The Demazure–Tits subgroup of a simple Lie group

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The Demazure–Tits subgroup of a simple Lie group **G** is the group of invariance of Clebsch–Gordan coefficients tables (assuming an appropriate choice of basis). The structure of the Demazure–Tits subgroups of A_n, B_n, C_n, D_n, and G_2 is described. Orbits of the permutation action of the DT group in any irreducible finite-dimensional representation space of A_2, C_2, and G_2 are decomposed into the sum of irreducible representations of the DT group.

I. INTRODUCTION

The purpose of this paper is to study a certain finite subgroup of any simple compact Lie group **G**. We call the subgroup the Demazure[1]–Tits[2] group and denoted it by **DT** or **DT**(**G**).

The maximal tori (called the Cartan subgroups) of a compact semisimple Lie group **G** are all conjugate. They are isomorphic to $\mathrm{U}(1)^l$, where l is the rank of **G**. The centralizer $\mathbf{C}_G(g)$ of g in **G** contains a Cartan subgroup; the elements $g \in \mathbf{G}$, whose centralizer is exactly a Cartan subgroup, are called regular. They form an open dense set in **G**.

Given a Cartan subgroup $\mathbf{H} \subset \mathbf{G}$, one considers its normalizer $\mathbf{N}_G(\mathbf{H})$ (the largest **G** subgroup containing **H** as an invariant subgroup). The quotient $\mathbf{N}_G(\mathbf{H})/\mathbf{H} = \mathbf{W}(\mathbf{G})$ is the Weyl group of **G**. This is a finite group with a natural action on the Cartan subalgebra **h** (the Lie algebra of **H**) of **G** generated by reflections along the simple roots. The importance of the Weyl group in the theory of Lie algebras, Lie groups, and their representations is well recognized. However, the exact sequence

$$1 \to \mathrm{U}(1)^l \to \mathbf{N}_G \mathrm{U}(1)^l \xrightarrow{\vartheta'} \mathbf{W}(\mathbf{G}) \to 1\,, \tag{1.1}$$

in general does not split, so **W** is not a subgroup of **G**, where **G** is simply connected compact. Among the finite subgroups of the normalizer $\mathbf{N}_G(\mathrm{U}(1)^l)$ that are mapped by ϑ onto **W** there is a natural one **DT**(G), defined by (2.15) below, that has been first pointed out by Demazure[1] and Tits.[2] Its intersection with $\mathrm{U}(1)^l$ is the group of square roots of 1, hence it is the extension

$$1 \to Z_2^l \to \mathbf{DT}(\mathbf{G}) \xrightarrow{\vartheta} \mathbf{W}(\mathbf{G}) \to 1\,, \tag{1.2}$$

which is naturally deduced from (1.1).

Physicists' interest in the Demazure–Tits group **DT**(**G**) is most likely to originate either from the similarity of its action in representation space to the action of the Weyl group in weight space, or from the fact that it permutes (with some changes of sign) the physical states of a **G**-irreducible space, thus making it possible to keep the same states even without the full Lie group symmetry. It is a finite subgroup of **G** that preserves the root space decomposition (Cartan decomposition) of the Lie algebra of **G**. The group **DT**(**G**) has occasionally appeared in mathematics literature; however, recognition of its usefulness in applied problems relevant to physics is quite recent (cf. Ref. 3, where the group **DT** is denoted by **N**). A systematic use of **DT**(**G**) has been made as the group of invariance of table of the Clebsch–Gordan coefficients (relative to an appropriate basis choice). In computing Clebsch–Gordan coefficients for $\mathbf{G} = \mathrm{SU}(5), \mathrm{O}(10)$, and E_6 (cf. Refs. 4–6) **DT** was used as a group of transformations among CGC of the same values. Practically it allows a small fraction of nonzero CGC to represent all.

In this article we give in Sec. II the structure of **DT**(**G**) for the classical groups A_l, B_l, C_l, D_l, and for G_2. Section III contains some examples of the **DT** group in lowest representations. In general, it is very interesting to decompose an irreducible **G**-representation space V_Λ (Λ is the highest weight) into a direct sum of subspaces irreducible with respect to **DT**(**G**). For groups **G** of rank $l = 2$ we describe **DT**(**G**) in detail in Secs. IV–VI. Namely, we find its character table, decompose any V_Λ into **DT**-invariant subspaces, and identify each **DT**-conjugacy class as a **G** class of elements of finite order (Sec. VII). The last step opens the possibility of using the powerful computing methods[7–10] with elements of finite order in **G** for the study of conjugacy classes of **DT** in all representations of **G**. The simple Lie group **G** in this article is always the simply connected one. Section VIII contains a summary of our results and some open problems. The Appendix contains a summation formula, which, as far as we know, does not appear in literature.

We denote a group (finite or continuous) by bold capital letters; for a Lie algebra we use lowercase bold symbols except for groups or algebras of specific types like A_2 or SU(3), etc. The symbols **W**(**g**) and **W**(**G**), **DT**(**G**) and **DT**(**g**), etc., where **g** is the Lie algebra of **G**, are used as synonyms.

II. THE STRUCTURE OF THE DEMAZURE–TITS SUBGROUPS OF THE SIMPLE SIMPLY CONNECTED LIE GROUPS

We denote by (λ,μ) the Cartan–Killing positive definite scalar product on the compact semisimple Lie algebra **g**, and let the roots be $\alpha_i \in \Delta$, its root system in a chosen Cartan subalgebra **h**; Δ is the root system of **g**. If l is the rank of **g**

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then the Weyl group $\mathbf{W}(\mathbf{g})$ is generated by the reflections r_i, $i = 1,...,l$, along the simple roots α_i,

$$r_i\lambda = \lambda - 2(\alpha_i,\lambda)(\alpha_i,\alpha_i)^{-1}\alpha_i \,. \tag{2.1}$$

When λ itself is a simple root, say α_i,

$$r_j\alpha_i = \alpha_i - \alpha_j A_{ij} \,, \tag{2.2}$$

where

$$A_{ij} = 2(\alpha_i,\alpha_j)(\alpha_j,\alpha_j)^{-1} \tag{2.3}$$

are the matrix elements of the Cartan matrix of $\mathbf{g}$.

We denote by $l = \dim \mathbf{h}$ the rank of $\mathbf{g}$. Let $\{r_i, 1 \leqslant i \leqslant l\}$ be a minimal set of generators of $\mathbf{W}(\mathbf{g})$ (the corresponding simple roots α_i form a base of $\mathbf{h}$); this group is completely characterized by the relations

$$1 \leqslant i,j \leqslant l, \quad (r_i r_j)^{m_{ij}} = I, \quad m_{ii} = 1, \quad 2 \leqslant m_{ij} = m_{ji} \leqslant 6 \,. \tag{2.4}$$

Note that $r_i r_j = r_j r_i$ when $m_{ij} = 2$. The list of possible values of m_{ij} was given by Coxeter and is summarized in the Coxeter–Dynkin diagram of $\mathbf{g}$. Namely, $m_{ij} = (1 - \theta_{ij}/\pi)^{-1}$, where θ_{ij} is the angle between α_i and α_j; it is 2, 3, 4, or 6 according to whether there are zero, one, two, or three lines joining vertices i and j. To specify the structure of $\mathbf{W}(\mathbf{g})$, we define first a family of matrix groups (see, e.g., Ref. 11).

A. The groups G(*m*, *p*, *n*)

Let m, p, n be integers with p dividing m; we denote by $\mathbf{A}(m,p,n)$ the group of diagonal $n \times n$ unitary matrices a that satisfy the relations

$$(a_{ii})^m = 1, \quad 1 \leqslant i \leqslant n, \quad \det(a)^{m/p} = 1 \,. \tag{2.5}$$

Let $\mathbf{\Pi}_n$ be the group of $n \times n$ permutation matrices; they have one 1 in each row and each column and zeros elsewhere. It is a faithful representation of $\mathbf{S}_n$, the group of permutations of n objects. The determinant of a permutation matrix is ± 1 according to the parity of the permutation. We denote by $\mathbf{G}(m,p,n)$ the matrix group generated by the groups $\mathbf{A}(m,p,n)$ and $\mathbf{\Pi}_n$. Obviously, $\mathbf{G}(m,p,n)$ is the semidirect product,

$$\mathbf{G}(m,p,n) = \mathbf{A}(m,p,n) \circledS \mathbf{\Pi}_n \,. \tag{2.6}$$

All the matrix groups $\mathbf{G}(m,p,n)$, except $\mathbf{G}(1,1,n) = \mathbf{\Pi}_n$ and $\mathbf{G}(2,2,2)$ are irreducible over $\mathbb{C}$. The only pair of conjugate groups is $\mathbf{G}(4,4,2)$ and $\mathbf{G}(2,1,2)$. For a finite group $\mathbf{G}$, we denote by $|\mathbf{G}|$ the number of its elements. Then

$$|\mathbf{G}(m,p,n)| = m^n p^{-1} n! \,. \tag{2.7}$$

The linear action of the Weyl group $\mathbf{W}(\mathbf{g})$ on the Cartan subalgebra $\mathbf{h}$ is represented by

$$\mathbf{W}(A_l) = \mathbf{G}(1,1,l+1), \quad \mathbf{W}(B_l) = \mathbf{W}(C_l) = \mathbf{G}(2,1,l) \,,$$
$$\mathbf{W}(D_l) = \mathbf{G}(2,2,l), \quad \mathbf{W}(G_2) = \mathbf{G}(6,6,2) \,. \tag{2.8}$$

Exceptionally, for $A_l \sim \mathrm{SU}_{l+1}$, we have used the Cartan algebra of U_{l+1}; in it the Cartan algebra of A_l is the hyperplane orthogonal to a vector with all coordinates equal.

For a matrix group $\mathbf{G}$ we denote by $\mathbf{SG}$, or sometimes by $\mathbf{G}^+$, its unimodular subgroup (i.e., the group of matrices with determinant 1). Note the isomorphism,

$$\mathbf{SG}(2,1,3) = \mathbf{W}(B_3)^+ \sim \mathbf{S}_4 \,. \tag{2.9}$$

We recall now, at least in a particular case, the definition of the *wreath product*: given a group $\mathbf{K}$, the wreath product by $\mathbf{S}_n$, which we denote by $K \uparrow n$, is the semidirect product

$$\mathbf{K}\uparrow n = \mathbf{K}^n \circledS \mathbf{S}_n \tag{2.10}$$

of $\mathbf{S}_n$ by n copies of $\mathbf{K}$, $\mathbf{S}_n$ acting by permutations on the n factors of $\mathbf{K}^n$. For a finite group $\mathbf{K}$,

$$|\mathbf{K}\uparrow n| = |\mathbf{K}|^n n! \,. \tag{2.11}$$

Let us point out that

$$\mathbf{G}(m,1,n) \sim \mathbf{Z}_m \uparrow n; \quad \text{e.g.,} \quad \mathbf{W}(B_l) \sim \mathbf{Z}_2 \uparrow l \,. \tag{2.12}$$

We will need the following properties of Weyl groups. The Lie algebras of types B_l and C_l have roots of two different lengths; the corresponding reflections form two conjugacy classes in $\mathbf{W}(B_l) = \mathbf{W}(C_l)$ with, respectively, l and $l(l-1)$ elements. The elements of the conjugacy class with l elements are the reflections of $\mathbf{A}(m,1,l)$. They commute and generate the Abelian group $\mathbf{A}(m,1,l)$. Here $\mathbf{W}(D_l)$ is an index 2 subgroup of $\mathbf{W}(B_l)$; when l is odd, $-I \notin \mathbf{W}(D_l)$. That is,

$$\mathbf{W}(B_l) = \mathbf{W}(D_l) \times \mathbf{Z}_2(-I), \quad \text{for } l \text{ odd} \,. \tag{2.13}$$

While the Weyl group $\mathbf{W}(\mathbf{g})$ is the same for all groups $\mathbf{G}$ that have the same Lie algebra $\mathbf{g}$, the Demazure–Tits group $\mathbf{DT}(G)$ does depend on the choice of $\mathbf{G}$; here we consider only simple simply connected compact Lie groups $\mathbf{G}$. We use the notation

$$\mathrm{prod}(n,x,y) = xyxy\cdots \,, \tag{2.14}$$

for a product of n factors, alternately x and y. Tits[2] defines $\mathbf{DT}(\mathbf{G})$ by its generators q_i and their relations

$$1 \leqslant i \leqslant l, \quad q_i^{\,4} = 1, \quad q_i^{\,2} q_j^{\,2} = q_j^2 q_i^{\,2} \,, \tag{2.15a}$$

$$\begin{aligned} &\mathrm{prod}(m_{ij},q_i,q_j) = \mathrm{prod}(m_{ij},q_j,q_i), \\ &q_i q_j^{\,2} q_i^{\,-1} = q_j^{\,2} q_i^{\,2A_{ij}} \,. \end{aligned} \tag{2.15b}$$

The q_i^2 are the square roots of 1 in the Cartan subgroup, they generate the kernel of ϑ in Eq. (1.2). The presence of the exponent $2A_{ij}$ in (2.15b) implies that $\mathbf{DT}(B_l)$ and $\mathbf{DT}(C_l)$ are different although $\mathbf{W}(B_l) = \mathbf{W}(C_l)$. Since we will use these relations often we give them more explicitly:

$$q_i^4 = 1, \quad q_i^2 q_j^2 = q_j^2 q_i^2 \,, \tag{E1}$$

$$m_{ij} = 2\colon \; q_i q_j = q_j q_i \,, \tag{E2}$$

$$m_{ij} = 3\colon \; q_i q_j q_i = q_j q_i q_j, \quad q_i q_j^{\,2} = q_j^{\,2} q_i^{\,-1} \,, \tag{E3}$$

$$\begin{aligned} &m_{ij} = 2k\colon \; (q_i q_j)^k = (q_j q_i)^k, \\ &q_i q_j^{\,2} q_i^{\,-1} = q_j^{\,2} q_i^{\,2A_{ij}} \,. \end{aligned} \tag{E4}$$

Consider two semisimple Lie groups $\mathbf{G}$ and $\mathbf{G}'$ both of rank l. If the Coxeter–Dynkin diagram of $\mathbf{G}$ is a subdiagram of the extended Coxeter–Dynkin diagram of $\mathbf{G}'$, then one has for the corresponding $\mathbf{DT}$ groups,

$$\mathbf{DT}(\mathbf{G}) \subset \mathbf{DT}(\mathbf{G}') \,. \tag{2.16}$$

Clearly $\mathbf{G}$ and $\mathbf{G}'$ have the same Cartan subgroup $\sim U_1^l$ and $\mathbf{N}_G(U_1^l) \subset \mathbf{N}_{G'}(U_1^l)$. Since the corresponding $\mathbf{DT}$ groups have the same kernel $\mathbf{Z}_2^l$, (2.16) holds. If the rank of $\mathbf{G}'$ is lower than l, (2.16) still holds provided the Coxeter–Dynkin diagram of $\mathbf{G}'$ is a subdiagram of the (nonextended) diagram of $\mathbf{G}$.

Let $\mathbf{C}(\mathbf{G})$ be the center of $\mathbf{G}$. The intersection $\mathbf{C}(\mathbf{G}) \cap \mathbf{DT}(\mathbf{G})$ is the group of square roots of $\mathbf{C}(\mathbf{G})$. We recall the nature of $\mathbf{C}(\mathbf{G})$ in Table I.

B. The DT subgroup of A_l

In the natural $(l+1)$-dimensional representation of SU_{l+1}, a Cartan subgroup is represented by diagonal matrices; its subgroup of square roots of the unit is $\mathbf{A}(2,2,l+1) \sim \mathbf{Z}_2^l$. The Weyl group $\sim \mathbf{S}_{l+1}$ permutes the elements of these diagonal matrices; it can be represented by the group of permutation matrices $\mathbf{\Pi}_{l+1}$. The reflections correspond to permutations of two elements, the r_i corresponding to the permutations of neighboring elements. In $\mathbf{\Pi}_{l+1}$ their determinant is -1. The unimodular matrices that represent them in $\mathbf{DT}(\mathrm{SU}_{l+1})$ have been given in Ref. 3 (where they are denoted R_i). They are

$$a_i = I_{i-1} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{l-i}\,, \tag{2.17}$$

where I_k is the $k \times k$ unit matrix.

Let us introduce the $(l+1)\times(l+1)$ diagonal matrices:

$$v_1 = -1 \oplus I_l\,, \tag{2.18}$$

$$v_i = I_{i-1} \oplus -1 \oplus I_{l-i+1} = v_1 \prod_{k=1}^{i-1} a_k^2\,, \quad 2 \leqslant i \leqslant l+1\,. \tag{2.19}$$

They are the reflections of the group $\mathbf{A}(2,1,l+1)$ that they generate. For $1 \leqslant i \leqslant l$, the matrices $v_i a_i$ belong to Π_{l+1} and generate it since they represent the permutations $(i,i+1)$. Hence we have shown that v_1 and the a_i's generate $\mathbf{G}(2,1,l+1)$. Since $\det(a_i) = 1 = -\det(v_1)$, the a_i's generate the unimodular subgroup $\mathbf{SG}(2,1,l+1)$. This proves that

$$\mathbf{DT}(A_l) = \mathbf{DT}(\mathrm{SU}_{l+1}) = \mathbf{SG}(2,1,l+1) \sim \mathbf{W}(B_{l+1})^{+}\,. \tag{2.20}$$

When l is even, $\det(-I_{l+1}) = -1$, so we obtain a unimodular representation $\mathbf{\Pi}_{l+1}$ of $\mathbf{S}_{l+1}$ by multiplying by -1 the matrices representing odd permutations; since $\mathbf{\Pi}_{l+1} \subset \mathbf{SG}(2,1,l+1)$, this shows that the exact sequence (1.2) splits for l even,

$$\mathbf{DT}(A_l) = \mathbf{Z}_2^l \circledS \mathbf{W}(A_l) \quad (l \text{ even})\,. \tag{2.21}$$

This is not the case for odd l; e.g., for $l=1$, $\mathbf{DT}(A_1) = \mathbf{Z}_4$ (see also at the end of this section). When l is even, we can write explicitly a choice of representatives $\tilde{a}_i$ of the a_i's that realizes the splitting (2.21). We define the a_i's using the sets of indices

$$\mathbf{F}(i,l) = \{k,(0<k \text{ odd} <i) \cup (i \leqslant k \text{ even} \leqslant l)\}\,, \tag{2.22a}$$

$$\tilde{a}_i = a_i \prod_{k \in F(i,l)} a_k^2\,. \tag{2.22b}$$

These $\tilde{a}_i$ generate a subgroup of $\mathbf{DT}(A_l)$ isomorphic to $\mathbf{W}(A_l) \sim \mathbf{S}_{l+1}$.

TABLE I. Structure of the center of a classical simple Lie group $\mathbf{G}$.

Algebra	A_l	B_l	C_l	D_l
$\mathbf{G}$	SU_{l+1}	Spin_{2l+1}	Sp_{2l}	Spin_{2l}
$\mathbf{C}(\mathbf{G})$	Z_{l+1}	Z_2	Z_2	$Z_2 \times Z_2$ (l even) Z_4 (l odd)

The center of A_l is the cyclic group $\mathbf{Z}_{l+1}$. When l is odd, the center has a nontrivial square root of unity that is in every Cartan subalgebra and therefore in $\mathbf{DT}(A_l)$. Indeed, the irreducible matrix group $\mathbf{SG}(2,1,l+1)$ has a nontrivial center $\mathbf{C}(\mathbf{SG}(2,1,l+1))$ only when it contains the $-I$ matrix, i.e., for odd l. Thus

$$\mathbf{C}(\mathbf{DT}(A_l)) = 1 \text{ or } Z_2(\alpha), \quad \text{for } l \text{ even or odd}\,;$$
$$\alpha = \prod_{k \text{ odd}} a_k^2\,. \tag{2.23}$$

C. The DT subgroup of C_l

Next we consider the $\mathbf{DT}$ of the symplectic group Sp_{2l}. We denote by c_i the generators of this group. The equations (E) applied to them become

$$\begin{aligned} &c_i^4 = 1, \quad c_i^2 c_j^2 = c^2{}_j c^2{}_i\,, \quad c_i {c_{i+1}}^2 = {c_{i+1}}^2 {c_i}^{-1}\,, \\ &c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1} \quad (1 \leqslant i \leqslant l-1)\,, \\ &c_{l-1} c_l c_{l-1} c_l = c_l c_{l-1} c_l c_{l-1}\,, \\ &c_{l-1} {c_l}^2 = {c_l}^2 {c_{l-1}}^{-1}, \quad c_l {c_{l-1}}^2 = {c_{l-1}}^2 c_l\,. \end{aligned} \tag{2.24}$$

According to (2.16), for $1 \leqslant i \leqslant l-1$, the c_i's generate $\mathbf{DT}(A_{l-1}) \subset \mathbf{DT}(C_l)$. In order to complete our study of C_l, our strategy is to consider its l elements s_i, $1 \leqslant i \leqslant l$, "above" the l *commuting* reflections r_i generating $\mathbf{A}(2,1,l) \subset \mathbf{W}(C_l)$, i.e.,

$$\begin{aligned} &\theta(s_i) = r_i\,, \quad s_l = c_l, \quad 1 \leqslant i \leqslant l-1, \\ &s_i = u_i s_l u_i^{-1} \quad \text{with } u_i = \prod_{k=i}^{l-1} c_k\,. \end{aligned} \tag{2.25}$$

(In the Π symbol, when the factors do not commute, they always are assumed to be placed in order of increasing index value: $u_i = c_i c_{i+1} \cdots c_{l-2} c_{l-1}$.) We know that these reflections commute among themselves. We now prove the following lemma.

Lemma 1: The elements s_i commute among themselves.

We first verify it for s_{l-1} and s_l. Indeed from (2.24) and (2.25), we compute

$$\begin{aligned} s_{l-1} s_l &= c_{l-1} c_l {c_{l-1}}^{-1} c_l \\ &= c_{l-1} c_l c_{l-1} c_l {c_{l-1}}^2 \\ &= c_l c_{l-1} c_l {c_{l-1}}^{-1} = s_l s_{l-1}\,. \end{aligned} \tag{2.26}$$

Because c_i and c_j commute when $|i-j| > 1$, with $\tilde{u}_i = \Pi_{k=i}^{l-1} c_k$, we have

$$\begin{aligned} s_i s_l &= \tilde{u}_i s_{l-1} \tilde{u}_i^{-1} s_l \\ &= \tilde{u}_i s_{l-1} s_l \tilde{u}_i^{-1} \\ &= \tilde{u}_i s_l s_{l-1} \tilde{u}_i^{-1} \\ &= s_l \tilde{u}_i s_{l-1} {\tilde{u}_i}^{-1} = s_l s_i \quad (i \leqslant l-2)\,. \end{aligned} \tag{2.27}$$

We need the relation [use (2.24) twice]

$$s_i = c_{i+1} s_i c_{i+1}{}^{-1} \quad (1 \leqslant i \leqslant l-2) \tag{2.28}$$

to prove by recursion that s_i and s_{i+1} commute. It is true for $i = l-2$:

$$\begin{aligned} s_{l-2} s_{l-1} &= c_{l-1} s_{l-2} s_l c_{l-1}{}^{-1} \\ &= c_{l-1} s_l s_{l-2} c_{l-1}{}^{-1} \\ &= c_{l-1} s_l c_{l-1}{}^{-1} s_{l-2} = s_{l-1} s_{l-2} \,. \end{aligned} \tag{2.29}$$

Assuming that it is true for $i = k$, we prove it for $i = k-1$,

$$\begin{aligned} s_{k-1} s_k &= c_k s_{k-1} c_k{}^{-1} s_k \\ &= c_k c_{k-1} s_k c_{k-1}{}^{-1} s_{k+1} c_k{}^{-1} \\ &= c_k c_{k-1} s_k s_{k+1} c_{k-1}{}^{-1} c_k{}^{-1} \\ &= c_k c_{k-1} s_{k+1} s_k c_{k-1}{}^{-1} c_k{}^{-1} \\ &= c_k s_{k+1} c_{k-1} s_k c_{k-1}{}^{-1} c_k{}^{-1} \\ &= c_k s_{k+1} s_{k-1} c_k{}^{-1} \\ &= c_k s_{k+1} c_k{}^{-1} s_{k-1} = s_k s_{k-1} \,. \end{aligned} \tag{2.30}$$

Finally when $i \leqslant j-2$, we define as before $u = u_i u_{j-1}{}^{-1}$. Then

$$\begin{aligned} s_i s_j &= u s_{j-1} u^{-1} s_j \\ &= u s_{j-1} s_j u^{-1} = u s_j s_{j-1} u^{-1} \\ &\qquad = s_j u s_{j-1} u^{-1} = s_j s_i \,. \end{aligned} \tag{2.31}$$

Using (2.24), we find

$$s_i{}^2 = \prod_{k=1}^{l} c_k{}^2 \,, \tag{2.32}$$

and remark that all the squares are different. Similarly,

$$c_i{}^2 = s_i{}^2 s_{i+1}{}^2, \quad c_l{}^2 = s_l{}^2 \quad (1 \leqslant i \leqslant l-1) \,. \tag{2.33}$$

Hence the s_i commute also with the $c_i{}^2$. They generate an Abelian group containing the kernel in (1.2) of $\mathbf{DT}(C_l)$. Moreover, the commutation of the s_i's shows that the covering of $\mathbf{A}(2,1,l) \subset \mathbf{W}(C_l)$ in $\mathbf{DT}(C_l)$ is

$$\vartheta^{-1}(\mathbf{A}(2,1,l)) = \mathbf{Z}_4{}^l \,. \tag{2.34}$$

When $1 \leqslant i \leqslant l-1$, we choose other representatives $\tilde{c}_i$ of the r_i's,

$$\begin{aligned} &\vartheta(\tilde{c}_i) = \vartheta(c_i) = r_i, \\ &\tilde{c}_i = s_i{}^2 c_i = c_i s_{i+1}{}^2 \quad (1 < i \leqslant l-1) \,, \end{aligned} \tag{2.35}$$

where the last equality is obtained by a repeated use of Eqs. (2.24). We verify that

$$\tilde{c}_i{}^2 = 1, \quad 1 \leqslant i \leqslant l-2, \quad (\tilde{c}_i \tilde{c}_{i+1})^3 = 1 \quad (1 \leqslant i \leqslant l-1) \,. \tag{2.36}$$

This shows that $\mathbf{DT}(C_l)$ contains a subgroup isomorphic to $\mathbf{W}(A_{l-1}) \sim \mathbf{S}_l$. We verify that it acts on the s_i by permutations

$$\begin{aligned} &\tilde{c}_i s_{i+1} \tilde{c}_i{}^{-1} = s_i, \quad \tilde{c}_i s_i \tilde{c}_i{}^{-1} = s_{i+1} \,, \\ &\tilde{c}_i s_j \tilde{c}_i{}^{-1} = s_j \quad (i < j \text{ or } i > j+1) \,. \end{aligned} \tag{2.37}$$

This completes the proof of the isomorphism

$$\mathbf{DT}(C_l) \sim Z_4 \wr l \sim \mathbf{G}(4,1,l) \,. \tag{2.38}$$

The center, $\mathbf{C}(\mathbf{DT}(C_l)) = \mathbf{Z}_4(s)$, of this group is the diagonal subgroup of $\mathbf{Z}_4{}^l$. It is generated by

$$s = \prod_{k=1}^{l} s_i \,. \tag{2.39}$$

Observe that

$$\mathbf{C}(\mathrm{Sp}_{2l}) \cap \mathbf{C}(\mathbf{DT}(C_l)) = Z_2(s^2) \,, \tag{2.40}$$

where α has been defined in (2.23),

$$s^2 = \prod_{k \text{ odd}} c_k{}^2 = \alpha \,. \tag{2.41}$$

The matrices representing c_i's in the $2l$-dimensional faithful representation of the symplectic group C_l are shown in Sec. III. All equations of this section can be thus verified.

D. The DT subgroup of B_l

Let us now consider the **DT** of Spin_{2l+1}. We denote by b_i its generators. For $1 \leqslant i \leqslant l-1$, like the c_i, these satisfy (2.24) and (E1). But the last line of Eq. (2.24) is replaced by

$$\begin{aligned} &b_{l-1} b_l b_{l-1} b_l = b_l b_{l-1} b_l b_{l-1} \,, \\ &b_l b_{l-1}{}^2 b_l = b_{l-1}{}^2 b_l{}^{-1}, \quad b_{l-1} b_l{}^2 = b_l{}^2 b_{l-1} \,, \end{aligned} \tag{2.42}$$

and $m_{ij} = 2$ when $|i-j| > 1$, so (E2) applies

$$b_i b_j = b_j b_i \quad (|i-j| > 1). \tag{2.43}$$

From these equations we obtain

$$\mathbf{Z}_2(\eta) \subseteq \mathbf{C}(\mathbf{DT}(B_l)), \quad \eta = b_l{}^2 \,. \tag{2.44}$$

Here $\mathbf{Z}_2(\eta)$ denotes the $\mathbf{Z}_2$ group generated by η. The group $\mathbf{Z}_2(\eta)$ is exactly $\mathbf{C}(\mathrm{Spin}_{2l+1})$. As we will see later, $\mathbf{C}(\mathbf{DT}(B_l))$ might be larger.

Since $\mathbf{W}(B_l) = \mathbf{W}(C_l)$, we follow the same strategy as for the study of $\mathbf{DT}(C_l)$: we introduce the representatives t_i of the $l-1$ reflections conjugate to b_l,

$$t_l = b_l, \quad t_i = b_i t_{i+1} b_i{}^{-1} = u_i b_l u_i{}^{-1} \quad (1 \leqslant i \leqslant l), \tag{2.45}$$

where the u_i are defined as in (2.25). This time we find that the t_i's all have the same square,

$$t_i{}^2 = \eta \,, \quad \eta^2 = 1 \,, \tag{2.46}$$

and, instead of commuting among themselves, we demonstrate that they "anticommute." More precisely their commutator is η,

$$t_i t_j t_i{}^{-1} t_j{}^{-1} = \eta \quad (1 \leqslant i, j \leqslant l) \,. \tag{2.47}$$

For this we follow the same path of computations as in Eq. (2.26)–(2.31):

$$\begin{aligned} t_{l-1} t_l &= b_{l-1} b_l b_{l-1}{}^{-1} b_l \\ &= b_{l-1} b_l b_{l-1} b_l b_{l-1}{}^2 \eta \\ &= b_l b_{l-1} b_l b_{l-1}{}^{-1} \eta = \eta t_l t_{l-1} \,. \end{aligned} \tag{2.48}$$

Replacing the s_i's by t_i's and (2.26) by (2.48), Eq. (2.27) carries through:

$$t_i t_l = \eta t_l t_i \quad (1 \leqslant i \leqslant l-2) \,. \tag{2.49}$$

Equation (2.28) depends only on (2.24) which is common for both $\mathbf{DT}(C_l)$ and $\mathbf{DT}(B_l)$. It reads for the latter group,

$$t_i = b_{i+1} t_i b_{i+1}{}^{-1} \quad (1 \leqslant i \leqslant l-2) \,. \tag{2.50}$$

To prove by recursion that t_i and t_{i+1} anticommute, we prove it first for $i=l-2$. For this we use (2.50), then (2.49),

$$\begin{aligned} t_{l-2}t_{l-1} &= b_{l-1}t_{l-2}t_l b_{l-1}{}^{-1} \\ &= \eta b_{l-1}t_l t_{l-2}b_{l-1}{}^{-1} \\ &= \eta b_{l-1}t_l b_{l-1}{}^{-1}t_{l-2} = \eta t_{l-1}t_{l-2} \,. \end{aligned} \tag{2.51}$$

We assume it true for $i+1$ and prove it for i. For this replace the s and c's of (2.30) by t and b's; use (2.51) instead of (2.29). An η will appear and this will conclude the proof of (2.47).

The group defined by Eqs. (2.46) and (2.47) is called a *Clifford group*. It is also called the *extra special two-group* in mathematics literature. We denote it by $\mathbf{CL}_l$. Its elements are the monomials of the symbolic polynomial $(1+\eta)\Pi_{i=1}^{l}(1+t_i)$. Thus its order is

$$|\mathbf{CL}_l| = 2^{l+1} \quad (1\leqslant i,j\leqslant l) \,. \tag{2.52}$$

The group $\mathbf{CL}_2$ is the quaternionic group, generated by two $i\sigma_k$, where the σ_k, $k=1,2,3$, are the three Pauli matrices. We define

$$t = \prod_{k=1}^{l} t_k \,. \tag{2.53}$$

From Eqs. (2.46) and (2.47) we get

$$\begin{aligned} &t_i t = t t_i \eta^{l-1}, \quad t^2 = \eta, \quad \text{for } l\equiv 1,2 \bmod 4\,; \\ &t^2 = 1, \quad \text{for } l\equiv 0,3 \bmod 4\,. \end{aligned} \tag{2.54}$$

We have seen that in $\mathbf{W}(B_l)$, the subgroup $\mathbf{W}(A_{l-1})$ generated by the r_k's, $1\leqslant k\leqslant l-1$, acts as the group of permutations $\mathbf{S}_l$ on the l reflections in $\mathbf{A}(2,1,l)\lhd\mathbf{W}(B_l)$ ($\lhd$ reads "invariant subgroup"). The corresponding action of b_k, $1\leqslant k\leqslant l-1$, on the t_i will be, by permutations modulo elements in Ker, $\mathbf{DT}(B_l) = \Pi_{i=1}^{l}\mathbf{Z}_2(b_i{}^2)$. By computation we find that this action is only modulo η; explicitly,

$$\begin{array}{lllll} b_i t_j b_i{}^{-1} = t_j, & \eta t_{j+1}, & t_{j-1}, & t_j, \\ \text{when} \quad j<i, & j=i, & j=i+1, & j>i+1\,. \end{array} \tag{2.55}$$

This also shows that $\mathbf{CL}_l\lhd\mathbf{DT}(B_l)$. Moreover, since the two subgroups $\mathbf{CL}_l$ and $\mathbf{DT}(A_{l-1})$ generate $\mathbf{DT}(B_l)$ and their intersection is only 1, this proves that

$$\mathbf{DT}(B_l) \sim \mathbf{CL}_l \rtimes \mathbf{DT}(A_{l-1}) \sim \mathbf{CL}_l \rtimes \mathbf{SG}(2,1,l) \,, \tag{2.56}$$

with the action defined in (2.55). From this equation we obtain the action of the b_i's on t defined in (2.53); it is trivial:

$$b_i t b_i{}^{-1} = t \,. \tag{2.57}$$

From (2.54), we see that when l is odd, $t\in\mathbf{C}(\mathbf{DT}(B_l))$. Finally, with (2.54) we obtain

$$\begin{array}{llll} \mathbf{C}(\mathbf{DT}(B_l)) = \mathbf{Z}_2(\eta), & \mathbf{Z}_4(t), & Z_2(\eta)\times\mathbf{Z}_2(t), \\ \quad l \bmod 4\equiv 0,2, & 1, & 3. \end{array} \tag{2.58}$$

We recall that for all values of l, $\mathbf{C}(B_l) = \mathbf{Z}_2(\eta)$.

In Sec. III we give an explicit representation of the b_i's in the 2^l-dimensional faithful representation of Spin_{2l+1}.

We denote by φ the homomorphism from Spin_{2l+1} onto $\text{SO}_{2l+1} \sim \text{Spin}_{2l+1}/\mathbf{Z}_2(\eta)$. These two groups are the images of the nontrivial irreducible representations of B_l. In the tensorial representations, $\mathbf{DT}(B_l)$ is represented by the splitting image

$$\varphi(\mathbf{DT}(B_l)) = \mathbf{Z}_2{}^{l-1}\rtimes\mathbf{W}(B_l) \sim (\mathbf{Z}_2{}^{l-1}\times\mathbf{Z}_2{}^{l})\rtimes\mathbf{S}_l \,. \tag{2.59}$$

E. The DT subgroup of D_l

We denote by d_i the generators of $\mathbf{DT}(D_l)\subset\text{Spin}_{2l}$. Since $D_l = \text{Spin}_{2l}$ is a maximal subgroup of $B_l = \text{Spin}_{2l+1}$ with the same rank l, we know from (2.16) that

$$\mathbf{DT}(D_l)\subset\mathbf{DT}(B_l) \,, \tag{2.60}$$

and that it is of index 2, i.e., the same as $\mathbf{W}(D_l)$ in $\mathbf{W}(B_l)$, since we pass from the latter group to the former one by replacing $\mathbf{A}(2,1,l)$ in it by its subgroup of unimodular matrices $\mathbf{A}(2,2,l) = \mathbf{SA}(2,1,l)$. It contains only the products of an even number of reflections r_i. We will write the generators w_i of $\vartheta^{-1}(\mathbf{SA}(2,1,l))$ as products of pairs of the t_i's. More generally, it follows from the structure of $\mathbf{W}$ that we can write the generators of $\mathbf{DT}(D_l)$ in terms of those of $\mathbf{DT}(B_l)$. Namely,

$$d_k = b_k, \quad d_l = b_l b_{l-1} b_l{}^{-1} \quad (1\leqslant k\leqslant l-1)\,. \tag{2.61}$$

We can verify that the d_i's satisfy the equations corresponding to (E2), and (E3). In particular,

$$d_{l-1}d_l = d_l d_{l-1} \,. \tag{2.62}$$

Since $\eta\in\mathbf{C}(\mathbf{DT}(B_l))$, it is also in $\mathbf{C}(\mathbf{DT}(D_l))$. It can now be defined by

$$\eta = d_{l-1}{}^2 d_l{}^2 \,. \tag{2.63}$$

We can choose for the generators of $\mathbf{SA}(2,1,l)$,

$$\begin{aligned} w_i &= t_i t_l = v_i d_{l-1}{}^{-1} d_l v_i{}^{-1} \,, \\ w_{l-1} &= t_{l-1}t_l = d_{l-1}{}^{-1}d_l \,, \\ v_i &= \prod_{k=i}^{l-2} d_k \quad (1\leqslant i\leqslant l-2)\,. \end{aligned} \tag{2.64}$$

From Eqs. (2.46) and (2.47) we find immediately that the $l-1$ w's satisfy the same equations so they generate a subgroup $\sim\mathbf{CL}_{l-1}$. This is an invariant subgroup of $\mathbf{DT}(D_l)$ that has a trivial intersection with the subgroup $\mathbf{DT}(A_{l-1})$. These two subgroups generate $\mathbf{DT}(D_l)$. Hence

$$\mathbf{DT}(D_l) = \mathbf{CL}_{l-1}\rtimes\mathbf{DT}(A_{l-1}) \,, \tag{2.65}$$

where the action of the d_i's on the w_j's is defined implicitly by (2.55) when the d_i's and the w_j's are expressed, respectively, as functions of b_i and t_j [see (2.61) and (2.64)].

Let us now consider the center of $\mathbf{DT}(D_l)$. As in (2.53) we define

$$\begin{aligned} w = \prod_{k=1}^{l-1} w_k &= t, && \text{for } l \text{ even}, \\ &= \eta t t_l \,, && \text{for } l \text{ odd}\,. \end{aligned} \tag{2.66}$$

Similarly to (2.54) we obtain

$$\begin{aligned} w w_i = w_i w\,, \quad & w^2 = 1, \quad \text{for } l\equiv 0,1 \bmod 4, \\ & w^2 = \eta, \quad \text{for } l\equiv 2,3 \bmod 4. \end{aligned} \tag{2.67}$$

When l is even,

$$\alpha = \prod_{k \text{ odd}} d_k{}^2 \,, \tag{2.68}$$

already defined in (2.23), is in $\mathbf{C}(\mathbf{DT}(A_{l-1}))$. It anticommutes with b_l, so it commutes with d_l. Hence it is in the

TABLE II. Structure of the center of the Demazure–Tits subgroup of the simple Lie group D_l and its intersection with the center of the Lie group. $\mathbf{Z}_k\,(y)$ denotes a cyclic group generated by y.

l (mod 4)	0	1	2	3
$\mathbf{C}(\mathbf{DT}(D_l))$	$\mathbf{Z}_2(\alpha)\times\mathbf{Z}_2(\eta)\times\mathbf{Z}_2(w)$	$\mathbf{Z}_2(\eta)\times\mathbf{Z}_2(w)$	$\mathbf{Z}_2(\alpha)\times\mathbf{Z}_4(w)$	$\mathbf{Z}_4(w)$
$\mathbf{C}(D_l)$	$\mathbf{Z}_2^{\,2}$	$\mathbf{Z}_4$	$\mathbf{Z}_2^{\,2}$	$\mathbf{Z}_4$
$\mathbf{C}(\mathbf{DT}(D_l))\cap(D_l)$	$\mathbf{Z}_2(\alpha)\times\mathbf{Z}_2(\eta)$	$\mathbf{Z}_2(\eta)$	$\mathbf{Z}_2(\alpha)\times\mathbf{Z}_2(\eta)$	$\mathbf{Z}_2(\eta)$

center of $\mathbf{DT}(D_l)$. We summarize the description of the center of $\mathbf{DT}(D_l)$ and its intersection with the center of G in Table II.

For l even, there are no faithful irreducible representations of D_l. We denote again by φ the homomorphism from Spin_{2l} onto $\mathrm{SO}_{2l}\sim\mathrm{Spin}_{2l+1}/\mathbf{Z}_2(\eta)$. In the tensorial representations, $\varphi(\mathbf{DT}(B_l))$ is represented by the splitting image,

$$\varphi(\mathbf{DT}(B_l))=\mathbf{Z}_2^{\,l-1}\otimes\mathbf{W}(D_l)\sim(\mathbf{Z}_2^{\,l-1}\times\mathbf{Z}_2^{\,l-1})\otimes\mathbf{S}_l\,. \tag{2.69}$$

F. The DT subgroup of G_2

The Weyl group of G_2 is the dihedral group of 12 elements isomorphic to $\mathbf{S}_3\times\mathbf{Z}_2$. Therefore the order of $|\mathbf{DT}(G_2)|$ is 48. From (2.16) we know that $\mathbf{DT}(\mathrm{SU}_3)\subset\mathbf{DT}(G_2)$ and it has index 2. Note that $\mathbf{DT}(\mathrm{SU}_3)$ is isomorphic to $\mathbf{S}_4$ [see (2.20) and (2.9)]; so it is complete. That means it has no center and no outer automorphism. Hence from a known theorem[11] one has the isomorphism

$$\mathbf{DT}(G_2)\sim\mathbf{S}_4\times\mathbf{Z}_2\,. \tag{2.70}$$

We have seen that $\mathbf{DT}(A_2)\sim\mathbf{Z}_2^{\,2}\times\mathbf{S}_3\sim\mathbf{S}_4$ splits. Since $\mathbf{W}(G_2)=\mathbf{S}_3\times\mathbf{Z}_2$, (2.70) implies that $\mathbf{DT}(G_2)$ also splits,

$$\mathbf{DT}(G_2)=\mathbf{Z}_2^{\,2}\times\mathbf{W}(G_2)\sim\mathbf{Z}_2^{\,2}\otimes\mathbf{S}_3\times\mathbf{Z}_2\,. \tag{2.71}$$

We recall that $\mathbf{C}(G_2)=1$; however, $\mathbf{C}(\mathbf{DT}(G_2))\sim\mathbf{Z}_2$.

In his paper Tits[2] asks the question: What is the smallest subgroup $\mathbf{W}'$ of $\mathbf{DT}(\mathbf{G})$ that covers $\mathbf{W}(\mathbf{G})$, i.e., $\vartheta(\mathbf{W}')=\mathbf{W}(\mathbf{G})$? With the knowledge of the explicit structure of the $\mathbf{DT}(\mathbf{G})$ groups we can give the answer. It is found in Table III.

To end this section we summarize in Table IV the information obtained on the structure of the $\mathbf{DT}(\mathbf{G})$ and their centers.

TABLE III. The smallest subgroups of the Demazure–Tits group $\mathbf{DT}(\mathbf{G})$ covering the Weyl group $\mathbf{W}(\mathbf{G})$. $\mathbf{K}=\ker\vartheta$. The exception for $\mathbf{DT}(A_l)$ is due to the solvability of $\mathbf{S}_4\sim\mathbf{Z}_2^{\,2}\times\mathbf{S}_3$; the result can be understood from $A_3\sim D_3$.

$\mathbf{G}$	rank l	$\mathbf{W}'$	$\mathbf{W}'\cap\mathbf{K}$
A_l	l even	$\sim\mathbf{W}$	1
	$3\neq l$ odd	$\mathbf{DT}(A_l)$	$\mathbf{Z}_2^{\,l}$
	$l=3$	$\mathbf{CL}_2\otimes\mathbf{S}_3$	$\mathbf{Z}_2(\alpha)$
C_l		$\mathbf{DT}(C_l)$	$\mathbf{Z}_2^{\,l}$
B_l		$\mathbf{CL}_l\otimes\mathbf{S}_l$	$\mathbf{Z}_2(\eta)$
D_l		$\mathbf{CL}_{l-1}\otimes\mathbf{S}_l$	$\mathbf{Z}_2(\eta)$
G_2		$\sim\mathbf{W}$	1

III. REPRESENTATIONS OF THE DEMAZURE–TITS GROUPS AND EXAMPLES

Let us underline some common features as well as differences between the well-known group $\mathbf{W}(\mathbf{G})$ and the group $\mathbf{DT}(\mathbf{G})$ that are used subsequently and provide some examples of elements R_i, $i=1,\ldots,l$, generating $\mathbf{DT}(\mathbf{G})$ in some low-dimensional representations of $\mathbf{G}$ of several types and many ranks. The rank $l=2$ cases are studied in much greater detail in Secs. IV–VI. Other properties of $\mathbf{DT}(\mathbf{G})$ can be found in Sec. III of Ref. 3.

The fundamental weights $\omega_1,\ldots,\omega_l$ are defined by

$$(\alpha_i,\omega_k)=\delta_{ik}(\alpha_i,\alpha_i)/2\,. \tag{3.1}$$

The weight lattice Q is the $\mathbb{Z}$ span of the fundamental weights of $\mathbf{G}$,

$$Q=\{\mu:=(a_1,\ldots,a_l)\,|\,\mu=a_1\omega_1+\ldots+a_l\omega_l,\ a_i\in\mathbb{Z}\}\,. \tag{3.2}$$

The sector of Q containing only dominant weights (all $a_i\geqslant 0$) is denoted Q^+. Each orbit of $\mathbf{W}$ in Q is a set of weights that contains precisely one dominant weight, say λ^+. By definition, the set of lattice points

$$\mathrm{O}(\lambda^+)=\{\mu\,|\,\mu=w\lambda^+,\ w\in\mathbf{W}\}\,, \tag{3.3}$$

is a $\mathbf{W}$ orbit, it is $\mathbf{W}$ invariant and is usually specified by its dominant weight λ^+. Subsequently, when no ambiguity could arise, we often use λ^+ for $\mathrm{O}(\lambda^+)$; similarly $\mathrm{O}(\lambda^+)$ is often denoted by $\mathbf{W}\lambda^+$. The number of elements of $\mathrm{O}(\lambda^+)$ is equal to the ratio

$$|\mathrm{O}(\lambda^+)|=|\mathbf{W}\lambda^+|=|\mathbf{W}|/|\mathrm{Stab}_w\lambda^+| \tag{3.4}$$

of the order of $\mathbf{W}$ to the order of the stabilizer of λ^+ in $\mathbf{W}$. It is tabulated in Ref. 13:

$$\mathrm{Stab}_W\,\lambda^+=\{w\,|\,w\lambda^+=\lambda^+\ \text{and}\ w\in\mathbf{W}\}\,. \tag{3.5}$$

$\mathrm{Stab}_W\,\lambda^+$ is the Weyl group of a (semisimple) Lie algebra obtained easily as follows. Take the Coxeter–Dynkin diagram of $\mathbf{G}$ ($\mathbf{W}$ is the Weyl group of $\mathbf{G}$) and attach the coordinates of the dominant weight λ^+ in the basis of the fundamental weights to the corresponding nodes of the Coxeter–Dynkin diagram. Remove nodes with nonzero coordinates. What remains is the diagram of a semisimple Lie subgroup of $\mathbf{G}$ whose Weyl group is $\mathrm{Stab}_W\,\lambda^+$.

An irreducible representation is specified up to $\mathbf{G}$ conjugacy by its highest weight $\Lambda\in Q^+$. Therefore a representation is usually denoted by Λ. An efficient algorithm for finding all λ^+ in $\Omega(\Lambda)$ is given in Refs. 12 and 13. For most cases of interest, λ^+ have been tabulated in Ref. 13 together with the multiplicity of their occurrences in $\Omega(\Lambda)$.

The weight system $\Omega(\Lambda)$ of a representation Λ is in-

TABLE IV. Structure of the Demazure–Tits subgroups of simple Lie groups. Symbols α, s, η, t, w, are, respectively, defined by the following equations: α: (2.23), (2.41), s: (2.39), η: (2.44), t:(2.53), w: (2.66). Here $\mathbf{Z}_n(y)$ denotes a cyclic group of order n generated by y. The Clifford group $\mathbf{CL}_l$ is defined by (2.53) and (2.54).

G	l mod 4	DT(G)	C(DT(G))	C(G)	C(DT(G))∩C(G)
A_l	0,2	$\mathbf{Z}_2{}^l \rtimes \mathbf{S}_{l+1}$	1	Z_{l+1}	1
	1,3	$\sim \mathbf{W}(B_{l+1})^+$	$\mathbf{Z}_2(\alpha)$	$\mathbf{Z}_{l+1}$	1
B_l	0,2	$\mathbf{CL}_l \rtimes \mathbf{DT}(A_{l-1})$	$\mathbf{Z}_2(\eta)$	$\mathbf{Z}_2(\eta)$	$\mathbf{Z}_2(\eta)$
B_l	1	$\mathbf{CL}_l \rtimes (\mathbf{Z}_2{}^l \rtimes \mathbf{S}_{l-1})$	$\mathbf{Z}_4(t)$	$\mathbf{Z}_2(\eta)$	$\mathbf{Z}_2(\eta)$
	3	$\mathbf{CL}_l \rtimes (\mathbf{Z}_2{}^l \rtimes \mathbf{S}_{l-1})$	$\mathbf{Z}_2(\eta) \times \mathbf{Z}_2(t)$	$\mathbf{Z}_2(\eta)$	$\mathbf{Z}_2(\eta)$
C_l		$\mathbf{Z}_4(s) \uparrow l$	$\mathbf{Z}_4(s)$	$\mathbf{Z}_2(\alpha)$	$\mathbf{Z}_2(\alpha)$
D_l	0	$\mathbf{CL}_{l-1} \rtimes \mathbf{DT}(A_{l-1})$	$\mathbf{Z}_2(\alpha) \times \mathbf{Z}_2(\eta) \times \mathbf{Z}_2(w)$	$\mathbf{Z}_2{}^2$	$\mathbf{Z}_2(\alpha) \times \mathbf{Z}_2(\eta)$
	1	$\mathbf{CL}_{l-1} \rtimes (\mathbf{Z}_2{}^l \rtimes \mathbf{S}_{l-1})$	$\mathbf{Z}_2(\eta) \times \mathbf{Z}_2(w)$	$\mathbf{Z}_4$	$\mathbf{Z}_2(\eta)$
	2	$\mathbf{CL}_{l-1} \rtimes \mathbf{DT}(A_{l-1})$	$\mathbf{Z}_2(\eta) \times \mathbf{Z}_4(w)$	$\mathbf{Z}_2{}^2$	$\mathbf{Z}_2(\alpha) \times \mathbf{Z}_2(\eta)$
	3	$\mathbf{CL}_{l-1} \rtimes (\mathbf{Z}_2{}^l \rtimes \mathbf{S}_{l-1})$	$\mathbf{Z}_4(w)$	$\mathbf{Z}_4$	$\mathbf{Z}_2(\eta)$
G_2		$\mathbf{S}_4 \times \mathbf{Z}_2$	$\mathbf{Z}_2$	1	1

variant under **W** and decomposes into several **W** orbits $\mathrm{O}(\lambda^+) = \mathrm{O}(\mathbf{W}\lambda^+)$:

$$\Omega(\Lambda) = \bigcup_{\lambda^+} \mathrm{O}(\lambda^+) . \tag{3.6}$$

The same orbit $\mathrm{O}(\lambda^+)$ often occurs with multiplicity $\mathrm{mult}_\Lambda(\lambda^+) > 1$ in $\Omega(\Lambda)$. We use n for the multiplicity $\mathrm{mult}_\Lambda(\lambda^+)$ of λ^+ in $\Omega(\Lambda)$ whenever there is no ambiguity as to what Λ and λ^+ are. The orbit $\mathrm{O}(\Lambda)$ of the highest weight Λ is always unique in $\Omega(\Lambda)$, i.e., $\mathrm{mult}_\Lambda(\Lambda) = 1$.

Consider the representation space V_Λ and its decomposition

$$V_\Lambda = \bigoplus_{\lambda^+ \in \Omega(\Lambda)} V_W(\lambda^+) = \bigoplus_{\lambda^+ \in \Omega(\Lambda)} \bigoplus_{\mu \in \mathrm{O}(\lambda^+)} V_\Lambda(\mu) \tag{3.7}$$

parallel to the decomposition (3.6) of $\Omega(\Lambda)$, where the subspace $V_W(\lambda^+)$ corresponds to $\mathrm{O}(\lambda^+)$. Indeed $V_W(\lambda^+)$ is the direct sum of weight subspaces $V_\Lambda(\mu)$, $\mu \in \mathrm{O}(\lambda^+)$. The dimensions are given by

$$\begin{aligned} \dim V_W(\lambda^+) &= |\mathbf{W}\lambda^+| \dim V_\Lambda(\mu) \\ &= |\mathbf{W}\lambda^+| \mathrm{mult}_\Lambda \lambda^+ . \end{aligned} \tag{3.8}$$

The permutation of weights

$$\mu' = r_i \mu, \quad \mu, \mu' \in \Omega(\Lambda), \quad r_i \in W,$$

by r_i's of (2.1) exactly corresponds to the permutation of weight subspaces $V_\Lambda(\mu)$ by the elements $R_i \in \mathbf{DT}$. Namely,

$$R_i V_\Lambda(\mu) = V_\Lambda(r_i \mu) = V_\Lambda(\mu'), \quad R_i \in \mathbf{DT}, \quad 1 \leqslant i < l. \tag{3.9}$$

In Ref. 3 the elements R_i are called *charge conjugation operators*. In practice one is more interested in the transformation properties of individual vectors $v_\mu \in V_\Lambda(\mu)$,

$$R_i v_\mu = v_{\mu'} = v_{r_i \mu}, \quad v_\mu \in V_\Lambda(\mu), \quad v_{r_i \mu} \in V_\Lambda(r_i \mu), \tag{3.10}$$

rather than in (3.9). Since there may be n, $n \geqslant 0$, linearly independent vectors v_μ, it turns out that the action of R_i on $V_\Lambda(\mu)$ is quite nontrivial even if r_i acts trivially on μ, i.e., if $r_i \mu = \mu$. Although one still has (3.9), it does not imply that $v_\mu = v_{\mu'}$. For examples see Ref. 3 and Appendix C of Ref. 14.

It follows from (3.9) and (3.10) that one can write symbolically

$$\mathbf{DT} V_W(\lambda^+) = V_W(\lambda^+) = \bigoplus_i m_i V(\Gamma_i), \quad m_i \in \mathbb{Z}_{>0} . \tag{3.11}$$

The action of **DT** is necessarily reducible in subspaces $V_W(\lambda^+)$ of V_Λ. Indeed, **DT**, being a finite group, has finitely many irreducible representations Γ_i, $i = 1,2,\ldots,k < \infty$, while the dimension of $V_W(\lambda^+)$ has no upper limit; it grows with Λ. The summation in (3.11) extends over the irreducible representations of **DT**.

Before turning to specific examples let us recall some notations and conventions. Consider l isomorphic copies of the complex Lie algebra $\mathrm{sl}(2,\mathbb{C})_i$, $1 \leqslant i \leqslant l$, in 1–1 correspondence with the simple roots of **G**. The basis elements e_i, f_i, h_i of each $\mathrm{sl}(2,\mathbb{C})_i$ are chosen to satisfy

$$\begin{aligned} &[e_i, f_i] = h_i, \quad [h_i, e_i] = 2e_i, \\ &[h_i, f_i] = -2f_i, \quad 1 \leqslant i \leqslant l . \end{aligned} \tag{3.12}$$

The generator of **G** can be written as linear combinations of $e_i - f_i$ and $\sqrt{-1}(f_i + e_i)$ for $i \in \{1,\ldots,l\}$ and their commutators. Since we make no direct use of these other generators, there is no need to write them down here. However, we *always* assume that a Chevalley basis[11] of **G** has been chosen. It amounts to having the structure constants integer.

The charge conjugation operators[3] $R_i \in \mathrm{G}$ can be written as

$$\begin{aligned} R_i &= \exp(f_i)\exp(-e_i)\exp(f_i) \\ &= \exp \tfrac{1}{2}\pi(f_i - e_i) , \quad 1 \leqslant i \leqslant l . \end{aligned} \tag{3.13}$$

They generate the Demazure–Tits group **DT**. It has been shown in Ref. 3 that

$$R_i{}^4 = 1, \quad R_i v_\lambda = (-1)^{(\Lambda - \lambda)/2} v_{-\lambda}, \quad v_\lambda \in V_\Lambda(\lambda) , \tag{3.14}$$

where Λ (= twice the angular momentum) denotes the irreducible representation of A_1 of dimension $\Lambda + 1$ and λ is a weight of its weight system $\Omega(\Lambda) = \{\lambda, \lambda - 2, \ldots, -\lambda\}$.

Let us consider examples of R_i in the lowest representations of simple Lie groups of different types.

(A_l) The faithful representation $\Lambda = (100\cdots0)$ of dimension $l + 1$

$$R_i = I_{i-1} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{l-i}, \quad 1 \leqslant i \leqslant l. \tag{3.15}$$

Here I_k is the $k \times k$ identity matrix. In matrixlike symbols we write negative signs over the digits.

(B_l) The matrices R_i, $1 \leqslant i \leqslant l-1$ (denoted by b_i in Sec. II) corresponding to $r_i \in \mathbf{W}$ in the (faithful) 2^l-dimensional spinor representation of Spin_{2l+1} are

$$\begin{aligned} R_i &= (\oplus^{i-1} I_2) \otimes P \otimes (\otimes^{l-i-1} I_2), \quad 1 \leqslant i \leqslant l-1, \\ R_l &= (\otimes^{l-1} I_2) \otimes \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix}, \end{aligned} \tag{3.16}$$

where P is the matrix

$$P = \tfrac{1}{2}(I_2 \otimes I_2 + \sigma_3 \otimes \sigma_3 + i\sigma_1 \otimes \sigma_2 - i\sigma_2 \otimes \sigma_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \bar{1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In particular, one has for $l = 3$ the B_3 representation of dimension 2^3 in a direct sum form, as

$$\begin{aligned} R_1 &= I_2 \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_2, \\ R_2 &= I_1 \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_2 \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus \mathrm{I}_1, \\ R_3 &= \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \bar{1} & 0 \\ 0 & 0 & 0 & \bar{1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{3.17}$$

Similarly one has the B_l representation of dimension $2l+1$ that is not faithful (trivial center),

$$\begin{aligned} R_i &= I_{i-1} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{2l-2i-1} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{i-1}, \\ &\quad 1 \leqslant i \leqslant l-1, \\ R_l &= I_{l-1} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & \bar{1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus I_{l-1}, \quad l \geqslant 2. \end{aligned} \tag{3.18}$$

(C_l) Representation of dimension $2l$,

$$\begin{aligned} R_i &= I_{i-1} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{2l-2i-2} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{i-1}, \\ R_l &= I_{l-1} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{l-1}. \end{aligned} \tag{3.19}$$

Note that, for $l = 2$, B_2 is identical to C_2 up to a renumbering $\alpha_1 \leftrightarrow \alpha_2$ of simple roots. In this case (3.18) and (3.19) refer to the same group in representations of dimension 5 and 4, respectively.

(D_l) When l is even no irreducible representation of $D_l = \text{Spin}_{2l}$ is faithful because the center is not cyclic, $C(D_l) = Z_2^{\,2}$. In order to have a faithful representation one can consider the direct sum of the two 2^{l-1}-dimensional spinor representations. It can be obtained from the 2^l-dimensional representation of $B_l = \text{Spin}_{2l+1}$. The matrices R_i corresponding to $r_i \in \mathbf{W}$ are

$$\begin{aligned} &R_i \text{ as in (3.16)}, \quad \text{for } 1 \leqslant i \leqslant l-1, \\ &R_l = (\oplus^{l-2} I_2) \otimes Q, \end{aligned} \tag{3.20}$$

with

$$Q = \tfrac{1}{2}(I_2 \otimes I_2 - \sigma_3 \otimes \sigma_3 + i\sigma_1 \otimes \sigma_2 - i\sigma_2 \otimes \sigma_1) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \bar{1} & 0 & 0 & 0 \end{pmatrix}.$$

The D_l representation of dimension $2l$ has

$$\begin{aligned} R_i &= I_{i-1} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{2l-2i-2} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_{i-1}, \\ R_l &= I_{l-2} \oplus \begin{pmatrix} 0 & 0 & \bar{1} & 0 \\ 0 & 0 & 0 & \bar{1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \oplus I_{l-2}. \end{aligned} \tag{3.21}$$

Somewhat special is the case $l = 4$. There are three representations of dimension 8. They differ by the following permutations of R_i's,

$$\begin{aligned} &10^{0}_{0} \quad \text{as in Eq. (3.21)}, \\ &00^{0}_{1} \quad R_1 \leftrightarrow R_4, \\ &00^{1}_{0} \quad R_1 \leftrightarrow R_3. \end{aligned} \tag{3.22}$$

(G_2) Representation of dimension 7,

$$\begin{aligned} R_1 &= I_1 \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_1 \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus I_1, \\ R_2 &= \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 \\ 0 & \bar{1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \bar{1} \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{3.23}$$

IV. THE DEMAZURE–TITS SUBGROUP of A_2

In Secs. IV–VI we consider each of the simple Lie groups of rank 2. The description of the Demazure–Tits group **DT** in these cases is carried much further than for higher ranks because one may expect that the lowest ranks will be used most frequently; also, the derivations and results are simpler. Our analysis serves as a model of what can be learned, at least in principle, about each case, besides being a particularly useful illustration.

Each of the three groups is specified up to an isomorphism by its simple roots α_1 and α_2 or, equivalently, by the Cartan matrix

$$(A_{ij}) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \begin{pmatrix} 2 & -A \\ -B & 2 \end{pmatrix}, \tag{4.1}$$

where

$$\begin{aligned} A = B = 1, &\quad \text{for } A_2, \\ A = 2B = 2, &\quad \text{for } B_2, \\ A = 3B = 3, &\quad \text{for } G_2. \end{aligned} \tag{4.2}$$

The Weyl group **W** acts on the weight lattice Q, which is the $\mathbb{Z}$ span of two fundamental weights ω_1 and ω_2. In particular,

$$\alpha_1 = 2\omega_1 - A\omega_2, \quad \alpha_2 = -B\omega_1 + 2\omega_2, \tag{4.3}$$

and therefore

$$\begin{aligned}\omega_1 &= [1/(4-AB)](2\alpha_1 + A\alpha_2),\\ \omega_2 &= [1/(4-AB)](B\alpha_1 + 2\alpha_2).\end{aligned} \tag{4.4}$$

The elements r_1 and r_2 generate **W** by their action (2.1) on the weights $\mu = a\omega_1 + b\omega_2 = (a,b) \in Q$, where $a,b \in \mathbb{Z}$. Namely,

$$r_1(a,b) = (-a, b + Aa), \quad r_2(a,b) = (a + Bb, -b). \tag{4.5}$$

In particular, one has for the simple roots, $r_1\alpha_1 = r_1(2,-A) = (-2,A) = -\alpha_1$, $r_2\alpha_2 = r_2(-B,2) = (B,-2) = -\alpha_2$. A weight is called dominant if $a,b \geqslant 0$.

The "lifting" of the action of **W** on Q to the action of **DT** on V_Λ, i.e., the homomorphism $\mathbf{DT} \to \mathbf{W}$, can be set up in several equivalent but not identical ways. To avoid possible ambiguities, we adopt from now on the following prescription. The elementary reflections $r_1, r_2 \in \mathbf{W}$ of (3) are lifted into R_1, R_2 as given in (3.13) and (3.14). Any other $w \in \mathbf{W}$ is expressed as a word $r_{i_1} r_{i_2} \cdots$ of minimal length in elementary reflections. Then as it is lifted we take the result to be $R_{i_1} R_{i_2} \cdots$. The group **W** also contains one element (opposite involution) of maximal length $k_{\max}$ = number of positive roots of **G**.

The decomposition of $V_W(\lambda^+)$ into **DT**-irreducible subspaces in the three cases of rank 2 is the main problem solved in the rest of this article. Our task is to find the multiplicities m_i of occurrence of the subspaces $V(\Gamma_i)$, irreducible with respect to the representations Γ_i of **DT** in the direct sum [cf. (3.11)],

$$V_W(\lambda^+) = \oplus_i m_i V(\Gamma_i), \quad m_i \in \mathbb{Z}_{>0}. \tag{4.6}$$

Unlike the **W** orbit $O(\lambda^+)$, which is independent of the rest of a weight system $\Omega(\Lambda)$ to which it may belong, the decomposition (4.6) depends on Λ and the multiplicity $n = \mathrm{mult}_\Lambda \lambda^+$. For simplicity of notation we write (4.6) as

$$\lambda^+ = \oplus_i m_i \Gamma_i. \tag{4.6$'$}$$

Let us now turn to the particular case of the Lie algebra A_2 [or Lie group SU(3)]. The multiplicity n of a dominant weight $\lambda^+ = (a,b)$ in an SU(3) representation $\Lambda = (p,q)$ is the coefficient of the term $P^p Q_q A^a B^b$ in the power expansion of the generating function[15]

$$\frac{1}{(1-PQ)^2}\left\{\frac{1}{(1-PA)(1-QB)(1-P^2B)} + \frac{Q^2A}{(1-PA)(1-QB)(1-Q^2A)} + \frac{P^3}{(1-PA)(1-P^2B)(1-P^3)} + \frac{Q^3}{(1-QB)(1-Q^2A)(1-Q^3)}\right\}. \tag{4.7}$$

From (4.7) we deduce that $n = 0$ unless $p - q + b - a = 0$ mod 3, $2p + q \geqslant 2a + b$, and $p + 2q \geqslant a + 2b$. Then the orbit multiplicity n is given by

$$n = \min[p, q, \tfrac{1}{3}(2p + q - 2a - b), \tfrac{1}{3}(p + 2q - a - 2b)] + 1. \tag{4.8}$$

The four expressions in the minimum symbol arise, respectively, from terms 4, 3,2, 1 in (4.7); there is no overlap (i.e., for given p,q,a,b at most one term contributes, namely the one giving the smallest value).

The Weyl group of A_2 is isomorphic to $\mathbf{S}_3$, the group of permutations of three objects. It is also the dihedral group $\mathbf{D}_3$. Its character table is given in Table V. That table contains as well the characters of the $\mathbf{DT}(A_2)$ group, the homomorphism between the classes of elements of W and DT groups, and the SU(3)-conjugacy classes of elements of **DT**.

TABLE V. The character table of the $\mathbf{DT}(A_2)$ and $\mathbf{W}(A_2)$ groups. Subscript of the class symbol indicates the order of its elements. EFO denotes the conjugacy class in SU(3) and IR means irreducible representation.

			Weyl group			
Class	1		3		2	Number of elements
	I		r_1		r_1r_2	Representative element
IR	C_1		C_2		C_3	
Γ_1	1	1	1	1	1	Γ_1
Γ_2	1	1	-1	-1	1	Γ_2
Γ_3	2	2	0	0	-1	Γ_3
	3	-1	-1	1	0	Γ_4
	3	-1	1	-1	0	Γ_5
	C_1	C_2	C_2'	C_4	C_3	IR
Representative element	I	R_1^2	$R_1R_2^2$	R_1	R_1R_2	
EFO	[100]	[011]	[011]	[211]	[111]	Class
Number of elements	1	3	6	6	8	
			Demazure - Tits group			

The character values afforded by the three conjugacy classes of **W** are easily deduced using the action of representative elements on the points of a generic orbit (a,b), illustrated on Fig. 1.

The decomposition of Weyl group orbits on the A_2 weight lattice into direct sums of irreducible representations of W is presented in Table VI.

The structure of the Demazure–Tits subgroup $\mathbf{DT} \subset \mathrm{SU}(3)$ is found either from the SU(n) case of Sec. II or by a direct computation.[3] It turns out to be the octahedral

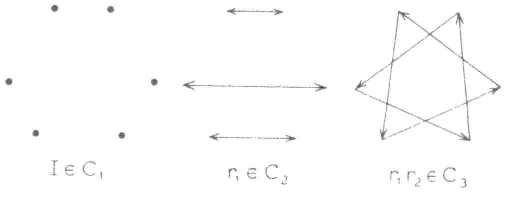


FIG. 1. Action of representative elements of conjugacy classes of the Weyl group of A_2 on weights of a generic orbit.

TABLE VI. Decomposition of the orbits of the Weyl group acting as a permutation group on the A_2 lattice. Character of each class on the orbits is shown.

W orbit	Shape	Characters E	C_2	C_3	W orbit decomposition
(a,b) $a,b>0$	hexagonal	6	0	0	$\Gamma_1 \oplus \Gamma_2 \oplus 2\Gamma_3$
$(a,0)$ or $(0,b)$ $a,b>0$	triangular	3	1	0	$\Gamma_1 \oplus \Gamma_3$
$(0,0)$	point	1	1	1	Γ_1

group. Its character table is in Table V. Each element of **W** corresponds to four elements of **DT**. The correspondences are shown in Table V. The irreducible representations Γ_1, Γ_2, and Γ_3 of **DT** coincide with Γ_1, Γ_2, Γ_3 of **W**. Our notations Γ_i, $i = 1,...,5$, for the representations of the octahedral group are taken from Ref. 16. Table V contains as well a sample element of each conjugacy class of **DT** and **W**, and its SU(3) conjugacy class is identified[7] in the case of **DT**.

Table VI contains the decomposition of **W** orbits in the weight lattice Q into direct sums of irreducible components. Let us point out that the action of **W** is reducible under a general linear transformation but cannot be further reduced when it is confined to permutations of the lattice points.

We now consider the decomposition of the **DT** orbits into direct sums of irreducible representations of **DT**. The results are summarized in Table VII.

The analysis is simplest for the generic (hexagonal) orbit; we need to consider only the classes C_1 and C_2 that correspond to Weyl class C_1. We use R_1^2 as the representative element for C_2. Its eigenvalue is $(-1)^{m_1}$, where m_1 is the SU(2) weight in the α_1 (horizontal) direction; thus the eigenvalue is $(-1)^a, (-1)^b, (-1)^{a+b}$ each for $2n$ states of the orbit and the trace (character) for C_2 is $6n$ for a,b both even, $-2n$ otherwise, as given in Table VII.

We can treat the two types of triangular orbit simultaneously by letting (b) stand for $(0,b)$ or $(b,0)$ according as b is positive or negative. Then b is the second weight component of the states of the orbit for which $m_1 = 0$. The classes C_1 and C_2 are treated as for the hexagonal orbit and have the characters given in Table VII. We must consider in addition the classes C_4 and C_2' whose representatives we take as R_1 and $R_1R_2^2$, respectively. Only the $m_1 = 0$ states contribute to their trace; for them the eigenvalue of R_2^2 is $(-1)^b$ and that of R_1 is $(-1)^{s_1/2}$, where s_1 is the representation label of the SU(2) group in the α_1 direction (s_1 is even for such states).

We will now derive a generating function for the characters of the classes C_4 and C_2'. The generating function for $SU(3) \supset SU(2) \times U(1)$ is

$$F(P,Q,S,Z) = [(1-PSZ)(1-PZ^{-2}) \times (1-QSZ^{-1})(1-QZ^2)]^{-1}. \quad (4.9)$$

In the expansion of (4.9) the coefficient of $P^pQ^qS^sZ^z$ is the multiplicity of the irreducible representation (s,z) of $SU(2) \times U(1)$ in (p,q) of SU(3). To convert (4.9) to a generating function for the C_4 characters we retain only the part even in S [only even s representations of SU(2) contain an $m = 0$ state], set $S^2 = -1$ [the eigenvalue of R_1 is $(-1)^{s/2}$], set $Z = \sqrt{B}$ and separate the result into non-negative and negative powers of B. The non-negative power part turns out to be

TABLE VII. Decomposition of orbits of the Demazure–Tits group in an SU(3) representation (p,q) into the direct sum of irreducible representations $\Gamma_1,..,\Gamma_5$ of **DT**. A **DT** orbit is specified by an SU(3) dominant weight (a,b); n is the multiplicity of (a,b) in (p,q). It is known that for (0,0) weight $n = 1 + \min\{p,q\}$; $k = p - q \bmod 2$.

	DT orbit in (p,q)					Decomposition					
Dominant weight	Characters					Multiplicities of irreps of **DT** group					
	C_1	C_2	C_2'	C_4	C_3	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Restrictions
(a,b) $a,b>0$	$6n$	$6n$	0	0	0	n	n	$2n$	...	...	a,b even
	$6n$	$-2n$	0	0	0	...	...	...	n	n	a,b not both even
$(0,b)$ for $b>0$	$3n$	$3n$	0	0	0	$n/2$	$n/2$	n	...	...	b,n even
	$3n$	$-n$	0	0	0	...	...	...	$n/2$	$n/2$	b odd, n even
	$3n$	$3n$	1	1	0	$(n+1)/2$	$(n-1)/2$	$(n+1)/2$	$(n+1)/2$	$(n+1)/2$	b even, n odd, $p-q$ even
	$3n$	$3n$	-1	-1	0	$(n-1)/2$	$(n+1)/2$	n	...	...	b even, n odd, $p-q$ odd
$(-b,0)$ for $b<0$	$3n$	$-n$	-1	1	0	...	...	...	$(n+1)/2$	$(n-1)/2$	b,n odd, $p-q$ odd
	$3n$	$-n$	1	-1	0	...	...	...	$(n-1)/2$	$(n+1)/2$	b,n odd, $p-q$ even
	n	n	0	0	0	$n/6$	$n/6$	$n/3$	...	...	$n=0 \bmod 6$
	n	n	0	0	-1	$(n-2)/6$	$(n-2)/6$	$(n+1)/3$	...	...	$n=2 \bmod 6$
	n	n	0	0	1	$(n+2)/6$	$(n+2)/6$	$(n-1)/3$	...	...	$n=4 \bmod 6$
$(0,0)$	n	n	1	1	1	$(n+5)/6$	$(n-1)/6$	$(n-1)/3$	...	...	$n=1 \bmod 6$, $k=0$
	n	n	-1	-1	1	$(n-1)/6$	$(n+5)/6$	$(n-1)/3$	...	...	$n=1 \bmod 6$, $k=1$
	n	n	1	1	0	$(n+3)/6$	$(n-3)/6$	$n/3$	...	...	$n=3 \bmod 6$, $k=0$
	n	n	-1	-1	0	$(n-3)/6$	$(n+3)/6$	$n/3$	...	...	$n=3 \bmod 6$, $k=1$
	n	n	1	1	-1	$(n+1)/6$	$(n-5)/6$	$(n+1)/3$	...	...	$n=5 \bmod 6$, $k=0$
	n	n	-1	-1	-1	$(n-5)/6$	$(n+1)/6$	$(n+1)/3$	...	...	$n=5 \bmod 6$, $k=1$

$$\frac{1}{(1-P^2Q^2)}\left(\frac{1}{(1+P^3)(1+P^2B)}\right.$$
$$\left.+\frac{QB}{(1+P^2B)(1-QB)}-\frac{Q^3}{(1-QB)(1+Q^3)}\right). \tag{4.10}$$

The coefficient of $P^pQ^qB^b$ in the expansion of (4.10) is the character of the class C_4 in the orbit $(0,b)$ in (p,q) of SU(3). The three terms in (4.10) never overlap (at most one contributes to the character in each case) and the character is $(-1)^{p-q+b}$ for n odd, 0 for n even, as shown in Table VII. To get the C_2' character, replace B by $-B$ in the generating function, or equivalently, multiply the C_4 character by $(-1)^b$. The characters for $(-b,0)$ orbits are obtained from the negative power (in B) part of the generating function with similar results, found in Table VII.

Finally we come to the (0,0) point orbit. The characters of C_1, C_2, C_2', C_4 are found as before. In addition we now get nonzero contributions from C_3. Since C_3 contributes nothing to the characters of other orbits, its character for the point orbit is equal to that for the whole irreducible representation of SU(3). It is given by the generating function[17]

$$(1-PQ)/(1-P^3)(1-Q^3), \tag{4.11}$$

i.e., 1 for $p=q=0$ mod 3, -1 for $p=q=1$ mod 3, 0 for $p=q=2$ mod 3, as shown in Table VII. There is no point orbit for $p-q\neq 0$ mod 3.

V. THE DEMAZURE–TITS SUBGROUP OF B_2

The irreducible representation (p,q) of the Lie algebra B_2 [or Lie group Sp(4) and also O(5)] has the highest weight $p\omega_1+q\omega_2$; in particular, (1,0) and (0,1) are the representations of dimensions 5 and 4, respectively. Similarly (a,b), $a,b\geqslant 0$, denotes a dominant weight or the Weyl group orbit of the B_2 lattice containing (a,b); the multiplicity of (a,b) in the weight system of (p,q) is denoted by n.

The multiplicity n of a dominant weight $\Lambda^+=(a,b)$ is the coefficient of the term $P^pQ^qA^aB^b$ in the power expansion of the generating function[15]

$$\frac{1}{(1-P)(1-PA)(1-Q^2)(1-QB)}$$
$$\times\left[\frac{1+PQB}{(1-P^2B^2)(1-P^2)}+\frac{Q^2}{(1-P)(1-Q^2)}\right.$$
$$\left.+\frac{Q^2A}{(1-Q^2)(1-Q^2A)}\right]. \tag{5.1}$$

TABLE VIII. The character table of the groups $\mathbf{DT}(B_2)$ and $\mathbf{W}(B_2)$. Subscripts of the class symbol indicate the order of its elements. Here *EFO* denotes a B_2-conjugacy class; IR is an irreducible representation.

Weyl group

Class / IR	1			2		2				1			2		Number of elements
	I			r_2		r_1				$r_1r_2r_1r_2$			r_1r_2		Repres. element
	C_1			C_2		C_2'				C_2''			C_4		
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Γ_1
Γ_2	1	1	1	-1	-1	1	1	1	1	1	1	1	-1	-1	Γ_2
Γ_3	1	1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	Γ_3
Γ_4	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	Γ_4
Γ_5	2	2	2	0	0	0	0	0	0	-2	-2	-2	0	0	Γ_5
	1	1	-1	-1	1	i	-i	i	-i	-1	-1	1	i	-i	Γ_6
	1	1	-1	-1	1	-i	i	-i	i	-1	-1	1	-i	i	Γ_7
	1	1	-1	1	-1	i	-i	i	-i	-1	-1	1	-i	-i	Γ_8
	1	1	-1	1	-1	-i	i	-i	i	-1	-1	1	i	-i	Γ_9
	2	-2	0	0	0	1+i	1-i	-1-i	-1+i	2i	-2i	0	0	0	Γ_{10}
	2	-2	0	0	0	1-i	1+i	-1+i	-1-i	-2i	2i	0	0	0	Γ_{11}
	2	-2	0	0	0	-1-i	-1+i	1+i	1-i	2i	-2i	0	0	0	Γ_{12}
	2	-2	0	0	0	-1+i	-1-i	1-i	1+i	-2i	2i	0	0	0	Γ_{13}
	2	2	-2	0	0	0	0	0	0	2	2	-2	0	0	Γ_{14}
	C_1	C_2	C_2'	C_2''	C_4	C_4'	C_4''	C_4'''	C_4^{iv}	C_4^{v}	C_4^{vi}	C_4^{vii}	C_8	C_8'	IR
Number of elements	1	1	2	4	4	2	2	2	2	1	1	2	4	4	
EFO	[100]	[010]	[001]	[001]	[110]	[201]	[201]	[021]	[021]	[110]	[110]	[110]	[111]	[111]	
Representative element	1	R_2^2	R_1^2	$R_1^2R_2$	R_2	R_1	R_1^3	$R_1R_2^2$	$R_2^2R_1^3$	$R_2R_1^3R_2R_1^3$	$R_1R_2R_1R_2$	$R_1^3R_2R_1R_2$	R_1R_2	$R_1^3R_2$	Class

Demazure–Tits group

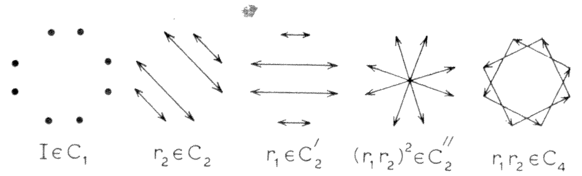


FIG. 2. Action of representative elements of conjugacy classes of the Weyl group of C_2 on weights of a generic orbit.

The character tables of the **W** and **DT** groups are given in Table VIII. The character values of the five conjugacy classes of **W** are found from the action of representative elements on the points of a generic orbit (a,b), $a>0$, $b>0$, illustrated in Fig. 2. Thus one finds the decomposition of the Weyl orbits into the direct sums shown in Table IX.

We turn to the decomposition of **DT** orbits of an arbitrary irreducible representation (p,q) of B_2. As usual the analysis is simplest for the generic (octagonal) orbit (a,b) with $a>0$ and $b>0$; only the classes C_1, C_2, C_2', which correspond to **W** class C_1 have nonzero characters. The weight vectors are eigenvectors of these classes' representative elements with the following eigenvalues:

$$I \rightarrow 1, \quad R_2{}^2 \rightarrow (-1)^{m_2}, \quad R_1{}^2 \rightarrow (-1)^{m_1}.$$

Here m_1 and m_2 are the SU(2) weights in the α_1 and α_2 directions. Thus for $R_1{}^2$ one has the eigenvalue $(-1)^a$ for the two top and two bottom states of each orbit, and $(-1)^{a+b}$ for the remaining four in the middle of the orbit. For $R_2{}^2$ one has the eigenvalue $(-1)^b$ for all eight states. In Table X one finds the decompositions.

For square representations [i.e., highest weights $(a,0)$ and $(0,b)$, $a>0$, $b>0$] the eigenvalues of representatives of the additional classes needed depend not only on the weights of the states, but also on labels s_1 and s_2 of the representation the SU(2) along the α_1,α_2 directions. We use generating functions to keep track of these additional labels.

First we consider the orbits $(a,0)$, squares with horizontal and vertical sides. The new classes are C_4 and C_2'' with representatives R_2 and $R_1{}^2R_2$, respectively. The characters of the classes C_1, C_2, C_2' are found as for the octagonal orbits. Only the upper right and lower left ($m_2=0$) states contribute to the characters of C_4 and C_2'', for them the eigenvalues of R_2 and $R_1{}^2R_2$ are, respectively, $(-1)^{s_2}$ and $(-1)^{a+s_2}$. We now derive a generating function for the characters of the classes C_4 and C_2''.

The generating function for $\mathrm{Sp}(4) \supset \mathrm{SU}(2) \times \mathrm{U}(1)$ branching rules is

$$F(P,Q;S_2,Z) = \frac{1}{(1-PZ^2)(1-PZ^{-2})(1-QS_2Z)(1-QS_2Z^{-1})} \left(\frac{1}{1-PS_2{}^2} + \frac{Q^2}{1-Q^2} \right). \tag{5.2}$$

In the expansion of (5.2) the coefficient of $P^pQ^qS_2{}^{s_2}Z^z$ is the multiplicity of the representation (s_2,z) of $\mathrm{SU}(2) \times \mathrm{U}(1)$ in (p,q) of Sp(4). To convert (5.2) to a generating function for half the C_4 character (because two states contribute), we retain the part even in S_2 [only odd-dimensional SU(2) representations have even valued weights, in particular, the weight $m_2=0$]. Then we set $S_2{}^2=-1$ [the eigenvalue of R_2 is $(-1)^{s_2/2}$], and set $Z^2=A$ and keep only the positive power part in A. The result is

$$\frac{1}{(1-P^2)(1+P)(1+Q^2A)} \left(\frac{1}{1-PA} + \frac{Q^4-PQ^2}{1-Q^4} \right). \tag{5.3}$$

Twice the coefficient of $P^pQ^qA^a$ is the character of C_4 for the orbit $(a,0)$. To get a generating function for half the C_2'' character substitute $A \rightarrow -A$ in (5.3) or, equivalently, multiply the C_4 character by $(-1)^a$. The coefficients of the expansions have been evaluated and the results are summarized in Table IX. We give below the multiplicity n of $(a,0)$ orbits, obtained from the generating function (5.1) with $B=0$, for all six cases q is even and $p+\frac{1}{2}q \geqslant a$:

$$\begin{array}{llll}
(1) & p,\ \frac{1}{2}q \geqslant a, & p-a \text{ even}, & n = 1 + \frac{1}{2}(pq+p+q-a^2), \\
(2) & p,\ \frac{1}{2}q \geqslant a, & p-a \text{ odd}; & n = \frac{1}{2}(pq+p+q-a^2+1), \\
(3) & p \geqslant a \geqslant \frac{1}{2}q, & p-a \text{ even}; & n = \frac{1}{4}q(\frac{1}{2}q+3) + \frac{1}{2}(p-a)(q+1) + 1, \\
(4) & p \geqslant a \geqslant \frac{1}{2}q, & p-a \text{ odd}; & n = \frac{1}{4}q(\frac{1}{2}q+3) + \frac{1}{2}((p-a)(q+1)+1), \\
(5) & \frac{1}{2}q \geqslant a \geqslant p; & & n = \frac{1}{2}(p+1)(p+q-2a+2), \\
(6) & a \geqslant \frac{1}{2}q,\ p; & & n = \frac{1}{2}(p+\frac{1}{2}q-a+1)(p+\frac{1}{2}q-a+2).
\end{array} \tag{5.4}$$

TABLE IX. Decomposition of the orbits of the Weyl group $\mathbf{W}(B_2)$ acting as a permutation group on the B_2 lattice. Characters of each class on the orbit are shown.

		Characters					
W orbit	Shape	C_1	C_2	C_2'	C_2''	C_4	**W** orbit decomposition
(a,b), $a,b>0$	octagon	8	0	0	0	0	$\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus 2\Gamma_5$
$(a,0)$, $a>0$	square	4	2	0	0	0	$\Gamma_1 \oplus \Gamma_3 \oplus \Gamma_5$
$(0,b)$, $b>0$	square	4	0	2	0	0	$\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_5$
$(0,0)$	point	1	1	1	1	1	Γ_1

TABLE X. Decomposition of the generic (octagonal) orbit of **DT**(B_2) into a direct sum of irreducible representations. n is the multiplicity of the orbit (a,b), $a,b>0$, in the representation (p,q) of B_2.

Nonzero characters			Orbit decomposition	Restrictions
C_1	C_2	C_2'		
$8n$	$8n$	$8n$	$n(\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus 2\Gamma_5)$	a,b even
$8n$	$-8n$	0	$n(\Gamma_{10} \oplus \Gamma_{11} \oplus \Gamma_{12} \oplus \Gamma_{13})$	b odd
$8n$	$8n$	$-8n$	$n(\Gamma_6 \oplus \Gamma_7 \oplus \Gamma_8 \oplus \Gamma_9 \oplus 2\Gamma_{14})$	a odd, b even

For the square orbit $(0,b)$, with diagonal sides, the classes with nonzero trace are C_1, C_2, C_2', C_4', C_4'', C_4''', and C_4^{iv}. The characters of C_1 C_2, C_2' are found as for the octagonal orbit. We take the representative elements of C_4', C_4'', C_4''', and C_4^{iv} to be, respectively, R_1, R_1^3, $R_1R_2^2$, $R_2^2R_1^3$. Only the top and bottom $(m_1=0)$ states of the orbit contribute to their characters; the eigenvalue of R_1 is $(-1)^{s_1}$ and that of R_2^2 is $(-1)^b$ for these states. We now derive a generating function for the characters of the classes in question.

The generating function for $\mathrm{Sp}(4) \supset \mathrm{SU}(2) \times \mathrm{SU}(2)$ branching rules is

$$F(P,Q;S_1,U)$$
$$=[(1-P)(1-PS_1U)(1-QS_1)(1-QU)]^{-1}. \tag{5.5}$$

In the expansion of (5.5) the coefficient of $P^pQ^qS_1^{s_1}U^u$ is the multiplicity of the representation (s_1u) of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ in (p,q) of $\mathrm{Sp}(4)$; here s_1 is the SU(2) representation label (highest weight) in the direction of α_1 and u is the representation label in the $\alpha_1+2\alpha_2$ (vertical) direction. To convert (5.5) into a generating function for half (because two states contribute) the C_4' character, we retain the part of (5.5) that is even in S_1 [only even s_1 representations of SU(2) have states with $m_1=0$]. Set $S_1^2=-1$ [the eigenvalue of R_1^2 is $(-1)^{s_1}$], multiply by $(1-U^{-2})(1-U^{-1}B)$ and keep the U^0 part (thereby retaining only positive u weights, which are just the orbit labels). The result is

$$\frac{1}{(1+P^2)(1+Q^2)}\left[\frac{1}{(1-P)(1+P^2Q^2)}\right.$$
$$\left.+\frac{QB}{(1+P^2Q^2)(1-QB)}+\frac{Q^2}{(1-QB)(1-Q^2)}\right]. \tag{5.6}$$

Twice the coefficient of $P^pQ^qB^b$ is the character of C_4' (and C_4'') for the orbit $(0,b)$. To get a generating function for half the characters of C_4''' (and C_4^{iv}) for the orbit, substitute $B \to -B$ in (5.6) or, equivalently, multiply the C_4' characters by $(-1)^b$. The coefficients have been evaluated (they take only the values ± 1 and 0) and the result is found in Table XII, along with the reduction of $(0,b)$ to the direct sum of irreducible representations of **DT**. We give below the multiplicity n for $(0,b)$ orbits, obtained from the generating function (5.1) with $A=0$. For each case $q-b$ is even and $p+q \geqslant b$.

(1) p even, $q \geqslant b$;

$$n=\tfrac{1}{2}[(p-\beta+2)(\beta+1)+(p-\gamma)(\gamma+1)$$
$$+(p+1)(q-b)],$$

(2) p odd, $q \geqslant b$;

$$n=\tfrac{1}{2}[(p-\delta+1)(\delta+1)+(p-\epsilon+1)(\epsilon+1)$$
$$+(p+1)(q-b)],$$

(3) p even, $q<b$;

$$n=\tfrac{1}{2}[(p-\beta-\xi+2)(\beta-\xi+1)$$
$$+(p-\gamma-\xi)(\gamma-\xi+1)],$$

(4) p odd, $q<b$;

TABLE XI. Decomposition of square orbit $(a,0)$ of **DT**(B_2) into the direct sum of its irreducible representations. Only nonzero characters are shown. The values of the multiplicity n are given in (5.4); $\alpha=(-1)^{q/2}(p+\frac{1}{2}q-a+2)$, $\beta=(-1)^{q/2}(p+\frac{1}{2}q-a+1)$, $\gamma=p+2$, $\delta=p+1$.

Characters					Decomposition	Restrictions		
C_1	C_2	C_2'	C_2''	C_4				
$4n$	$4n$	$4n$	α	α	$(\frac{1}{2}n+\frac{1}{4}\alpha)(\Gamma_1 \oplus \Gamma_3) \oplus (\frac{1}{2}n-\frac{1}{4}\alpha)(\Gamma_2 \oplus \Gamma_4) \oplus 2\Gamma_5$	$a \geqslant \frac{1}{2}q$,	a even,	$p+\frac{1}{2}q$ even
$4n$	$4n$	$4n$	$-\beta$	$-\beta$	$(\frac{1}{2}n-\frac{1}{4}\beta)(\Gamma_1 \oplus \Gamma_3) \oplus (\frac{1}{2}n+\frac{1}{4}\beta)(\Gamma_2 \oplus \Gamma_4) \oplus 2\Gamma_5$	$a \geqslant \frac{1}{2}q$,	a even,	$p+\frac{1}{2}q$ odd
$4n$	$4n$	$-4n$	β	$-\beta$	$(\frac{1}{2}n-\frac{1}{4}\beta)(\Gamma_6 \oplus \Gamma_7) \oplus (\frac{1}{2}n+\frac{1}{4}\alpha)(\Gamma_8 \oplus \Gamma_9) \oplus 2\Gamma_{14}$	$a \geqslant \frac{1}{2}q$,	a odd,	$p+\frac{1}{2}q$ even
$4n$	$4n$	$-4n$	$-\alpha$	α	$(\frac{1}{2}n+\frac{1}{4}\alpha)(\Gamma_6 \oplus \Gamma_7) \oplus (\frac{1}{2}n-\frac{1}{4}\alpha)(\Gamma_8 \oplus \Gamma_9) \oplus 2\Gamma_{14}$	$a \geqslant \frac{1}{2}q$,	a odd,	$p+\frac{1}{2}q$ odd
$4n$	$4n$	$4n$	γ	γ	$(\frac{1}{2}n+\frac{1}{4}\gamma)(\Gamma_1 \oplus \Gamma_3) \oplus (\frac{1}{2}n-\frac{1}{4}\gamma)(\Gamma_2 \oplus \Gamma_4) \oplus 2\Gamma_5$	$a<\frac{1}{2}q$,	a even,	$p+\frac{1}{2}q$ even, p even
$4n$	$4n$	$-4n$	p	$-p$	$(\frac{1}{2}n-\frac{1}{4})(\Gamma_6 \oplus \Gamma_7) \oplus (\frac{1}{2}n+\frac{1}{4}p)(\Gamma_8 \oplus \Gamma_9) \oplus 2\Gamma_{14}$	$a<\frac{1}{2}q$,	a odd,	$p+\frac{1}{2}q$ even, p even
$4n$	$4n$	$4n$	p	p	$(\frac{1}{2}n+\frac{1}{4}p)(\Gamma_1 \oplus \Gamma_3) \oplus (\frac{1}{2}n-\frac{1}{4})(\Gamma_2 \oplus \Gamma_4) \oplus 2\Gamma_5$	$a<\frac{1}{2}q$,	a even,	$p+\frac{1}{2}q$ odd, p even
$4n$	$4n$	$-4n$	γ	$-\gamma$	$(\frac{1}{2}n-\frac{1}{4}\gamma)(\Gamma_6 \oplus \Gamma_7) \oplus (\frac{1}{2}n+\frac{1}{4}\gamma)(\Gamma_8 \oplus \Gamma_9) \oplus 2\Gamma_{14}$	$a<\frac{1}{2}q$,	a odd,	$p+\frac{1}{2}q$ odd,p even
$4n$	$4n$	$4n$	$-\delta$	$-\delta$	$(\frac{1}{2}n-\frac{1}{4}\delta)(\Gamma_1 \oplus \Gamma_3) \oplus (\frac{1}{2}n+\frac{1}{4}\delta)(\Gamma_2 \oplus \Gamma_4) \oplus 2\Gamma_5$	$a<\frac{1}{2}q$,	a even,	p odd
$4n$	$4n$	$-4n$	$-\delta$	δ	$(\frac{1}{2}n+\frac{1}{4}\delta)(\Gamma_6 \oplus \Gamma_7) \oplus (\frac{1}{2}n-\frac{1}{4}\delta)(\Gamma_8 \oplus \Gamma_9) \oplus 2\Gamma_{14}$	$a<\frac{1}{2}q$,	a odd,	p odd

TABLE XII. Decomposition of square orbit $(0,b)$ of $\mathbf{DT}(B_2)$ into irreducible representations of $\mathbf{DT}(B_2)$. Characters not shown are 0. Values of the multiplicity n are given in (5.7). For $p \geqslant b$, we have

$\alpha = +1$, for (p mod 4, q mod 4, b mod 4) = (0,0,0),(0,1,1),(0,1,3),(0,2,2),(1,0,0),(2,2,2);
$\alpha = -1$, for (p mod 4, q mod 4, b mod 4) = (1,2,0),(2,0,2),(2,1,1),(2,1,3),(2,2,0),(3,0,2);
$\alpha = 0$, otherwise.

For $p < b$, we have

$\alpha = +1$, for (p mod 4, q mod 4, b mod 4) = (0,0,0),(0,1,1),(0,2,2),(0,3,3);
$\alpha = -1$, for (p mod 4, q mod 4, b mod 4) = (2,2,0),(2,3,1),(2,0,2),(2,1,3);
$\alpha = 0$, otherwise.

Characters						
C_1	C_2	C_2'	C_4', C_4''	C_4''', C_4^{iv}	Decomposition	Restriction
$4n$	$4n$	$4n$	2α	2α	$\frac{1}{2}(n+\alpha)(\Gamma_1 \oplus \Gamma_2) + \frac{1}{2}(n-\alpha)(\Gamma_3 \oplus \Gamma_4) + n\Gamma_5$	b even
$4n$	$-4n$	0	2α	-2α	$\frac{1}{2}(n+\alpha)(\Gamma_{10} \oplus \Gamma_{11}) + \frac{1}{2}(n-\alpha)(\Gamma_{12} \oplus \Gamma_{13})$	b odd

$$n = \tfrac{1}{2}[(p-\delta-\xi+1)(\delta-\xi+1) + (p-\epsilon-\xi+1)(\epsilon-\xi+1)]\,. \tag{5.7}$$

In the above

$$\beta = \mathrm{Min}\left\{\left[\frac{b}{2}\right], \frac{p}{2}\right\}, \quad \gamma = \mathrm{Min}\left\{\left[\frac{b-1}{2}\right], \frac{p}{2}-1\right\},$$

$$\delta = \mathrm{Min}\left\{\left[\frac{b}{2}\right], \frac{p-1}{2}\right\}, \quad \epsilon = \mathrm{Min}\left\{\left[\frac{b-1}{2}\right], \frac{p-1}{2}\right\},$$

$$\xi = \tfrac{1}{2}(b-q)\,.$$

Finally we turn to the (0,0) point orbit. The characters C_1, C_2, C_2', C_4', C_4'', C_4''', and C_4^{iv} are found as before. In addition we now get nonzero characters for C_4^{v}, C_4^{vi}, C_4^{vii}, C_8, and C_8'. Since their characters are zero for the other orbits, their characters on the point orbit are equal to those on the whole representation of the B_2 algebra. Thus they are given by the generating functions of Ref. 17 (replacing the variables A and B by Q and P, respectively):

$$\frac{(1+P)(1+PQ^2)}{(1-P^2)^2(1+Q^2)^2}, \quad \text{for } C_4^{v},\ C_4^{vi},\ C_4^{vii};$$

$$\frac{(1-P)(1+PQ^2)}{(1-P^4)(1+Q^4)}, \quad \text{for } C_8,\ C_8'\,.$$

For C_4^{v}, C_4^{vi}, C_4^{vii} we find the characters,

$$(-1)^{q/2}(\tfrac{1}{2}p + \tfrac{1}{4}q + 1), \quad \text{for } p \text{ even},$$

$$(-1)^{q/2}\tfrac{1}{2}(p+1), \quad \text{for } p \text{ odd}.$$

For C_8 and C_8' we find the characters

$$\begin{aligned} &(-1)^{q/4}, && \text{for } p=0 \bmod 4, \quad q=0 \bmod 4;\\ -&(-1)^{q/4}, && \text{for } p=1 \bmod 4, \quad q=0 \bmod 4;\\ &(-1)^{(q-2)/4}, && \text{for } p=1 \bmod 4, \quad q=2 \bmod 4;\\ -&(-1)^{(q-2)/4}, && \text{for } p=2 \bmod 4, \quad q=2 \bmod 4;\\ &0, && \text{otherwise}\,. \end{aligned}$$

There is no point orbit for q odd. The generating function for the multiplicity of the point is

$$(1+PQ^2)/(1-P)(1-P^2)(1-Q^2)^2, \tag{5.8}$$

which implies

$$\begin{aligned} n &= \tfrac{1}{2}(pq+p+q)+1, \quad \text{for } p \text{ even},\\ n &= \tfrac{1}{2}(p+1)(q+1), \quad \text{for } p \text{ odd}\,. \end{aligned} \tag{5.9}$$

The decomposition of the point orbit into irreducible representations of **DT** is given in Table XIII.

VI. THE DEMAZURE–TITS SUBGROUP OF G_2

As in the previous two cases, $(p,q) = p\omega_1 + q\omega_2$ is the highest dominant weight denoting an irreducible representation of G_2. In particular, (1,0) and (0,1) are the representations of dimensions 14 and 7, respectively. A dominant weight $(a,b) = a\omega_1 + b\omega_2$ denotes the **W** orbit in the G_2-weight (and also root) lattice containing it, as well as the **DT** orbit of subspaces in the representation space labeled by the highest weight (p,q). Naturally one assumes that $(a,b) \in \Omega(p,q)$, otherwise our problem is trivial.

TABLE XIII. Decomposition of the point orbit of $\mathbf{DT}(B_2)$ into its irreducible representations. The values of n are given in (5.9). $\alpha = (-1)^{q/2} \times (\frac{1}{2}p + \frac{1}{4}q + 1)$, $\beta = (-1)^{q/2}\frac{1}{2}(p+1)$, $\gamma = 2(-1)^{q/4}$, $\delta = 2(-1)^{(q-2)/4}$.

Nonzero multiplicities of irreducible $\mathbf{DT}(B_2)$ representations					
Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	(p,q) mod 4
$\frac{1}{8}(n+p+\alpha+\gamma+4)$	$\frac{1}{8}(n-p+\alpha-\gamma)$	$\frac{1}{8}(n+p+\alpha-\gamma)$	$\frac{1}{8}(n-p+\alpha+\gamma-4)$	$\frac{1}{4}(n-\alpha)$	(0,0)
$\frac{1}{8}(n+p+\alpha)$	$\frac{1}{8}(n-p+\alpha)$	$\frac{1}{8}(n+p+\alpha)$	$\frac{1}{8}(n-p+\alpha)$	$\frac{1}{4}(n-\alpha)$	(0,2)
$\frac{1}{8}(n-p+\beta-\gamma+1)$	$\frac{1}{8}(n+p+\beta+\gamma+3)$	$\frac{1}{8}(n-p+\beta+\gamma-3)$	$\frac{1}{8}(n+p+\beta-\gamma-1)$	$\frac{1}{4}(n-\beta)$	(1,0)
$\frac{1}{8}(n-p+\beta+\delta-3)$	$\frac{1}{8}(n+p+\beta-\delta-1)$	$\frac{1}{8}(n-p+\beta-\delta+1)$	$\frac{1}{8}(n+p+\beta+\delta+3)$	$\frac{1}{4}(n-\beta)$	(1,2)
$\frac{1}{8}(n+p+\alpha+2)$	$\frac{1}{8}(n-p+\alpha-2)$	$\frac{1}{8}(n+p+\alpha+2)$	$\frac{1}{8}(n-p+\alpha-2)$	$\frac{1}{4}(n-\alpha)$	(2,0)
$\frac{1}{8}(n+p+\alpha-\delta-2)$	$\frac{1}{8}(n-p+\alpha+\delta-2)$	$\frac{1}{8}(n+p+\alpha+\delta+2)$	$\frac{1}{8}(n-p+\alpha-\delta+2)$	$\frac{1}{4}(n-\alpha)$	(2,2)
$\frac{1}{8}(n-p+\beta-1)$	$\frac{1}{8}(n+p+\beta+1)$	$\frac{1}{8}(n-p+\beta-1)$	$\frac{1}{8}(n+p+\beta+1)$	$\frac{1}{4}(n-\beta)$	(3,0),(3,2)

TABLE XIV. Character table of the $\mathbf{DT}(G_2)$ and $\mathbf{W}(G_2)$ groups. Representative element of each conjugacy class is shown. Subscript on class symbol is the order of its elements. Conjugacy classes of G_2 are given as EFO. IR is an irreducible representation.

Weyl group											
Class	I		$(r_1r_2)^3$		r_2		r_1		$(r_1r_2)^2$	r_1r_2	Representative element
	1		1		3		3		2	2	Number of elements
IR	C_1		C_2		C_2'		C_2''		C_3	C_6	
Γ_1	1	1	1	1	1	1	1	1	1	1	Γ_1
Γ_2	1	1	-1	-1	-1	-1	1	1	1	-1	Γ_2
Γ_3	1	1	1	1	-1	-1	-1	-1	1	1	Γ_3
Γ_4	1	1	-1	-1	1	1	-1	-1	1	-1	Γ_4
Γ_5	2	2	-2	-2	0	0	0	0	-1	1	Γ_5
Γ_6	2	2	2	2	0	0	0	0	-1	-1	Γ_6
	3	-1	-3	1	1	-1	-1	1	0	0	Γ_7
	3	-1	3	-1	-1	1	-1	1	0	0	Γ_8
	3	-1	-3	1	-1	1	1	-1	0	0	Γ_9
	3	-1	3	-1	1	-1	1	-1	0	0	Γ_{10}
	C_1	C_2	C_2'	C_2''	C_2'''	C_4	C_2^{IV}	C_4'	C_3	C_6	IR
Number of elements	1	3	1	3	6	6	6	6	8	8	
EFO	[100]	[001]	[001]	[001]	[001]	[110]	[001]	[201]	[101]	[111]	
Representative element	I	R_1^2	$(R_1R_2)^3$	$R_1R_2R_1^3R_2R_1R_2$	$R_1^2R_2$	R_2	$R_1R_2^2$	R_1	$(R_1R_2)^2$	R_1R_2	Class
Demazure–Tits group											

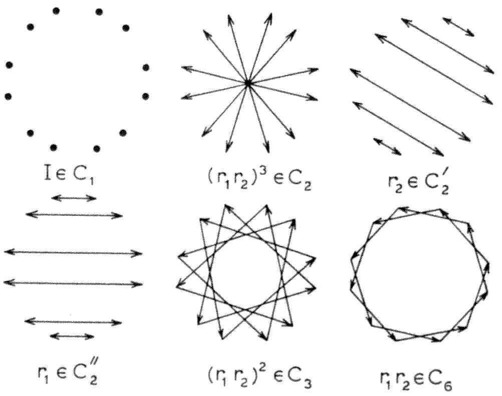


FIG. 3. Action of representative elements of conjugacy classes of the Weyl group of G_2 on weights of a generic orbit.

The multiplicity $n = \text{mult}_{(p,q)}\,(a,b)$ of a weight (a,b) in the weight system $\Omega(p,q)$ is also the multiplicity of the **DT** orbit. It can be found either in the tables of Ref. 13 (for the lowest 100 representations) or it can be calculated using the G_2 character generator, Eq. (2.7) of Ref. 18. There in order to conform to present notation the following substitutions should be made: $A \to Q, B \to P, \eta \to AB^{-3/2}, \xi \to B^{1/2}$; then the coefficient of the term $P^pQ^qA^aB^b$ (a,b non-negative) in the power expansion of the generating function is the multiplicity n.

The character table of the Weyl group $\mathbf{W}(G_2)$ and the Demazure–Tits group $\mathbf{DT}(G_2)$ are found in Table XIV.

First consider $\mathbf{W}$ acting on the G_2 weight lattice. Representative elements of the $\mathbf{W}$-conjugacy classes are

$$C_1\colon\ I,\quad C_2\colon\ (r_1r_2)^3,\quad C_2'\colon\ r_2,\qquad C_2''\colon\ r_1,\quad C_3\colon\ (r_1r_2)^2,\quad C_6\colon\ r_1r_2\,. \tag{6.1}$$

The subscript on a class symbol is the order of its elements; r_1 and r_2 are the elementary reflections (2.1). The traces of classes of each type are easy to determine as before: each point of the orbit that is not moved by the representative element contributes 1 to the trace. Hence it suffices to see the action of the representative of each class on $Q(G_2)$. It is shown in Fig. 3.

Consider the generic, or dodecagonal, orbit $(a,b), a>0, b>0$, of the Weyl group in the G_2 weight lattice Q. The class C_1 has trace 12, while all other classes have trace 0. Hence one has the decomposition $(a,b) = \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus 2\Gamma_5 \oplus 2\Gamma_6$ as shown in Table XV. Similarly for the hexagonal orbit $(a,0), a>0$, the class C_1 has trace 6, the class C_2' has trace 2, and all other classes have trace 0. We find the decomposition $(a,0) = \Gamma_1 \oplus \Gamma_4 \oplus \Gamma_5 \oplus \Gamma_6$ (cf. Table XV). For the other hexagonal orbit, $(0,b), b>0$, the class C_1 has trace 6, the class C_2'' has trace 2, and the others are 0. The decomposition is $(0,b) = \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_5 \oplus \Gamma_6$. Finally for the point orbit $(0,0)$ each class has trace 1 so that its decomposition is $(0,0) = \Gamma_1$. The decomposition of Weyl group orbits of $Q(G_2)$ is summarized in Table XV.

Next let us consider the **DT** group acting on the weight vector basis of V_Λ, $\Lambda = (p,q)$ and let us find the decomposition (3.11).

We consider first the generic orbit $(a,b), a>0, b>0$, which appears with multiplicity n in $V_{(p,q)}$. The classes with nonzero traces are C_1 and C_2. The trace of C_1 is $12n$. For C_2 we have the representative element R_1^2; its eigenvalue is $(-1)^{m_1}$, where m_1 is the SU(2) weight in the α_1 direction. The values of $|m_1|$ at the 12 points of the orbit are a, $a+b$,

TABLE XV. Decomposition of the Weyl group orbits of the G_2 lattice.

W orbit on G_2 lattice	Shape	Characters of classes C_1	C_2	C_2'	C_2''	C_3	C_6	Orbit decomposition
(a,b) $a,b>0$	dodecagonal	12	0	0	0	0	0	$\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus 2\Gamma_5 \oplus 2\Gamma_6$
$(a,0)$ $a>0$	hexagonal	6	0	0	2	0	0	$\Gamma_1 \oplus \Gamma_4 \oplus \Gamma_5 \oplus \Gamma_6$
$(0,b)$ $b>0$	hexagonal	6	0	2	0	0	0	$\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_5 \oplus \Gamma_6$
$(0,0)$	point	1	1	1	1	1	1	Γ_1

$2a+b$, each $4n$ times. Hence the trace for C_2 is $12n$ for a,b both even, and $-4n$ otherwise. Hence one has the decomposition as given in Table XVI.

The hexagonal orbit $(a,0)$, $a>0$, has two horizontal sides; the classes with nonzero character are C_1, C_2, C_2''', C_4, as follows from Fig. 3. The trace of C_1 is $6n$. For C_2 the trace is $6n$ if a is even, and $-2n$ if a is odd. We will derive generating functions for traces of C_2''' and C_4. Orient the $SU(2)\times SU(2)$ subgroup of G_2 so that α_2 points in the direction of the second SU(2) root. The states not moved by R_2 and $R_1{}^2R_2$, the representative elements of C_4 and C_2''', respectively, are those with dominant weight $(a,0)$ and opposite weight $(-a,0)$. On these states the eigenvalue of R_2 is $(-1)^{t/2}$, and that of $R_1{}^2$ is $(-1)^a$; $|m_s|$ takes the value $2a$, where (s,m_s) are the representation label and weight of the first SU(2) subgroup and (t,m_t) those of the second.

The even-even part of the $G_2\supset SU(2)\times SU(2)$ branching rules generating function is found from Ref. 18, Eq. (3.1) (to conform to our present notations, the substitutions $A\rightarrow Q$ and $B\rightarrow P$ should be made):

$$F(P,Q;S^2,T^2)=\frac{1}{(1-P^2)(1-PS^2)(1-QT^2)(1-Q^2S^2T^2)}\left[\frac{1+PQ^3S^2+Q^3S^2T^2+PQ^3S^2T^2}{(1-Q^3S^2)(1-Q^2)}\right.$$
$$\left.+\frac{PT^2+PQT^2+P^2QS^2T^4+PQ^2S^2T^2}{(1-Q^2)(1-PT^2)}+\frac{P^2S^2T^6+P^3S^2T^6+PQS^2T^4+P^4QS^4T^{10}}{(1-PT^2)(1-P^2S^2T^6)}\right]. \tag{6.2}$$

Because $R_2=(-1)^{t/2}$, we set $T^2=-1$. The result is

$$F'(P,Q;S^2)=\frac{1}{(1-P^2)(1+Q)}\left[\frac{1}{(1-PS^2)(1-Q^2)(1+Q^2S^2)}-\frac{P}{(1+P)(1-PS^2)(1-Q^2)}\right.$$
$$-\frac{PQ}{(1+P)(1-Q^2)(1+Q^2S^2)}+\frac{PQS^2}{(1+P)(1-PS^2)(1+Q^2S^2)}$$
$$\left.-\frac{P^2S^2+P^3QS^4}{(1+P^2S^2)(1-PS^2)(1+Q^2S^2)}\right]. \tag{6.3}$$

Finally we convert this generating function for SU(2) representations to the corresponding one for non-negative SU(2) weights (or G_2 orbit labels, since $a=\frac{1}{2}m_s$) by computing

$$G(P,Q;A)=\left.\frac{F'(P,Q;S^2)}{(1-S^{-2})(1-S^{-2}A)}\right|_{S^0}$$
$$=\frac{1}{1+Q}\left[\frac{1}{(1-P)(1-P^2)(1-Q^4)(1-PA)}\right.$$
$$-\frac{Q^2A}{(1-P^2)(1-Q^4)(1-PA)(1+Q^2A)}-\frac{P}{(1-P^2)^2(1-Q^2)(1-PA)}$$
$$-\frac{PQ}{(1-P^2)(1+P)(1-Q^4)(1+Q^2A)}+\frac{PQ}{(1-P^2)^2(1+Q^2)(1+Q^2A)}$$
$$+\frac{PQA}{(1-P^2)^2(1-PA)(1+Q^2A)}-\frac{P^2+P^3Q}{(1-P^4)(1-P)(1+Q^2)(1-PA)}$$
$$\left.-\frac{P^2A+P^3QA^2}{(1-P^4)(1+P^2A)(1-PA)(1+Q^2A)}+\frac{P^2Q^2A-P^3QA^2}{(1-P^4)(1+Q^2)(1-PA)(1+Q^2A)}\right]. \tag{6.4}$$

TABLE XVI. Decomposition of G_2 orbits of the Demazure–Tits group **DT** in a representation (p,q) into a direct sum of irreducible representations $\Gamma_1,\ldots,\Gamma_{10}$ of **DT**. An orbit is given by a G_2 dominant weight (a,b); n is the multiplicity of (a,b) in (p,q). Notation: c, d, e, f, g are the coefficients of the term $P^pQ^qA^aB^b$ in the power series of Eqs. (6.5), (6.8), (6.9), (6.10), (6.11), respectively; $X_\pm=(n\pm e)/12$, $Y_\pm=(d\pm c)/4$, $Z_\pm=(f\pm g)/6$.

DT orbit in (p,q)												Decomposition									
Dominant weight		Characters										Multiplicities of irreps of DT(G_2)									
		C_1	C_2	C_2'	C_2''	C_2'''	C_2^{IV}	C_4	C_4'	C_3	C_6	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9	Γ_{10}
(a,b)	a,b even	12n	12n	0	0	0	0	0	0	0	0	n	n	n	n	2n	2n	0	0	0	0
a,b>0	otherwise	12n	-4n	0	0	0	0	0	0	0	0	0	0	0	0	0	0	n	n	n	n
(a,0)	a even	6n	6n	0	0	2c	0	2c	0	0	0	$\frac{n+c}{2}$	$\frac{n-c}{2}$	$\frac{n-c}{2}$	$\frac{n+c}{2}$	n	n	0	0	0	0
	a odd	6n	-2n	0	0	-2c	0	2c	0	0	0	0	0	0	0	0	0	$\frac{n-c}{2}$	$\frac{n+c}{2}$	$\frac{n+c}{2}$	$\frac{n-c}{2}$
(0,b)	b even	6n	6n	0	0	0	2d	0	2d	0	0	$\frac{n+d}{2}$	$\frac{n+d}{2}$	$\frac{n-d}{2}$	$\frac{n-d}{2}$	n	n	0	0	0	0
	b odd	6n	-2n	0	0	0	-2d	0	2d	0	0	0	0	0	0	0	0	$\frac{n+d}{2}$	$\frac{n+d}{2}$	$\frac{n-d}{2}$	$\frac{n-d}{2}$
(0,0)		n	n	e	e	c	d	c	d	f	g	$X_++Y_++Z_+$	$X_-+Y_-+Z_-$	$X_+-Y_++Z_+$	$X_--Y_-+Z_-$	$2X_--Z_-$	$2X_+-Z_+$	0	0	0	0

The symbol $|_{S^0}$ indicates that only the 0th power of S term of (6.4) should be retained. The power series expansion of $G(P,Q;A)$,

$$G(P,Q;A) = \sum_{pqa} P^p Q^q A^a c_{pqa} , \tag{6.5}$$

states that the trace of class C_4 is $2c_{pqa}$ for the orbit $(a,0)$ in (p,q); for C_2''' the trace is $2(-1)^a c_{pqa}$. The factor 2 appears because two states contribute to the trace. The decomposition of the $(a,0)$ orbit is shown in Table XVI.

For the hexagonal orbit $(0,b)$, $b>0$ (two vertical sides), the classes with nonzero trace are C_1, C_2, C_2^{iv}, C_4'. For C_1 the trace is $6n$. For C_2 it is $6n$ for b even, $-2n$ for b odd. We derive generating functions for the trace of C_2^{iv} and C_4', using the representative elements $R_2{}^2R_1$ and R_1, respectively. Orient the SU(2)×SU(2) subgroup with the first SU(2) root along α_1 of G_2. The states not moved by R_1 and $R_2{}^2R_1$ are those with weights $(0, \pm b)$. On these states the eigenvalue of R_1 is $(-1)^{s/2}$ [s is the first SU(2) representation label] and that of $R_2{}^2$ is $(-1)^b$; $|m_t|$ takes the value $2b$ [m_t is the second SU(2) weight]. Because $R_1 = (-1)^{s/2}$, we set $S^2 = -1$ in the generating function (6.2) with the result

$$F''(P,Q;T^2) = \frac{1}{(1-P^2)(1+Q^2T^2)} \left[\frac{1}{(1-Q^2)(1+Q^3)} + \frac{QT^2}{(1+Q^3)(1-QT^2)} - \frac{P^2T^6}{(1-QT^2)(1+P^2T^6)} - \frac{P-PT^2-P^2T^4}{(1+P^2T^6)(1+P)} - \frac{PQ^2}{(1+P)(1-Q^2)} \right] . \tag{6.6}$$

Finally we convert this generating function for SU(2) representations into the corresponding one for non-negative weights (or G_2 orbits labels, since $b = \frac{1}{2}m_t$) by computing

$$\begin{aligned} H(P,Q;B) &= \frac{F''(P,Q;T^2)}{(1-T^{-2})(1-T^{-2}B)} \bigg|_{T^0} \\ &= \frac{1}{1+Q^2B} \left[\frac{1+Q+Q^2}{(1-P^2)(1+Q^3)(1-Q^4)} + \frac{P^2}{(1+P)(1-P^4)(1+Q^2)} \right. \\ &\quad - \frac{P^2}{(1-P^4)(1-Q)(1+Q^2)} + \frac{PQ^2}{(1+P)(1-P^2)(1-Q^4)} \\ &\quad - \frac{P^2B+P^2B^2+P^4B^3}{(1-P^4)(1-Q)(1+P^2Q^3)} + \frac{PB+P^2B+P^2B^2+P^3B^3+P^4B^3-P^3-P^3B}{(1+P)(1-P^4)(1+P^2B^3)} \\ &\quad \left. + \frac{QB}{(1-P^2)(1-Q)(1+Q^3)(1-QB)} - \frac{P^2B^3}{(1-P^2)(1-Q)(1-QB)(1+P^2B^3)} \right] . \end{aligned} \tag{6.7}$$

The power series expansion of $H(P,Q;B)$,

$$H(P,Q,B) = \sum_{pqb} P^p Q^q B^b d_{pqb} , \tag{6.8}$$

gives the trace of the class C_4 for the orbit $(0,b)$ in the G_2 representation (p,q) as $2d_{pqb}$; for C_2''' the trace is $2(-1)^b d_{pqb}$. The decomposition of the orbit $(0,b)$ is given in Table XVI.

Finally we deal with the point orbit (0,0). All classes can now have nonzero trace. The traces of classes C_1, C_2, C_2''', C_2^{iv}, C_4, C_4' are computed as above for the hexagonal orbits. Thus the trace of C_1 and C_2 is n, the multiplicity of the orbit. A generating function for n is obtained from (6.2) by setting $S = T = 1$, since each even SU(2)×SU(2) representation has just one state at the origin; n for (p,q) is the coefficient of P^pQ^q in the power series expansion. For C_2^{iv} and C_4' the trace is $c = c_{pq0}$, the coefficient of $P^pQ^qA^0$ in the expansion of (6.4). For C_2''' and C_4 the trace is $d = d_{pq0}$, the coefficient of $P^pQ^qB^0$ in the expansion of (6.7). Since the remaining classes have zero trace for all but the point orbit, their trace for the point orbit is their character in the whole irreducible representation (p,q). Accordingly we can get it from the known generating functions for the characters of the corresponding G_2-conjugacy class of elements of finite order in G_2, Ref. 17. For C_2' and C_2'' we have

$$\begin{aligned} \sum_{pq} P^pQ^q e_{pq} = {} & \frac{1}{(1+P)^2(1-P^2)^2(1+Q)^2(1-Q^2)^2} \\ & \times [1+P-2PQ-P^2Q-PQ^2 \\ & + Q^3+2P^3Q-2P^2Q^2+2PQ^3 \\ & + P^4Q-P^3Q^2-P^2Q^3-2P^3Q^3 \\ & + P^3Q^4+P^4Q^4] . \end{aligned} \tag{6.9}$$

For C_3 we have

$$\begin{aligned} \sum_{pq} P^pQ^q f_{pq} = {} & \frac{1}{(1-P^3)^2(1+Q+Q^2)^3} \\ & \times [1+P+2Q+2Q^2+PQ^2+PQ \\ & + Q^3+P^4Q+P^3Q^2 \\ & + 2P^4Q^2+P^3Q^3+2P^4Q^3 \\ & + P^3Q^4+P^4Q^4] . \end{aligned} \tag{6.10}$$

For C_6 we have

$$\sum_{pq} P^pQ^q g_{pq} = \frac{(1-Q^2)(1-P+Q-P^4Q+P^3Q^2-P^4Q^2)}{(1-P^6)(1-Q^6)} . \tag{6.11}$$

Our result, the decomposition of the point orbit, is given in Table XVI.

VII. CONJUGACY CLASSES OF ELEMENTS GENERATING THE DEMAZURE–TITS GROUPS

In this section we consider the elements R_k, $k \in \{1,2,...,l\}$, which generate the Demazure–Tits group $\mathbf{DT}(\mathbf{G})$ up to equivalence transformation by the simple connected Lie group $\mathbf{G}$, and identify the $\mathbf{G}$-conjugacy classes to which they belong. Since part of that has been done already in Ref. 7, here we just complete Table III of that article.

First let us show that R_k, $k \in \{1,2,...,l\}$, are rational elements in any $\mathbf{G}$. (An element is rational if its character values for any representation of $\mathbf{G}$ are integers.) Consider $R_k \in \mathrm{SU}_k(2) \in \mathbf{G}$, and the subgroup $\mathrm{SU}_k(2)$ whose simple root is α_k. The character value of R_k for any representation $\Lambda(\mathbf{G})$ of $\mathbf{G}$ is by definition its character for the subgroup representation $\Lambda(\mathrm{SU}_k(2)) \subset \Lambda(\mathbf{G})$. Then recalling[3,17] that R_k is a rational element of $\mathrm{SU}_k(2)$, it has to be rational also in $\mathbf{G}$.

We know[3] that all R_k are of order 4 and that those R_k corresponding to simple roots α_k of the same length are $\mathbf{G}$ conjugate, while any two R_k corresponding to roots of different lengths are not $\mathbf{G}$ conjugate. Therefore here we have to identify one conjugacy class of elements of order 4 in D_l, E_6, E_7, and E_8 and two such conjugacy classes in F_4. For all other cases the conjugacy classes were found.[7] All the conjugacy classes of R_k are shown in Table XVII.

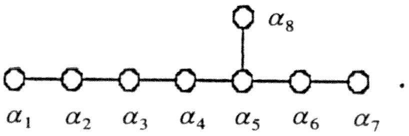
From now on we assume the conventions and results of Ref. 7. In particular, elements of finite order in $\mathbf{G}$ are denoted by relatively prime non-negative integers attached to the nodes of extended Coxeter–Dynkin diagram; we use the Dynkin numbering of the nodes (cf., for instance, Ref. 7 or Ref. 13). It is not difficult to list all conjugacy classes of elements of order 4 in any $\mathbf{G}$. Thus, for example, there are only seven such classes of elements in E_8. Since this is clearly the most complicated case we have to face, we illustrate in this example how one can proceed.

Let $g \in E_8$ belong to one of the seven E_8-conjugacy classes of elements of order 4, $g^4 = 1$. Note that all E_8 representations are self-contragredient. Therefore g and $g^{-1} = g^3$ are conjugate, $g \sim g^3$. That is, all powers of g relatively prime to 4 are conjugate to g. Consequently,[7] the character $\chi_\Lambda(g)$ of any element of our seven conjugacy classes is an integer in any representation Λ of E_8.

TABLE XVII. $\mathbf{G}$-conjugacy classes of elements generating the Demazure–Tits group and their second powers. Subscript short (long) corresponds to short (long) simple roots of a simple Lie algebra.

$\mathbf{G}$	R_{long}	R^2_{long}	R_{short}	R^2_{short}
A_1	[11]	[01]	$\cdots$	$\cdots$
A_l $l \geqslant 2$	$[210\cdots01]$	$[010\cdots01]$	$\cdots$	$\cdots$
B_l $l \geqslant 2$	$[2010\cdots0]$	$[0010\cdots0]$	$[110\cdots0]$	$[010\cdots0]$
C_l $l \geqslant 3$	$[210\cdots0]$	$[010\cdots0]$	$[2010\cdots0]$	$[0010\cdots0]$
D_l $l \geqslant 4$	$[2010\cdots0]$	$[0010\cdots0]$	$\cdots$	$\cdots$
E_6	[2000001]	[0000001]	$\cdots$	$\cdots$
E_7	[21000000]	[01000000]	$\cdots$	$\cdots$
E_8	[210000000]	[010000000]	$\cdots$	$\cdots$
F_4	[2100]	[0100]	[2001]	[0001]
G_2	[210]	[010]	[101]	[010]

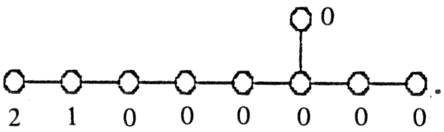
Since all eight R_k, $k \in \{1,2,...,8\}$, are E_8 conjugate, it suffices to consider only, say, R_1. In adopted conventions the E_8 simple roots are numbered as



Let us find the character value $\chi_{\mathrm{Ad}(E_8)}(R_1)$ of R_1 on the 248-dimensional (adjoint) representation $\mathrm{Ad}(E_8)$. But h_1 is orthogonal to the diagram with α_1 removed. That is, the diagram of E_7 (its dimension is 133) on which the SU(2) with simple root α_1 acts trivially. Otherwise R_1 sends h_1 to $-h_1$ and merely transposes the remaining root vectors in pairs which contributes nothing to the character. Therefore one has

$$\chi_{\mathrm{Ad}(E_8)}(R_k) = 132, \quad k \in \{1,2,...,8\}.$$

Next we find which of the seven elements of order 4 in E_8 has that character value on $\mathrm{Ad}(E_8)$. It turns out that there is just one such element [21000000]. Using the extended diagram, it is given as



In order to verify that its character is indeed 132, one can consult the table of positive roots of E_8 (pp. 62 and 63 of Ref. 13), this time reading the roots in the simple root basis (α basis). We need to know only the α_1 coordinate of each root. That coordinate takes only five values ± 2, ± 1, 0, negative values occurring for negative roots only. An E_8 root with the α_1 coordinate m contributes[7] to the character value $\exp(2\pi i m/4)$. Moreover since the character must be integer, the values $m = 1$ and 3 can be disregarded; they must cancel out. Among the positive roots one finds 63 times $m = 0$ and once $m = 2$; the negative roots contribute similarly. Adding the eight zero weights of the adjoint representation as another $m = 0$ eight times, one gets the character as 132. In the same way, but much more quickly, one can determine the rest of the conjugacy classes of R_k in any other simple $\mathbf{G}$.

VIII. CONCLUDING REMARKS

The Weyl group has been the most important device in virtually any extensive work with representations of high rank ($\gg 1$) simple Lie algebras/groups. The higher the rank the more difficult it is to proceed without it.

Physical states "live" in representation spaces rather than in spaces populated by roots of an algebra or weights of its representations. Consequently, the symmetries of the Weyl group are no more than an (homomorphic) image of the general symmetries of physical states. Moreover, interesting problems at any period of time are usually at (or beyond) the limits of what one can calculate with present day methods. Therefore using only the Weyl group is helpful but one can often proceed much more effectively.

A motivation to carry out large scale computations is often present in physics but only rarely in mathematics. That is perhaps the reason that a tool of prime importance like the Demazure–Tits group has been relatively little studied by mathematicians.

This independent sequel to Ref. 3 is an attempt to partially rectify the situation. The principal results are the following: Description of the **DT** in the classical series of simple Lie groups and G_2; identification of the conjugacy classes (under the Lie group action) of the elements generating **DT**; finding the character table of **DT** in simple Lie groups of rank 2; and decomposition of all finite-dimensional representations of rank 2 Lie groups into direct sums of irreducible components of **DT**.

There remain unsolved other equally interesting problems involving **DT**. We name a few.

The character tables of **DT** group in simple Lie groups of rank >2. An extension of known character tables of W to those of **DT**, as exemplified here for rank 2, is possible and it may not even be difficult.

The structure of **DT** in E_6, E_7, E_8, and F_4. The following appears to be true: $\mathbf{DT}(E_k) \subset \mathbf{DT}(E_{k+1})$ for $k = 6$ and 7. The homomorphism $\mathbf{DT}(E_k) \rightarrow \mathbf{W}(E_k)$ is nonsplit.

Branching rules for Lie groups of rank >2 to **DT**. The multiplicities of Weyl group orbits in corresponding weight systems are either known[13] or can easily be found right now for every case which may conceivably ever be needed.

Integrity bases of invariants and covariants of **DT**. Their description along the lines, for instance, Ref. 16 is possible at least for lower ranks.

Let us finish the article with a remark concerning the action of $\mathbf{DT}(\mathbf{G})$ on a generic orbit $V_W(\lambda^+)$. Its dominant weight $\lambda^+ = (\lambda_1,...,\lambda_l)$ has only trivial stabilizer in $\mathbf{W}$; equivalently, λ^+ has no zero coordinates in the basis of fundamental weights, $\lambda_j > 0$ for any $1 \leqslant j \leqslant l$. The decomposition (3.11) in this case depends only on the values λ_j mod 2, $1 \leqslant j \leqslant l$ and not on the highest weight Λ of any representation of $\mathbf{G}$.

The only elements of $\mathbf{DT}(\mathbf{G})$ which have nonzero trace on $V_W(\lambda^+)$ are the 2^l elements which are mapped under ϑ' of (1.2) to the identity element of $\mathbf{W}$. All other elements of **DT** move every vector of $V_W(\lambda^+)$. The 2^l elements are of the form

$$\prod_{i=1}^{l} (R_i^2)^{\delta_i}, \quad \delta_i = 0 \quad \text{or } 1 .$$

The eigenvalue of R_i^2 acting on any vector of weight $\Sigma_k m_k \omega_k$ is just $(-1)^{m_i}$. The weight component m_i is also the SU(2) weight in the α_i direction.

The eigenvalues of all elements of **DT** with nonzero trace thus depend only on the weights of the orbit. Their characters and hence their orbit decomposition, therefore depend only on the parity of λ_j's.

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APPENDIX: A SUMMATION FORMULA

Here we derive the following identity:

$$\sum_{x=0}^{[q/2]} \frac{(-1)^{q-x}(q-x)!}{x!(q-2x)!} = (q+2) \bmod 3 - 1 , \qquad \text{(A1)}$$

which we have not been able to find in the literature. The right-hand side is the character of the conjugacy class [111] of elements of finite order in SU(3) on the irreducible representation (p,q), $p \geqslant q$, $p - q = 0$ mod 3, as given in Ref. 17, and used in Sec. IV of this paper. We may represent the EFO by R_1R_2, an element of $\mathbf{DT} \subset \mathrm{SU}(3)$ belonging to the **DT** class C_3. Since it has trace 0 on all but the point orbit, its trace for the point orbit is also given by the right-hand side of (A1). We show below that it is also given by the left-hand side of (A1).

The zero-weight space $V_{(p,q)}(0,0)$ is of dimension $q+1$. It is spanned by the $q+1$ vectors which can be written[19] as

$$|x\rangle = (\eta\eta^*)^x(\xi\xi^*)^{q-x}(\eta\xi\zeta)^{(p-q)/3}, \quad x = 0,1,...,q , \qquad \text{(A2)}$$

where η, ξ, ζ are the three weight vectors of the SU(3) representation (1,0) of weights (1,0), $(-1,1)$, $(0,-1)$, respectively; η^*, ξ^*, ζ^* are the weight vectors of the representation (0,1) with weights $(-1,0)$, $(1,-1)$, (0,1), respectively. We eliminate ζ^* of weight (0,1) by means of the syzygy $\eta\eta^* + \xi\xi^* + \zeta\zeta^* = 0$ (the scalar $\eta\eta^* + \xi\xi^* + \zeta\zeta^*$ never appears in these states). The action of R_1R_2 is to permute $\eta\xi\zeta$ and $\eta^*\xi^*\zeta^*$ cyclically. Thus (A2) becomes

$$\begin{aligned} R_1R_2|x\rangle &= (\xi\xi^*)^x(-\eta\eta^* - \xi\xi^*)^{q-x}(\eta\xi\zeta)^{(p-q)/3} \\ &= (-1)^{q-x}(\eta\xi\zeta)^{(p-q)/3} \sum_{\alpha=0}^{q-x} (\xi\xi^*)^{q-\alpha} \\ &\quad \times (\eta\eta^*)^{\alpha} \frac{(q-x)!}{\alpha!(q-\alpha-x)!} \\ &= (-1)^{q-x} \sum_{\alpha} |\alpha\rangle \frac{(q-x)!}{\alpha!(q-\alpha-x)!} . \qquad \text{(A3)} \end{aligned}$$

The contribution of $|x\rangle$ to the trace is the coefficient of $|x\rangle$ on the right-hand side of (A3) and the complete trace is hence the left-hand side of (A1).

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