SOME USE OF METABELIAN GROUPS IN PHYSICS

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ABSTRACT

Metabelian groups are those whose structure differ the least from Abelian groups. We explain the classification of finite metabelian groups and give some examples of physics problems where they play an important role.

This paper is written for physicists, with some pedagogical purpose. So we have first to define and explain some simple basic notions. If X, Y are subgroups of a group G, we denote by $[X,Y]$ the G-subgroup generated by the commutators $[x,y] \equiv xyx^{-1}y^{-1}$, $x \in X$, $y \in Y$. Then we define a sequence of subgroups of G : ($\leq$ reads "subgroup")

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \ldots \geq G^{(n-1)} \geq G^{(n)} \geq \ldots \qquad (1)$$

with $$G^{(1)} = \left[G^{(0)}, G^{(0)}\right], \ldots, G^{(n)} = \left[G^{(0)}, G^{(n-1)}\right] \qquad (1')$$

Note that $G^{(1)} \lhd G = G^{(0)}$ ($\lhd$ reads "invariant subgroup") and $G/G^{(1)}$ is Abelian. We say that G is nilpotent of class k :

$$G \text{ is } N_kG \Leftrightarrow G^{(k)} = 1 \quad , \quad G^{(k-1)} \neq 1 \qquad (2)$$

Note that "G is N_1G" simply means G is Abelian. The nilpotent groups share many properties with the Abelian groups and the N_2G are the least different from them. They are usually called metabelian groups in the literature although this term is sometimes used for a larger class of groups. Here we will use "N_2G" as an adjective equivalent to metabelian or sometimes as a noun.

Every finite Abelian group is the direct product of cyclic groups

$$A = \mathop{X}_{j}^{J} Z_{n_j} \tag{3}$$

Among all such possible decompositions there is a most refined one (up to an automorphism of A) that we will give below (see (5'). Let us denote by $|G|$ the order of the finite group G (i.e. $|G|$ is the number of its elements). The decomposition of $|G|$ into prime factors reads ($\mathcal{P}$ is the set of prime numbers)

$$|G| = \prod_{p \in \mathcal{P}} p^{r_p} \tag{4}$$

where only a finite number of r_p is different from zero; if only one of them, r_p, is different from zero, G is called a p-group. In any group G with order given by (4), Sylow showed last century that there exists subgroups of order p^{r_p} for each $p \in \mathcal{P}$, and for a given p, all these Sylow subgroups are conjugated. [See e.g. [1], Vol.II, §54]. So for finite Abelian groups, for each p, there is a unique Sylow subgroup G_p. Since for $p \neq q$, $G_p \cap G_q = \{1\}$ in any group, it is easy to see that for finite Abelian groups

$$G = \mathop{X}_{p \in \mathcal{P}} G_p \quad , \quad |G_p| = p^{r_p} \tag{5}$$

Finally, there is for each Abelian G_p of (5) a most refined decomposition :

$$G_p = \mathop{X}_{k=1}^{K} Z_{p^{r_k}} \tag{5'}$$

Note that each G_p in (5) is invariant by any automorphism of G, A subgroup with such a property is said to be <u>characteristic</u> and one usually uses $H \lhd\!\lhd G$, for H is a characteristic subgroup of G.

The property (5) extends to finite nilpotent groups (FNG) ; we sketch here the proof. When $H \leq G$, the normalizer of H in G, $N_G(H)$ is the set of $x \in G$, $xHx^{-1} \subset H$: It is the largest subgroup of G which contains H as invariant subgroup. If H is a strict subgroup, $H < G$, of the nilpotent group G, then $H < N_G(H)$. Indeed if $r_o > 0$ is the smallest r such that $G^{(r)} \subseteq H$ $[H, G^{(r_o-1)}] \subseteq [G, G^{(r-1)}] = G^{(r)}$; hence there is an $x \in G^{(r-1)}$ such that $x \notin H$, $xHx^{-1} \subset H$. So for instance $G_p \lhd N_G(G_p) \lhd N_G(N_G(G_p))$.

However, here also $G_p \triangleleft N_G(G_p)$ so $G_p \triangleleft N_G(N_G(G_p))$; the contradiction is avoided if $N_G(G_p) = G$, i.e. $G_p \triangleleft G$. Every Sylow subgroup of a nilpotent group is an invariant subgroup. If p and q are distinct primes $G_p \cap G_q = 1$ and $x \in G_p$, $y \in G_q \Rightarrow [x,y] = xyx^{-1}y^{-1} \in G_p \cap G_q = 1$ so $[G_p,G_q] = 1$. It is then not difficult(*) to verify that equation (5) holds in that case and that every G_p is $N_{k'}G$, $k' \leq k$ when G is N_kG .

Hence every nilpotent group is a direct product of its Sylow subgroups.

For N_2G groups, $1 = G^{(2)} = [G,G^{(1)}]$ so $G^{(1)} \leq C(G)$, the center of G . Since $G/G^{(1)}$ is Abelian, we have to study what is called the <u>central extension problem</u> for Abelian groups: given two Abelian groups A,B find <u>all</u> groups G such that

$$A \leq C(G) \ , \ B = G/A \ , \tag{6}$$

We denote by ϕ the group homomorphism :

$$G \overset{\phi}{\to} B \ , \quad \text{Image } \phi = B \quad , \text{Kernel } \phi = A \tag{6'}$$

It is convenient to adopt here for a while the additive notation for Abelian groups. Then for a given integer n , $a \to na$ is a group-homomorphism $A \overset{n}{\to} A$. Its image is denoted by nA and its kernel by $_nA$ ($_nA$ is the largest subgroup H of A satisfying $nH = 0$); it is customary to use the notation $A/nA = A_n$.

If G is a central extension (6,6') , its elements can be labeled by pairs (a,b) , $a \in A$, $b \in B$ where $(a,0) = a \in A$ and, when $b \neq 0$, $(0,b)$ is an <u>arbitrary</u> element of the coset of A which is sent by ϕ on b . The map $B \overset{\sigma}{\to} b$ defined by $\sigma(b) = (0,b)$ is called a section over B . It satisfies. $\phi_o\sigma = I_B$ the identity map on B . The section σ can be a group homomorphism only for a direct product. In general the group law of G reads :

$$(a_1,b_1)(a_2,b_2) = (a_1+a_2+\omega(b_1,b_2),b_1+b_2) \tag{7}$$

(*) Verify that evey element of G can be written as a product $x = \Pi x_p$, $x_p \in G_p$ and that the decomposition of 1 is unique.

where, with our conventions

$$\forall b \in B \qquad \omega(0,b) = \omega(b,0) = 0 \tag{7'}$$

and, due to the associativity of the group law

$$\delta\omega(b_1,b_2,b_3) \equiv \omega(b_1,b_2) - \omega(b_2,b_3) + \omega(b_1+b_2,b_3) - \omega(b_1,b_2+b_3) = 0 \tag{8}$$

Every function $B \times B \overset{\omega}{\to} A$ which satisfies (8) is called a 2-cocycle, and defines an extension. By adding their value in A, the 2-cocycle form a group, denoted $Z^2(B,A)$. If in the same extension G, we had made another choice $\sigma'(b) = (0,b)'$ of section over B, the two choices define a function $B \overset{\psi}{\to} A$ defined by $\sigma'(b) = \psi(b)\sigma(b)$ or $(0,b)' = (\psi(b),b)$.

The corresponding cocycle ω' would differ from ω by

$$\omega'(b_1,b_2) - \omega(b_1,b_2) = \varphi(b_1,b_2) \tag{9}$$

with

$$\varphi(b_1,b_2) = \psi(b_1) - \psi(b_1+b_2) + \psi(b_2) \tag{9'}$$

A 2-cocycle which satisfies (9') is called a 2-coboundary. The 2-coboundaries form a subgroup $B^2(B,A) \subset Z^2(B,A)$. Since a given extension (6) is defined by a cocycle modulo the coboundaries, the set of solutions of the central extension problem form a group

$$H^2(B,A) = Z^2(B,A)/B^2(B,A) \tag{10}$$

which is called the second(*) (central) cohomology of B by A. The direct product corresponds to $0 \in H^2(B,A)$. Two groups G,G' which correspond to different elements of $H^2(B,A)$ are called inequivalent extensions of B by A; there is no isomorphism $G \overset{i}{\to} G'$ which would reduce to the identity on the subgroup A and/or whose induced iso-

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(*) All other cohomology groups $H^n(B,A)$ exist; we do not need them here. However we want to tell the reader wo would not know it that, for the central case, $H^1(B,A) = \mathrm{Hom}(B,A)$ the group of homomorphisms $B \to A$.

morphism $B = G(A) \stackrel{\hat{1}}{\to} G'(A) = B$ would be the identity on B . However G and G' might be isomorphic! This is the case when they are on the same orbit of the natural action of $\mathrm{Aut}\, B \times \mathrm{Aut}\, A$ on $H^2(B,A)$.

Let $Z^2_s(B,A)$ the subgroup of symmetrical cocycles : $\omega(b_1,b_2) = \omega(b_2,b_1)$. Since the coboundary is symmetrical, one can define

$$H^2_s(B,A) = Z^2_s(B,A)/B^2(B,A) \tag{11}$$

From the group law (7) we see that extensions which correspond to elements of $H^2_s(B,A)$ are Abelian. So the FN_2G correspond to the elements of $H^2(B,A)$ which are <u>not</u> in the subgroup $H^2_s(R,A)$. We can also say that for the cocycles of the FN_2G's satisfying the central extension problem (6), the function $\lambda(b_1,b_2)$ defined by

$$\omega(b_1,b_2) - \omega(b_2,b_1) = \lambda(b_1,b_2) \tag{12}$$

does not vanish identically. Note that, from (7') and (12)(*)

$$\lambda(0,b) = \lambda(b,0) = \lambda(b,b) = 0 \ , \quad \lambda(b_1,b_2) = -\lambda(b_2,b_1) \qquad \text{(12a,b,c)}$$

Moreover equation (9') implies that the $\lambda(b_1,b_2)$ are independent from the arbitrary choice of coset representatives so they are canonical in the extension G .

Indeed

$$\lambda(b_i,b_j) = [(0,b_i),(0,b_j)] = (0,b_i)(0,b_j)(0,b_i)^{-1}(0,b_j)^{-1} \tag{13}$$

where

$$(0,b)^{-1} = (-\omega(b,b^{-1}),b^{-1}) \tag{13'}$$

and we use

$$\omega(b,b^{-1}) = \omega(b^{-1},b) \tag{13''}$$

easily obtained from (7') and (8).

(*) Beware that (12c) for $b_1 = b_2$ is $2\lambda(b,b) = 0$ and this does not imply $\lambda(b,b) = 0$!

Define $\widetilde{\omega}(b_2,b_1) = \omega(b_1,b_2)$ and verify that $\delta\omega = 0$ implies $\delta\widetilde{\omega} = 0$. So, as a difference of cocycles, $\lambda(b_1,b_2)$ is a cocycle. Let $\Lambda(B,A)$ be the subgroup of $Z^2(B,A)$ generated by the λ's . Note that if $_2A \neq 0$, $Z^2_s(B,A) \cap \Lambda(B,A)$ may not be trivial. However $\omega(b_1,b_2) \overset{\sigma}{\to} \lambda(b_1,b_2)$ is a group homomorphism $Z^2 \overset{\sigma}{\to} \Lambda$ with Ker $\sigma = Z^2_s$. So we have the induced homomorphism $H^2(B,A) \overset{\sigma}{\to} \Lambda(B,A)$ of Kernel $H^2_s(B,A)$:

$$H^2(B,A) \overset{\sigma}{\to} H^2(B,A)/H^2_s(B,A) = \Lambda(B,A) \tag{14}$$

By a repeated use of the cocycle equation (8) one proves that the λ's on group homomorphisms from B to A for each of these arguments :

$$\lambda(b_1,b_2+b_2') = \lambda(b_1,b_2)+\lambda(b_1,b_2'),\lambda(b_1+b_1',b_2) = \lambda(b_1,b_2)+\lambda(b_1',b_2) \tag{15}$$

As a particular case we note that

$$\lambda(n_1b_1,n_2b_2) = n_1n_2\lambda(b_1,b_2) \tag{16}$$

This combined with (12b) yields

$$\lambda(n_1b,n_2b) = 0 \tag{17}$$

This proves that when B is a cyclic group

$$H^2(Z_n,A) = H^2_s(Z_n,A) \tag{17'}$$

or equivalently :

<u>Lemma 1.</u> The central extensions of a cyclic group $B = Z_n$ by an Abelian group A are Abelian.

The extension problem will yield FN_2G groups only when B is not cyclic. We also remark that (17) for $n_1,1$ and for $1,n_2$ yields

$$n_1b_1 = 0 = n_2b_2 \Rightarrow (n_1,n_2)\ (b_1,b_2) = 0 \tag{18}$$

where $(n_1,n_2,...)$ is the greatest common divisor of the integers inside the brackets.

Given the decomposition (3) for A , the cocycle ω can be decomposed as a sum of its components ω_i in each A_i . This corresponds to

$$H^2(B, X_{j=1}^J Z_{n_j}) = X_{j=1}^J H^2(B, Z_{n_j}) \tag{19}$$

The non Abelian central extension of B by Z_n are special FN_2G that we call CK FN_2G , for finite class 2 nilpotent group with cyclic kernel. We remark that the image of a FN_2G irreducible complex linear representation whose dimension is greater than one, is CK FN_2G .

If one chooses J extensions G_j $1 \leq j \leq J$ of B by Z_{n_j} , with cocycles ω_i , the corresponding extension G of B by A , with cocycle $\omega = \Sigma_j \omega_j$ can be obtained by taking first the direct product $X_{j=1}^J G_j$ whose elements are $(\alpha_1, b_1)(\alpha_2 b_2) \ldots (\alpha_J, b_J)$, and after making all b_i's equal. Then G is called the diagonal subdirect product of the G_j's .

We have still to study the structure of the $CKFN_2G$'s . We know from Lemma 1 that B cannot be a cyclic group. So we consider a non trivial decomposition (3) for B . We prove in Appendix

$$H^2(X_{k=1}^k Z_{n_k}, A) = (X_{k=1}^K A_{n_k}) \times (\underset{i \leq k < \ell \leq K}{X} (n_k, n_\ell) A) \tag{20}$$

We need to study only the case where A is Abelian. Since the cohomology vanishes when $|B|$ and $|A|$ are relatively prime, we consider only the case of p-groups. Then

$$H^2(X_{j=1}^J Z_{p^{n_j}}, Z_{p^{n_o}}) = (X_{j=1}^J Z_p[n_o, n_j]) \times (X_{1 \leq k < \ell \leq J} Z_p[n_o, n_k, n_\ell]) \tag{21}$$

where $[n_o, n_1, \ldots, n_j]$ has the value of the smallest integer inside the square brackets. Using the proof of the appendix, we can write the genetic code of the groups corresponding to an element of the cohomology group in (21) : We obtain

$$1 = b_o^{p^{n_o}} = [b_o, b_i] = [b_i, b_j] b_o^{-n_{ij} p^{n_o - [n_o, n_i, n_j]}} , \quad b_i^{p^{n_i}} = b_o^{m_i p^{n_o - [n_o, n_i]}} \tag{22}$$

with

$$n_{ii} = 0 \ , \ n_{ij} = -n_{ji} \ ; \ i < j \ , \ 0 \leq n_{ij} < p^{[n_o,n_i,n_j]} \ ,$$
$$0 \leq m_i < p^{[n_o,n_i]} \ , \ n_i \leq n_{i+1} \qquad (22')$$

The group is Abelian if all n_{ij} vanish. Two groups with non identical sets of n_i's are non isomorphic. When they have some n_i in common, the groups with different sets of m_i's and n_{ij}'s are non equivalent extensions but they can be isomorphic as we already remarked.

To summarize we have obtained a classification of FN_2G's . They are a direct product of their Sylow groups G_p . Each G_p is the diagonal subproduct of p-$CKFN_2G$'s , which are non commutative central extensions of Abelian p-groups by a cyclic p-group. And each p-$CKFN_2G$ has a genetic code given by the equations (22).

Before showing some physical examples, we make a last remark. If $X \triangle G$, $Y \triangle G$ then it is easy to show $X.Y = Y.X \lhd G$ (X.Y is the set $\{xy, x \in X, y \in Y\}$. It is slightly more tricky ([2] § 7.4) to prove that if moreover X and Y are nilpotent, X.Y is nilpotent. Hence by making the product elements by elements of all nilpotent invariant subgroups of G , one obtains the largest invariant nilpotent G-subgroup. It is G itself if G is nilpotent. When it is not the case, it is a proper subgroup called the Fitting subgroup of G [2,§7.4].

An elegant way to study the irreducible representations of the 230 crystallographic space groups requires the building of families of $CKFN_2G$ groups and some extensions of Z_2, Z_3, D_3 by such groups. We have published yet only a part of this work [3] . A crystallographic space group G has an invariant subgroup $T = Z^3$ which forms a lattice of translations; the corresponding quotient $G/T \cong P$ is called a crystallographic point group. There are 32 of them, defined up to a conjugation into the orthogonal group O(3) of rotations and reflections. They fall into 18 isomorphic classes : 9 are Abelian, 2 are $CKFN_2G$; the other 7 are non nilpotent (they are supersolvable) and their Fitting group is Abelian. We give all relevant information in Table 1.

Table 1.

Structure of crystallographic point groups. P.G. = point group, I.C. = isomorphic class. MAISG = Maximal Abelian invariant subgroups, CQ = Corresponding quotient. S_n permutation group of n object, A_n , its subgroup of even permutations.

Abelian IC	Abelian PG		Non Abelian IC	Non Abelian PG	MAISG	their IC	CQ
1	C_1			D_4	C_4, D_2		
Z_2	C_2, C_s, C_i	$CKFN_2G$	D_4	C_{4v}	C_4, C_{2v}	Z_4, Z_2^2	Z_2
Z_3	C_3			D_{2d}	$S_4; D_2, C_{2v}$		
Z_4	C_4, S_4	$CKFN_2G$	$Z_2 \times D_4$	D_{4h}	C_{4h}, D_{2h}	$Z_2 \times Z_4, Z_2^3$	Z_2
$Z_2 \times Z_3$	C_6, S_6, C_{3h}		$S_3 = D_3$	D_3, C_{3v}	C_3	Z_3	Z_2
Z_2^2	D_2, C_{2v}, C_{2h}			C_{6v}, D_6	C_6		
Z_2^3	D_{2h}		D_6	D_{3d}	S_6	$Z_2 \times Z_3$	Z_2
$Z_2 \times Z_4$	C_{4h}			C_{3h}	C_{3h}		
$Z_2^2 \times Z_3$	C_{6h}		$Z_2 \times D_6$	D_{6n}	C_{6h}	$Z_2^2 \times Z_3$	Z_2
			A_4	T	D_2	Z_2^2	Z_3
			$Z_2 \times A_4$	T_h	D_{2h}	Z_2^3	Z_3
			S_4	O, T_d	D_2	Z_2^2	D_3
			$Z_2 \times S_4$	O_h	D_{2h}	Z_2^3	D_3

Non Abelian nilpotent point groups have several maximal Abelian invariant subgroups. Non nilpotent point groups have a unique maximal Abelian invariant subgroup : it is their Fitting group.

The set of equivalence classes of irreducible representations of a group G is often denoted by $\hat{G}$ and it is called the dual of G . For an Abelian group A , the dual $\hat{A}$ is a group : its multiplication law is the tensor product of the (one dimensional) representations. $\hat{T}$, the dual of the translation $T \sim Z^3$ has the topology of a three dimensional torus; it can be identified with the Brillouin zone and its elements $k \in \hat{T}$ are the wave vectors. The T representation corresponding to k is

$$k(t) = e^{i\underline{k}.\underline{t}} \tag{23}$$

$$\ker k = \{t \in T \,,\ e^{ikt} = 1\} \quad , \ \mathrm{Im}\, k = \{e^{ikt},\ t \in T\} \tag{24}$$

It could have been easy to extend our paper to finitely generated nilpotent groups. For simplicity we consider here only the $k \in \hat{T}$ with rational coordinates $0 \leq k_1, k_2, k_3 < 1$. Then $\mathrm{Im}\, k$ is a finite, and therefore cyclic group Z_{n_k} . The actions of G and P on T define their action on $\hat{T}$ (T act trivially, so G acts through the quotient $P : G/T$). We denote by G_k and $P_k = G_k/T$ the isotropy groups (= little groups) of k . The irreducible representations of G are built from those of G_k by induction [4,5]. Let Γ_{G_k} an irrep of G_k ; the different representations corresponding to the k of a same G-orbit $G(k)$ yield the same irreducible representation $\Gamma_G^{\alpha,(k)}$ of G . Its dimension is

$$\dim \Gamma_G^{\alpha,(k)} = (\dim \Gamma_{G_k}^{\alpha})(G:G_k) = (\dim \Gamma_{G_k}^{\alpha})(P:P_k) \tag{25}$$

when $(G:b_k)$ and $(P:P_k)$ are respectively the index of G_k in G , of P_k in P , i.e. $(P:P_k) = |P|/|P_k|$.

It is easy to see that the Herring group [6] :

$$P(k) = G/\ker k \tag{26}$$

is a central extension of P_k by $\mathrm{Im}\, k \leq C(P(k))$. The irreducible representations of G_k correspond to the allowed ones of $P(k)$ i.e. those irreps whose kernel does not contain elements of $\mathrm{Im}\, k$ different from one. We have constructed all $P(k)$'s . The finite ones correspond to

$$\mathrm{Im}\, k = Z_{n_k} \quad . \tag{27}$$

When P_k is Abelian or nilpotent, the corresponding $P(k)$ is $CKFN_2G$ or FN_3G . The other finite $P(k)$'s are extensions of Z_2, Z_3, D_3 by the preceding ones.

The simplest and in a certain sense "elementary" metabelian matrix groups appear in a natural way in a quite different physical context, namely in diagonalization of dynamical matrix $D(n)$ of a system of n molecules vibrating harmonically on a circle without restriction to the closest neighbour approximation (details in [7]). For any $n \geq 3$, the corresponding metabelian matrix group $K(n)$ is generated by the following two matrices

$$A(n) = \begin{pmatrix} 0 & 1 \\ I_{n-1} & 0 \end{pmatrix} , \quad B(n) = (B(n)_{ij}) , \tag{28}$$

where

$$B(n)_{rs} = \varepsilon_n^r \delta_{rs} , \quad \varepsilon_n^r = e^{2\pi i r/n}, \quad r,s = 0,1,\dots,n-1 , \tag{29}$$

and I_m denotes the $m \times m$ unit matrix. The group $K(n)$ generated by $A(n)$ and $B(n)$ is a metabelian group

$$Z_n(A(n)) \times Z_n(B(n)) = K(n)/Z_n(\varepsilon_n^{-1} I) \tag{30}$$

where

$$\varepsilon_n^{-1} I = [A(n),B(n)] \equiv A(n)B(n)A(n)^{-1}B(n)^{-1} \tag{30'}$$

($Z_n(X)$ means the cyclic group of order n generated by the matrix X)

A complete classification (up to an isomorphism) of irreducible metabelian groups which are maximal in $GL(n,\mathbb{C})$, the group of $n \times n$ inversible matrices with complex elements, is given by Suprunenko [8]. Generators of such maximal subgroups are presented as ordinary or tensor products of $A(n)$, $B(n)$, λI_r matrices with $n \geq 3$, $r \geq 2$. So the maximal irreducible metabelian groups can be built from the $K(n)$ groups. In that sense, these $K(n)$ groups can be considered as "elementary" metabelian groups.

Finally, we would like to note that a particular family of metabelian groups $C(m;k)$ appears if one takes the defining relations of the so-called generalized Clifford algebras $C_m^{(k)}$, [9]-[11], as the genetic code for $C(m;k)$:

$$C(m;k) = \langle \omega, c_1, \ldots, c_m | \omega^k = c_i^k = [\omega, c_i] = \omega[c_i, c_j] = E \ ; \ 1 \leq i < j \leq m \rangle , \tag{31}$$

where $\omega, c_1, \ldots, c_m$ are generators of $C(m;k)$ and E denotes the unit element of $C(m;k)$; A,B denotes $ABA^{-1}B^{-1}$. From (28) and (27) it follows that $K(n) \cong C(2;n)$.

All irreducible representations of $C_m^{(k)}$ have been determined explicitly by Morris, [10], by giving the matrix generators in the form of products of matrices $A(n)$ and $B(n)$. Generalized Clifford groups $C(m;k)$ appear in other physical applications, mainly through the generalized Clifford algebra; these are relevant in supersymmetry physics [12].

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Appendix.

We give here the proof of equation (20). We consider a central extension G of Z_n by A , an arbitrary Abelian group : $Z_n = G/A$, $A \leq C(G)$. Let z be a generator of Z_n . Using as in (7) the additive notation, we has

$$(0,Z)^n = (\rho(z),0) \quad \text{with} \quad \rho(z) = \sum_{k=o}^{n-1} \omega(x^n,z) = \sum_{x \in Z_n} \omega(x,z) \ .$$

With another choice of section $(\psi(z),z)$, we obtain $(\psi(z),z)^n = (n\psi(z)+\rho(z),0)$. So $\rho(z)$ is defined up to an element of the group nA ; hence it defines an element of $A/nA = A_n$ i.e.

$$H_o^2(Z_n,A) = A_n \tag{A1}$$

Let z_i be the generator of Z_{n_i} . We can choose the elements $(0,z_i)$ such that for $i < j : (0,z_i)(0,z_j) = (0,z_i+z_j)$. Then $(0,z_j)(0,z_i) = [(0,z_j),(0,z_i)] \ (0,z_i)(0,z_j)$ and we have seen that the commutator $(z_j,z_i) = [(0,z_j)(0,z_i)]$ is independent from any convention in the choice of representatives. So the set of

$\lambda(z_i,z_j) = -\lambda(z_j,z_i)$, $i < j$ complete the definition of

$H^2_o(\overset{k}{\underset{i=1}{X}} Z_{n_i}, A)$. From (A1) and (18) we obtain (20).

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