WHAT IS POLARIZATION? HOW TO COMPARE ITS MEASUREMENT WITH BEAM AND TARGET AND WITH COLLIDING BEAMS?

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Polarization is what has to be measured in order to completely specify the state of a particle whose energy and momentum are known. In quantum mechanics it has to be the expectation value of an operator commuting with E and $\vec{P}$ the energy and momentum operators. What is this operator? It was found probably by Pauli at least forty years ago and was known by oral tradition before to appear in print (Lubanski Physica 2 (1942) 310 quotes Pauli unpublished). Let us look for it.

Euclidian invariance of Physics imposes momentum and angular momentum conservation. The commutation relations between the corresponding operators $\vec{P}$ and $\vec{J}$ are

$$[P_i,P_j] = 0 \ , \ [J_i,J_j] = i\varepsilon_{ijk}J_k \ , \ [P_i,P_j] = i\varepsilon_{ijk}P_k \qquad . \qquad (1)$$

For a particle, $\vec{J}$ can be decomposed into orbital and spin angular momentum:

$$\vec{J} = \vec{L} + \vec{S} \qquad \text{where} \qquad \vec{L} = \vec{X} \times \vec{P} \qquad (2)$$

Since the P_i's commute, one can measure completly the momentum. The operators commuting with P_i are obviously the functions of $\vec{J}^2$ and $\vec{J}.\vec{P} = \vec{P}.\vec{J}$ (true equality, although $\vec{J}$ and $\vec{P}$ do not commute). This last operator is a pseudo scalar; it is the helicity. Since

$$\vec{L}.\vec{P} = \vec{P}.\vec{L} = 0 \qquad (3)$$

the helicity operator depends only on spin $\vec{J}.\vec{P} = \vec{S}.\vec{P}$ and it is a good candidate for the polarization operator. The only trouble is that in special relativity, helicity cannot be defined for one particle with non vanishing mass. (As we shall see, one needs two particles states to extend the helicity concept in special relativity to all particles).

In special relativity energy and momentum are the time component P^o and space components P^i of a four vector P^λ . Similarly J^k is the space past $J^k = M^{ij} = - M^{ji}$ (ijk = circular permutation of 1,2,3) of a shew symmetric tensor $M^{\mu\nu} = -M^{\nu\mu}$. The P^λ and $M^{\mu\nu}$ are the generators of the Poincaré group and any function of them is a kinematical observable of special relativity. The P^λ commute

$$[P^\lambda, P^\mu] = 0 \qquad (4)$$

but the P^λ and $M^{\mu\nu}$ and the $M^{\mu\nu}$ among themwelves do not commute. Moreover the separation of the total angular momentum $M^{\mu\nu}$ into an orbital and a spin part is not easy because a particle at rest for an observer is in motion for an equivalent observer and may have an orbital angular momentum for the later but not for the former. However one must not confuse spin and polarization ; they are two different concepts which coïncide only in the rest system of a non zero mass particle. The relativistic spin operator, part of the skew symmetric tensor $M^{\mu\nu}$ does not commute with the P^{λ}'s . The polarization operator W does.

Let $\tilde{M}$ the dual tensor of M

$$\tilde{M}_{\lambda\mu} = \frac{1}{2} \epsilon_{\lambda\mu\nu\rho} M^{\nu\rho} \qquad . \tag{5}$$

The polarization operator is the axial vector

$$\underline{W} = \underline{\underline{\tilde{M}}}.\underline{P} \qquad \text{i.e.} \quad W_{\lambda} = \tilde{M}_{\lambda\mu} P^{\mu} \qquad . \tag{6}$$

It does commute with the P^{λ}'s :

$$[P^{\lambda}, W^{\mu}] = 0 \qquad . \tag{7}$$

Moreover, the only operators which commute with all P^{λ}'s and $M^{\mu\nu}$ are $\underline{P}^2 = P^{\lambda}P_{\lambda}$ and $\underline{W}^2 = W^{\mu}W_{\mu}$. On the Hilbert space of a single particle states (i.e. for an irreducible unitary representation of the Poincaré group) these operators are multiple of the identity :

$$\underline{P}^2 = m^2 I \quad , \quad W^2 = -m^2 j(j+1) I \quad . \tag{8}$$

where m and j are the mass and spin of the particle.

From the antisymmetry of $\tilde{M}$, $P.\tilde{M}.P = 0$, i.e.

$$\underline{P}.\underline{W} = 0 \qquad . \tag{9}$$

Note that the W_{μ} do not commute among each other in general. Indeed

$$[W_{\mu}, W_{\nu}] = i\epsilon_{\mu\nu\rho\sigma} P^{\rho} W^{\sigma} \qquad . \tag{10}$$

Note also that $\underline{P}$ and $\underline{W}$ chosen here have same dimension ; this was done to deal with both $m = 0$ and $m \neq 0$ cases.

<u>1. Case</u> $m = 0$.

Then $\underline{P}^2 = 0 = \underline{W}^2$, $\underline{P}.\underline{W} = 0$. Two orthogonal light vectors are colinear so

$$\underline{W} = \lambda \underline{P} \tag{11}$$

when the pseudo scalar λ is the helicity of the mass zero particle. It is quantized $\lambda = -\frac{1}{2}$ for neutrinos, $\lambda = \frac{1}{2}$ for anti-neutrinos, $|\lambda| = 1$ for photons. We remark that in that case the components of

$\underline{W}$ commute among each other as equations (11) and (10) or (4) show.

2. Case $m \neq 0$

In that case, a complete set of commuting observables is composed of :

$$\text{the four components of } \underline{P}\text{, one component of } \underline{W},\ \underline{W}^2 \qquad (12)$$

Consider a particle of energy momentum $\underline{p}$ with

$$\underline{p}^2 = m^2$$

(of course we have chosen the unit system in which $\hbar = c = 1$). Let $\underline{n}^{(o)} = \underline{p}/m$ and $\underline{n}^{(i)}$ a set of four orthogonal 4-vectors

$$\alpha,\beta = 0,1,2,3,\ \underline{n}^{(\alpha)}.\underline{n}^{(\beta)} = g^{\alpha\beta} \quad . \qquad (13)$$

We call such a set a tetrad of $\underline{p}$. Consider the operators.

$$S^{(\alpha)} = -\frac{1}{m}\,\underline{W}.\underline{n}^{(\alpha)} = -\frac{1}{m}\,W^{\lambda}n^{(\alpha)}{}_{\lambda} \quad . \qquad (14)$$

Equations (9) requires $S^{(o)} = 0$. From (10), (13), (14) one finds that the three other operators $S^{(i)}$ satisfy :

$$[S^{(i)}, S^{(j)}] = i\varepsilon_{ijk}S^{(k)} \quad . \qquad (15)$$

In the rest frame of the particle, the expectation values of these different operators are

$$< \underline{P} > = (m,\vec{o}),\ < \frac{1}{m}\underline{W} > = (o,\vec{w}) \text{ with } w^k = < S^{(k)} > = < J^k > = < M^{ij} > . \qquad (16)$$

However one must not confuse these different operators. While $\frac{1}{m}W^k, S^{(k)}, M^{ij} = J^k$ are, for a particle at rest, the generators of the rotation group, their covariant meaning are quite different : W^k are the components of the polarization operator, $S^{(k)}$ are the generator of the little group of $\underline{p}$, i.e. the subgroup of the Lorentz group which leave $\underline{p}$ invariant, $M^{\mu\nu}$ the generators of the Lorentz group, and therefore the operators for the relativistic angular momentum components.

For a particle of spin j , mass $m \neq 0$, the polarization state is given by the expectation value of the irreducible symmetrized power of $\underline{W}$ up to degree $2j$. , i.e.

$$W^{\lambda} = \frac{1}{m}< W^{\lambda}>, W^{\mu\nu} = \frac{1}{2m^2} < W^{\mu}W^{\nu} + W^{\nu}W^{\mu} - \frac{1}{2}\,g^{\mu\nu}\underline{W}^2 > \text{, etc...} \qquad (17)$$

are the dipole, quadrupole, etc... polarization tensors :

$W^{\lambda\mu\nu\cdots}$ is completly symmetrical in its indices,

$$p_\lambda W^{\lambda\mu\nu\cdots} = 0 \ , \ W^{\lambda\mu\nu\cdots} = 0 \qquad (18)$$

In appendix, several applications of this covariant polarization formalism are outlined. Here we consider only the case of $m \neq 0$, spin $\frac{1}{2}$ particles. Such particles have only a dipole polarization. Their state is completly described by its energy momentum $\underline{p}$ and its polarization $\underline{w}$. They satisfy

$$\text{spin } \frac{1}{2} : \underline{p}^2 = m^2 \quad \underline{p}.\underline{w} = 0 \quad -\underline{w}^2 = (\text{degree of polarization})^2 \qquad (19)$$

It is convenient to use the normalization $\underline{w} = \frac{2}{m} < \underline{W} >$ so the polarization degree is 1 for a completly polarized state and 0 for the unpolarized state.

In a given system, fixed by a time axis (the lab system, the center of mass system, etc...) one can choose the tetrad.

$$\underline{n}^{(0)} = \frac{\underline{p}}{m} = (\gamma, \gamma\vec{v}) \quad \text{where} \quad \gamma = E/m = (1-\vec{v}^2)^{-\frac{1}{2}}$$
$$\underline{n}^{(3)} = \underline{\ell} = (\gamma v, \gamma\frac{\vec{v}}{v}) \ , \ \underline{n}^{(1)} = (0,\vec{s}_1) \quad \underline{n}^{(2)} = (0,\vec{s}_2) \ . \qquad (20)$$

Then we can define 3 pseudo scalars ζ_i

$$\zeta_i = -\underline{W}.\underline{n}^{(i)} \Longleftrightarrow \underline{W} = \sum_{i=1}^{3} \zeta_i \underline{n}^{(i)} \qquad (21)$$

and ζ_3 is the longitudinal polarization. This of course is not a covariant concept. For instance in the $\pi^{\pm}$ decay the $\mu^{\pm}$ is totally polarized ; this polarization is pure longitudinal in the π rest frame but not for a π decay in flight (see appendix). Of course $\underline{\ell}$ is the helicity axis. Since $\underline{P}.\underline{W} = 0$, in term of operators we can define the helicity λ as

$$\vec{P}.\vec{W} = 2\lambda |W^o P^o| \qquad (22)$$

this definition depends on the choice of a time axis (for $m \neq 0$). However we are not interested by isolated particles, we use them for making collisions; given two particles with energy momenta $\underline{p}'$, $\underline{p}''$ we can define the corresponding for vectors $\underline{\ell}'$, $\underline{\ell}''$ by

$$\underline{\ell}' = \frac{1}{sh\chi}(\underline{\beta}' ch\chi - \underline{\beta}'') \quad \underline{\ell}'' = \frac{1}{sh\chi}(\underline{\beta}'' ch\chi - \underline{\beta}') \qquad (23)$$

where $\underline{\beta}' = \underline{p}'/m'$, $\underline{\beta}'' = \underline{p}''/m''$, $m^2 = (\underline{p}'+\underline{p}'')^2$,

$$ch\chi = \underline{\beta}'.\underline{\beta}'' = \frac{m^2-m'^2-m''^2}{2m'm''} \ , sh\chi = \frac{\sqrt{\Delta(m^2,m'^2,m''^2)}}{2m'm''} > 0 \qquad (23')$$

$$\text{with } \Delta(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \ . \qquad (23'')$$

By definition ℓ', ℓ'' are the longitudinal polarization = helicity quantization vectors of the particles in the rest system of $p = p' + p''$. In a collision $a+b$, ℓ_a, ℓ_b are the s-channel helicity vectors.

If, in this collision particle a goes to particle c, we can define the t-channel helicity vectors. They are the longitudinal polarization vectors in the rest frame of $p_a + p_c$. This is usually called the Breit frame $(\vec{p}_a = -\vec{p}_c)$.

Similarly one can define the u-channel helicity vectors.

Example : A two body elastic collision in the center of mass $a+b \longrightarrow c+d$ when c is the final particle of a. The vector n orthogonal to the three plane p_a, p_b, p_c, p_d, chosen according the Basel convention ($\det n, p_a, p_b, p_c$) > 0, is common to the four tetrads p, ℓ, s, n. We draw on the picture the space part of these four vectors in the s-channel Helicity reference frames use the ℓ vectors of the tetrad as quantization axis. Transversity reference frames use n as quantization axis ($n = n^{(3)}$ for each tetrad, in each channel.)

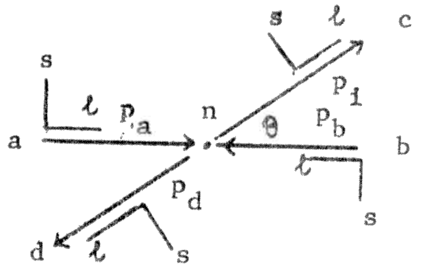

Figure 1.

To pass from the tetrad of one channel to that of another channel for a given particle, one has to do a rotation in the ℓ, s 2-plane about the crossing angle ; this angle goes to zero in the forward direction limit.

If $a+b \longrightarrow c+d$ is time reversal invariant, then

$$A_{ij} = C_{ij} \tag{24}$$

i.e. initial and final state correlation parameters coïncide.

The "Ann Arbor conventions" advocate s-channel tetrad in the c.m.system with the nice feature that the tetrads of the initial (or the final) particles are transformed into each other by a rotation R of π around $\vec{n}$. But for the study of the same reaction in the laboratory system $(\vec{p}_b = 0)$ the advocated choice of tetrad might be natural for an experimental setting but it makes relation (24) no longer valid except for $A_{nn} = C_{nn}$. Indeed, if the advocated tetrad of the initial particles are obtain in the lab. system from those of the c.m system by the boost Λ putting the b particle at rest, an additional rotation around $\vec{n}$ must be made for the tetrad of the final particles. However the initial state correlation parameters A_{ij} measured in the c.m. and lab. frame could be directly compared if it were not for the very akward choice of sign of $\vec{L}_b$ ("for historical season") the two initial tetrads are not transformed into each other by $\Lambda R \Lambda^{-1}$, the conjugated of the rotation of π.

Appendix . Other examples of application of the covariant polarization formalism.

1) $\pi \to \mu\nu$ decay. The polarization $\underline{w}$ of μ is orthogonal to $\underline{p}_\mu$ and depends only of $\underline{p}_\pi$ since $\underline{p}_\nu = \underline{p}_\pi - \underline{p}_\mu$; using $\underline{p}_\nu^2 = 0$, one finds

for $\mu^\pm$, $\underline{w} = \pm \frac{1}{\cos 2\omega} \left(\frac{\underline{p}_\pi}{m_\pi} \sin 2\omega - \frac{\underline{p}_\mu}{m_\mu} \right)$, $\frac{m_\mu}{m_\pi} = \mathrm{tg}\,\omega$, $\omega = .648..$

2) Density matrix of a Dirac particle with non vanishing mass.

Dirac equation $(\not{p}-m)u = 0$, $\rho(\underline{p}) = \frac{(1+\gamma^5 \not{w})(\not{p}+m)}{4m}$

first introduced in L.MICHEL, A.S. WIGHTMAN Phys. Rev. 98 (1955) 1190 where it is also extended to zero mass particles.

The covariant relation between 2×2 density matrix formalism and Dirac matrices has been established for instance in C. BOUCHIAT, L. MICHEL Nucl. Phys. 5 (1958) 416. If τ_k are the three Pauli matrices, ϵ , $\epsilon' = \pm 1$

$$u_\alpha(\underline{p},\epsilon\underline{w})\bar{u}_\beta(\underline{p},\epsilon'\underline{w}) = \frac{1}{4}\left[(\delta_{\epsilon\epsilon'} + \gamma^5 \not{n}^{(k)}(\tau_k)_{\epsilon'\epsilon})(\not{p}+m)\right]_{\alpha\beta}$$

where $\underline{n}^{(k)}$ form with $\underline{p}/m$ the tetrad defined in (13).

3) Covariant density matrix of a particle with arbitrary spin $j = \frac{n}{2}$; $m \neq 0$. (L. MICHEL N. Cim. Supl. 14 (1959) 95

$$\rho(p) = \frac{I}{2j+1}\, w_{\lambda_1} \frac{W^{\lambda_1}}{m} + w_{\lambda_1\lambda_2} \frac{W^{\lambda_1}}{m}\frac{W^{\lambda_2}}{m} - \dots + (-1)^n w_{\lambda_1\lambda_2\cdots\lambda_n} \frac{W^{\lambda_1}}{m}\frac{W^{\lambda_2}}{m}\cdots\frac{W^{\lambda_n}}{m}$$

where the $w_{\lambda_1\lambda_2\cdots\lambda_k}$ are completly symmetrical tensors, orthogonal to $\underline{p}$: and traceless ; they are the covariant form of the polarization multipoles.

$$w_{\lambda_1\cdots\lambda_k} = w_{\lambda_{i_1}\cdots\lambda_{i_k}} , \quad w^\lambda{}_{\lambda\,\lambda_3\cdots\lambda_k} = 0, \quad p^\lambda w_{\lambda\,\lambda_2\cdots\lambda_k} = 0 \quad .$$

The $w_{\lambda_1\cdots\lambda_k}$ are all even under parity since $\underline{W}$ is an axial vector. For instance a parity conserving two body decay of a spin j-particle measures only the even multipoles of the polarization when the polarization of the decay product is not observed. Indeed the observables are of the form $\mathrm{tr}\,\rho(\underline{p})A$, i.e. linear in $\rho(\underline{p})$ and Lorentz invariant. In the decay $p \to p_1 + p_2$ there is only one four vector linearly independent from p , e.g. $q = p_1 - p_2$. When k is odd

$w_{\lambda_1 \ldots \lambda_k} q^{\lambda_1} \ldots q^{\lambda_k}$ is a pseudo scalar.

4) Polarization effects in Møller scattering.

In the Born approximation, which is good for not too high momentum transfer, $A^a = A^b = P^c = P^d = 0$, no analyzing power or polarizing power because it is a one photon exchange and all amplitudes are relatively real (in helicity).

All other effects can be computed in a completely covariant manner: cf C. BOUCHIAT, L. MICHEL. Compt Rend. Acad. Sci. Paris 243 (1956) 692 and Nucl. Phys. 5 (1958) 416; we also computed the spin effects for Bhabha scattering. We gave the results with arbitrary tetrads $\underline{n}^{(i)}$ for each particles, in terms of the invariants $\varkappa, \lambda, \mu$ traditionaly used since the thirties. Here is the value of A_{ij} for Møller scattering in terms of the new fashionable $s = 2m^2(1+\varkappa)$ $t = 2m^2(1-\lambda)$, $u = 2m^2(1-\mu)$; $s+t+u = 4m^2$

$$A_{ij} = N_{ij}/D$$

$$\begin{aligned} N_{ij} = &- n_a^{(i)} \cdot \underline{n}_b^{(j)}\, 2(t^2u^2 - 4m^2tu(m^2-t-u)) \\ &- m^{-2}(\underline{n}_a^{(i)} \cdot \underline{p}_c)(\underline{n}_b^{(j)} \cdot \underline{p}_d)\, 4u^2(t+m^2) \\ &- m^{-2}(\underline{n}_a^{(i)} \cdot \underline{p}_d)(n_b^{(j)} \cdot \underline{p}_c)\, 4t^2(u+m^2) \\ &- [(\underline{n}_a^{(i)} \cdot \underline{p}_c)(\underline{n}_b^{(j)} \cdot \underline{p}_c) + (\underline{n}_a^{(i)} \cdot \underline{p}_d)(\underline{n}_b^{(j)} \cdot \underline{p}_d)]\, 4tu \end{aligned}$$

$$D = t^2(2t^2+2ut+u^2-8m^2t+8m^4)+u^2(2u^2+2ut+t^2-8m^2u+8m^4)+2ut\,[(u+t)^2-4m^4]$$

When we use the s-helicity tetrad defined in the text (see fig. 1; this is the Ann Arbor convention in the center of mass) in the limit $\frac{s}{m^2} \gg 1$ we need only one parameter $\eta = \frac{tu}{s-4m^2} = \frac{1}{4}(\sin\theta)^2$ to express

$$A_{\ell\ell} = \frac{\eta(2-\eta)}{(1-\eta)^2} \quad , \quad A_{3s} = \frac{\eta^2}{(1-\eta)^2} = A_{nn}$$

All other A_{ij} vanishes for all s and t except $A_{\ell s} = A_{s\ell}$ which decreases as $\frac{m}{\sqrt{s}}$ when $\frac{s}{m^2} \to \infty$. Since $\sigma_{-\ell,\ell} = \sigma_{\ell,-\ell}$, $A_{\ell\ell} = A_{-\ell-\ell} = \frac{\sigma_{\ell\ell} - \sigma_{-\ell\ell}}{\sigma_{\ell\ell} + \sigma_{\ell\ell}}$

This yields $\frac{\sigma_{\ell\ell}}{\sigma_{-\ell\ell}} = \frac{1}{1-4\eta+2\eta^2}$ which increases from 1 to 8 when θ goes from 0° to 90°. (This is used in F. Low's report). The cross section of electrons with some helicity is always larger than that with opposite helicity.

$$\frac{\sigma_{nn}}{\sigma_{-nn}} = \frac{\sigma_{ss}}{\sigma_{-ss}} = \frac{(1-\eta)^2+\eta^2}{(1-\eta)^2-\eta^2} = \frac{1-2\eta+2\eta^2}{1-\eta}$$ which increases from 1 to 5/4

In the lab system, the A_{ij} have same value except that the strange Ann Arbor convention requires that one changes the sign of $A_{\ell\ell}$ and A_{ss} .