) \ , \ \underline{n}^{(3)} = (\gamma v, \gamma\hat{v})$$

with

$$\hat{v} = \vec{v}/v \ , \ \vec{n}^{(1)2} = \vec{n}^{(2)2} = 1 \ , \ \vec{n}^{(1)}\cdot\vec{n}^{(2)} = 0 \quad \hat{v} = \vec{n}_1 \times \vec{n}_2 \quad .$$

The components of $\vec{\omega}_T$ are

$$\omega_T^k = \underline{n}^{(i)} \cdot \underline{\underline{F}} \cdot \underline{n}^{(j)}$$

with $\omega_T^1 = \gamma \, \vec{B}\cdot\vec{n}^{(1)}$, $\omega_T^2 = \gamma \, \vec{B}\cdot\vec{n}^{(2)}$, $\omega_T^3 = \vec{B}\cdot\vec{v}$.

With the definitions

$$\vec{B}_\perp = (\vec{B}\cdot\vec{n}^{(1)})\vec{n}^{(1)} + (\vec{B}\cdot\vec{n}^{(2)})\vec{n}^{(2)} \quad , \quad \vec{B}_{/\!/} = (\vec{B}\cdot\hat{v})\hat{v}$$

we obtain

$$\vec{\omega}_T = \vec{B}_{/\!/} + \gamma\vec{B}_\perp \, .$$

which is exactly the term propositional to $\frac{e}{m\gamma}\frac{g-2}{2}$ in (1) .