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Spontaneous symmetry breaking and bifurcations from the Maclaurin and Jacobi ellipsoids

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Résumé. — L'état d'équilibre d'un fluide tournant soumis à son interaction gravitationnelle est déterminé par des équations non linéaires. Les solutions d'équilibre, paramétrées par le carré du moment cinétique, présentent des bifurcations accompagnées de brisures de symétrie. D'hypothèses très générales on déduit des règles de sélection concernant les brisures de symétrie qui peuvent apparaître dans ce problème. Les bifurcations sont du même type que celles à la Landau qui apparaissent dans les transitions de phase du second ordre. La méthode est illustrée par l'exemple simple d'un fluide incompressible animé d'une rotation globale et une nouvelle famille infinie de bifurcations est trouvée. Cependant les règles de sélection sont plus générales ; elles s'appliquent aussi aux modèles qui représentent la rotation d'une étoile de façon plus réaliste.

Abstract. — The equilibrium of a rotating self-gravitating fluid is governed by non-linear equations. The equilibrium solutions, parametrized in terms of the angular momentum squared, exhibit the phenomenon of bifurcation, accompanied by spontaneous symmetry breaking. Under very general assumptions, a set of selection rules can be derived, which drastically restrict the patterns of symmetry breaking that are allowed to appear. Bifurcations of this kind are similar to second-order phase transitions à la Landau. The method is illustrated by the simple example of an incompressible fluid in rigid rotation. However, the selection rules are more general ; they apply also to models which approximate a rotating star more realistically.

1. Introduction. — The phenomenon of *bifurcation* of solutions, encountered in non-linear eigenvalue problems, is closely related to that of *spontaneous symmetry breaking*. Numerous examples in bifurcation theory [1] suggest that, when a bifurcation occurs in a stationary problem, the symmetry of the new solution is lower than the symmetry of the solution it bifurcates from, even though the symmetry of the governing equations remains unchanged [2].

We propose to examine this connection within the

framework of an old problem of astrophysical interest : the equilibrium of rotating fluid masses held together by gravitation [3]. In the particular case of an incompressible, homogeneous fluid in rigid rotation, the equations of hydrostatic equilibrium form a set of *non-linear* equations for the surface which bounds the fluid at equilibrium. These equations depend upon the parameter J^2 , the square of the angular momentum, and are invariant under the group $D_{\infty h}$ [4]. For $0 < J^2 \leq 0.384\,436$ [5], the equilibrium surface is also invariant under $D_{\infty h}$ (Maclaurin ellipsoids). As the angular momentum squared increases beyond the critical value 0.384 436, new solutions appear, whose invariance groups are subgroups of $D_{\infty h}$. The first of these is the set of Jacobi ellipsoids, inva-

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riant under the subgroup D_{2h} of $D_{\infty h}$. Other solutions are known, bifurcating both from the Maclaurin and Jacobi sequences, but a rigorous classification of all possible solutions is still missing.

The aim of this paper is to give such a classification, using as a criterion the type of symmetry breaking that accompanies the bifurcation. More precisely, we address ourselves to the following problem : given a solution of the equations of hydrostatic equilibrium and its isotropy group, find the possible isotropy groups of the solutions which bifurcate from it, and the values of J^2 at which the bifurcations appear. Our investigation is strictly limited to the *equilibrium* of the fluid mass ; *stability* is not examined here [6].

The plan of the paper is the following. In section 2 we write the equation of hydrostatic equilibrium, then derive from it a linearized equation for the deformation by which, starting from a given equilibrium solution, new solutions may be obtained. Polynomial deformations of ellipsoidal solutions are discussed in detail. In section 3 the group-theoretical description of symmetry breaking is used to obtain, under very general assumptions, the list of all possible patterns of symmetry breaking that may accompany bifurcations from the Maclaurin and Jacobi sequences. These results are used in section 4 to compute explicitly the bifurcations corresponding to polynomial deformations of the lowest degree associated with each type of allowed symmetry breaking. In addition to the known bifurcations, a new infinite family of bifurcations from the Maclaurin sequence is found, corresponding to a D_{mh} breaking. The connection with the theory of second-order phase transitions and the general applicability of the method to cases which approximate more realistically a rotating star (compressible fluid with differential rotation) are discussed in section 5. The parametrization of ellipsoidal solutions is described in the appendix.

2. Basic equations ; polynomial solutions. — Consider a homogeneous incompressible fluid of given mass and volume, rotating rigidly about a fixed axis with angular momentum \mathbf{J} . We will assume that in the coordinate system in which the fluid is at rest the only forces are the gravitational and centrifugal forces. The fluid is assumed to occupy a connected volume, bounded by the surface

$$S(\mathbf{x}) = 0. \quad (2.1)$$

We adopt the convention that $S(\mathbf{x}) > 0$ inside the fluid.

2.1 EQUILIBRIUM EQUATION. — In the corotating system, the equations of hydrodynamics reduce to the equations of hydrostatic equilibrium [5]

$$\nabla(p - \varphi) = 0, \quad (2.2)$$

where $p(\mathbf{x})$ is the pressure, and

$$\varphi(\mathbf{x}) = \varphi_g(\mathbf{x}) + \varphi_c(\mathbf{x}) \quad (2.3)$$

is the sum of the gravitational and centrifugal potentials. The gravitational potential, satisfying Poisson's equation

$$\Delta\varphi_g = -\rho, \quad \rho(\mathbf{x}) = \begin{cases} 1, & S(\mathbf{x}) \geq 0 \\ 0, & S(\mathbf{x}) < 0 \end{cases} \quad (2.4)$$

and the boundary condition $\varphi_g(\mathbf{x}) \rightarrow 0, |\mathbf{x}| \rightarrow \infty$, is given by

$$\varphi_g[S](\mathbf{x}) = \frac{1}{4\pi} \int_{S(\mathbf{y}) \geq 0} \frac{d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|}. \quad (2.5)$$

Taking the 3rd axis along \mathbf{J} , the centrifugal potential is

$$\varphi_c[S](\mathbf{x}; J^2) = \frac{J^2}{2I^2} (x_1^2 + x_2^2), \quad (2.6)$$

where

$$I[S] = \frac{15}{8\pi} \int_{S(\mathbf{x}) \geq 0} (x_1^2 + x_2^2) d\mathbf{x} \quad (2.7)$$

is the corresponding moment of inertia.

Equation (2.2) must be integrated subject to the boundary condition

$$p(\mathbf{x})|_{S(\mathbf{x})=0} = 0, \quad (2.8)$$

whence it follows that the fluid boundary is an equipotential :

$$\varphi[S](\mathbf{x}; J^2)|_{S(\mathbf{x})=0} = \text{constant}. \quad (2.9)$$

This is an equation for the function S which determines the boundary at equilibrium, dependent upon the parameter J^2 . We shall call equation (2.9) the *equilibrium equation* and its solutions $S(\mathbf{x}; J^2)$ *equilibrium solutions*.

Let us show that an *ellipsoid* is an equilibrium solution. We set

$$S(\mathbf{x}; J^2) = 1 - \sum_{i=1}^3 \frac{x_i^2}{\varepsilon_i}, \quad (2.10)$$

where

$$\varepsilon_3 \leq \varepsilon_2 \leq \varepsilon_1, \quad \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1. \quad (2.11)$$

Then, in terms of the functions $\alpha_{i_1 \dots i_n}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ defined by equation (A.3),

$$\varphi_g(\mathbf{x}) = \frac{1}{4} \left[\alpha - \sum_{i=1}^3 \alpha_i x_i^2 \right], \quad (2.12)$$

$$I = \frac{1}{2} (\varepsilon_1 + \varepsilon_2),$$

and equation (2.9) yields

$$\left[\alpha_1 - \frac{8J^2}{(\varepsilon_1 + \varepsilon_2)^2} \right] \varepsilon_1 = \left[\alpha_2 - \frac{8J^2}{(\varepsilon_1 + \varepsilon_2)^2} \right] \varepsilon_2 = \alpha_3 \varepsilon_3, \quad (2.13)$$

from which we deduce

$$J^2 = \frac{1}{8}(\varepsilon_1 + \varepsilon_2)(\varepsilon_1 \alpha_1 + \varepsilon_2 \alpha_2 - 2 \varepsilon_3 \alpha_3), \quad (2.14)$$

$$\varepsilon_3 \alpha_3 = \varepsilon_1 \varepsilon_2 \alpha_{12}. \quad (2.15)$$

Equation (2.15) always admits the trivial solution $\varepsilon_1 = \varepsilon_2$. However, if the angular momentum squared exceeds the critical value 0.384 436, a new solution, with $\varepsilon_1 \neq \varepsilon_2$, appears. The continuous set of equilibrium solutions obtained by varying continuously the angular momentum [by equation (2.14) the ε 's are functions of J^2] form the Maclaurin ($\varepsilon_1 = \varepsilon_2$) and Jacobi ($\varepsilon_1 \neq \varepsilon_2$) sequences. These are illustrated in figure 1, where the squares of the polar and equatorial eccentricities

$$e^2 = 1 - \frac{\varepsilon_3}{\varepsilon_1}, \quad \eta^2 = 1 - \frac{\varepsilon_2}{\varepsilon_1} \quad (2.16)$$

of the equilibrium ellipsoid are plotted *versus* the angular momentum squared.

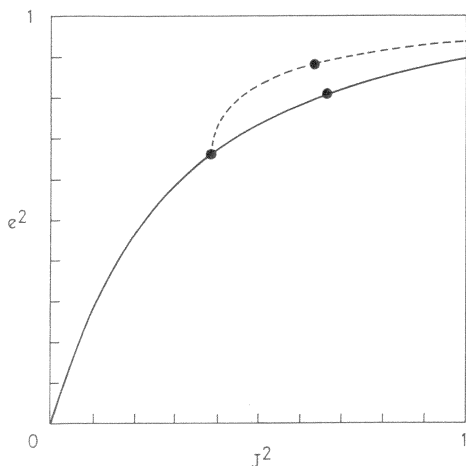


FIG. 1a

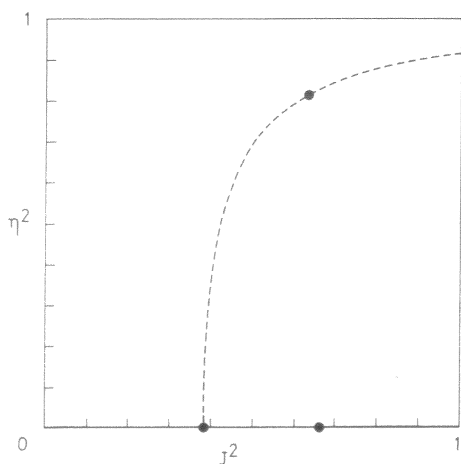


FIG. 1b

Fig. 1. — Ellipsoidal equilibrium solutions. Polar eccentricity squared (a) and equatorial eccentricity squared (b) *versus* angular momentum squared. Solid curves : Maclaurin sequence. Dashed curves : Jacobi sequence. The dots indicate the Maclaurin-Jacobi bifurcation (see table II) and the next bifurcation on each branch (see table III).

2.2 BIFURCATION EQUATION. — We want to find whether new solutions of the equilibrium equation, different from ellipsoids, exist, and to determine the values of J^2 at which they bifurcate from the Maclaurin and Jacobi sequences.

Given an equilibrium solution $S(\mathbf{x}; J^2)$, new solutions may be obtained from it by applying to the fluid a static deformation. Such a deformation is conveniently described in terms of a vector field $\xi(\mathbf{x})$, giving the displacement of the fluid element at point \mathbf{x} :

$$\mathbf{x} \rightarrow \mathbf{x} + \lambda \xi(\mathbf{x}), \quad S(\mathbf{x}; J^2) \geq 0, \quad (2.17)$$

where λ is a real parameter. In particular, the fluid surface (2.1) will be deformed into

$$\tilde{S}(\mathbf{x}; J^2) = 0, \quad (2.18)$$

where

$$\tilde{S}(\mathbf{x}; J^2) = S(\mathbf{x} + \lambda \xi(\mathbf{x}); J^2). \quad (2.19)$$

Only first-order deformations will be considered here then

$$\tilde{S}(\mathbf{x}; J^2) = (1 + \lambda \xi \nabla) S(\mathbf{x}; J^2). \quad (2.20)$$

We require that the deformations leave invariant the density, the centre of mass and the angular momentum of the fluid. To first order in λ , the displacement must then satisfy the conditions

$$\nabla \cdot \xi = 0, \quad (2.21)$$

$$\int_{S(\mathbf{x}; J^2) \geq 0} \xi \, d\mathbf{x} = 0, \quad (2.22)$$

$$\int_{S(\mathbf{x}; J^2) \geq 0} [2 \mathbf{n}(\mathbf{x}\xi) - \mathbf{x}(\mathbf{n} \cdot \xi) - \xi(\mathbf{n} \cdot \mathbf{x})] \, d\mathbf{x} = 0, \quad (2.23)$$

\mathbf{n} being the unit vector along the rotation axis. A displacement field satisfying equations (2.21)-(2.23) will be called *admissible*. Since the defining equations are linear, the set of admissible ξ 's is a vector space \mathcal{U}_a .

Under the displacement (2.17) the potential changes according to

$$\varphi(\mathbf{x}; J^2) \rightarrow \varphi(\mathbf{x}; J^2) + \lambda(\xi \cdot \nabla \varphi - \nabla \cdot \mathbf{F}[\xi])(\mathbf{x}; J^2), \quad (2.24)$$

where

$$\mathbf{F}[\xi](\mathbf{x}; J^2) = \frac{1}{4\pi} \int_{S(\mathbf{y}; J^2) \geq 0} \frac{\xi(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, d\mathbf{y}. \quad (2.25)$$

The term $\nabla \mathbf{F}$ represents the change in the gravitational potential due to the deformation of the surface. There is no corresponding term for the centrifugal potential since, to first order in λ , the moment of inertia remains constant as a consequence of eq. (2.23).

In order that the deformed fluid configuration be in equilibrium, eq. (2.9) must hold for the modified potential of eq. (2.24) and the deformed surface (2.20).

This yields the condition

$$B[\xi](\mathbf{x}; J^2) |_{S(\mathbf{x}; J^2)=0} = \text{constant}, \quad (2.26)$$

where

$$B[\xi](\mathbf{x}; J^2) = (\xi \cdot \nabla \varphi - \nabla \cdot \mathbf{F}[\xi])(\mathbf{x}; J^2). \quad (2.27)$$

Equation (2.26) represents a *linear* equation for the first-order displacement field ξ which deforms the initial equilibrium configuration of the fluid, $S(\mathbf{x}; J^2) = 0$, into a new equilibrium configuration, $\tilde{S}(\mathbf{x}; J^2) = 0$. Equation (2.26) will be referred to as the *bifurcation equation*.

An admissible displacement field ξ will be called *trivial* if, on the surface $S(\mathbf{x}; J^2) = 0$, it lies in the tangent plane to this surface, i.e. if

$$\xi \cdot \nabla S(\mathbf{x}; J^2) |_{S(\mathbf{x}; J^2)=0} = 0. \quad (2.28)$$

The set of trivial ξ 's forms a subspace \mathcal{U}_t of \mathcal{U}_a . We can now prove the following.

Theorem : If $\xi \in \mathcal{U}_t$, then $\nabla \cdot \mathbf{F}[\xi] = 0$.

Indeed, integrating by parts and using eqs. (2.21) and (2.28), we have

$$\begin{aligned} 4 \pi \nabla \cdot \mathbf{F}[\xi](\mathbf{x}; J^2) &= \nabla \cdot \int_{S(\mathbf{y}; J^2) \geq 0} \frac{\xi(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} = - \int_{S(\mathbf{y}; J^2) \geq 0} \xi(\mathbf{y}) \cdot \nabla \frac{1}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} \\ &= - \int_{S(\mathbf{y}; J^2)=0} \frac{\xi(\mathbf{y}) \cdot d\sigma(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} + \int_{S(\mathbf{y}; J^2) \geq 0} \frac{\nabla \cdot \xi(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d\mathbf{y} = 0. \end{aligned} \quad (2.29)$$

From this theorem it follows immediately that a trivial displacement satisfies identically the bifurcation equation. Such trivial solutions do not deform the original surface and will be discarded. More precisely, we will look for solutions of eq. (2.26) in the quotient space $\mathcal{U}_a/\mathcal{U}_t$. Such solutions will be called *bifurcation solutions*.

2.3 POLYNOMIAL DEFORMATIONS OF ELLIPSOIDS. —

We will now examine in detail the case in which the fluid surface is an ellipsoid (Maclaurin or Jacobi) given by eq. (2.10), and the displacement field ξ is a *polynomial* in \mathbf{x} . We denote by

$$P(\mathbf{x}; J^2) = \xi \cdot \nabla S(\mathbf{x}; J^2), \quad S(\mathbf{x}; J^2) \geq 0, \quad (2.30)$$

a polynomial scalar field which, on the ellipsoid $S(\mathbf{x}; J^2) = 0$, is proportional to the normal component of ξ . Equation (2.19) then reads

$$\tilde{S}(\mathbf{x}; J^2) = (S + \lambda P)(\mathbf{x}; J^2). \quad (2.31)$$

so the polynomial P is the surface deformation. We remark that for the ellipsoidal surface (2.10), equation (2.30) implies $P(0; J^2) = 0$.

We now want to find a vector deformation ξ corresponding to a given surface deformation P . Since every polynomial P has a unique decomposition as a sum of homogeneous polynomials P_n of degree n we choose

$$\xi_i = - \sum_{n>0} \frac{\varepsilon_i}{2n} \frac{\partial P_n}{\partial x_i} = \Xi_i[P]. \quad (2.32)$$

Like the ξ 's, the surface deformations P have to satisfy a set of admissibility conditions :

$$\sum_{n>0} \sum_{i=1}^3 \frac{\varepsilon_i}{2n} \frac{\partial^2 P_n}{\partial x_i^2} = 0 \quad (2.33)$$

$$\int \nabla P d\mathbf{x} = 0 \quad (2.34)$$

$$\int \left\{ x_3 \frac{\partial P}{\partial x_i} + x_i \frac{\partial P}{\partial x_3} \right\} d\mathbf{x} = 0 \quad i = 1, 2 \quad (2.35)$$

$$\int \sum_{i=1}^2 x_i \varepsilon_i \frac{\partial P}{\partial x_i} d\mathbf{x} = 0. \quad (2.36)$$

The integrals are over the domain $S(\mathbf{x}_i; J^2) \geq 0$. One can verify that eqs. (2.33)-(2.36) are equivalent to eq. (2.21)-(2.23). In particular the two eqs. (2.35), (2.36) express the conservation of angular momentum.

Equation (2.30) associates to every trivial vector information ξ_t a trivial surface deformation P_t which vanishes on the surface $S(\mathbf{x}; J^2) = 0$. Hence

$$P_t = QS. \quad (2.37)$$

One can now prove the following.

Theorem : If P is a polynomial of given degree and parity, then $\nabla \mathbf{F}[\Xi P]$ is also a polynomial of the same degree and parity.

The proof is elementary, but rather involved ; we sketch below the main steps. Since $\mathbf{F}[\xi]$ is a linear functional, it is sufficient to prove the theorem for the case in which the components of ξ are monomials of degree $n - 1$, viz. of the form

$$\xi_r(\mathbf{x}) = \prod_{i=1}^3 x_i^{n_i}, \quad \sum_{i=1}^3 n_i = n - 1. \quad (2.38)$$

Then, the theorem holds for the first few values of $n \geq 1$ [7], which suggests a proof by induction. So, assuming it to be true for a degree $n - 1$, we will show that the same follows for degree n . We use the identity

$$\prod_{i=1}^3 \left(\varepsilon_i \frac{\partial}{\partial k_i} \right)^{n_i} S^n = C_n \prod_{i=1}^3 x_i^{n_i} + \sum_{j=1}^{E[n/2]} R_{n-2j} S^j, \quad (2.39)$$

where C_n is a coefficient, $R_{n-2j}(\mathbf{x})$ is a polynomial of degree lower than or equal to $n - 2j$, and $E[n/2]$ denotes the largest integer smaller than or equal to $n/2$. This enables us to write

$$C_n F_r \left[\prod_{i=1}^3 x_i^{n_i} \right] = F_r \left[\prod_{i=1}^3 \left(\varepsilon_i \frac{\partial}{\partial x_i} \right)^{n_i} S^n \right] - \sum_{j=1}^{E[n/2]} F_r [R_{n-2j} S^j]. \tag{2.40}$$

Now we use in the right-hand side the equation [8]

$$F_r \left[\frac{\partial}{\partial x_k} (S^n f) \right] = \frac{\partial}{\partial x_k} F_r [S^n f] \tag{2.41}$$

[here $f(\mathbf{x})$ is an arbitrary function], and the fact [9] that $F_r[S^n]$ is a polynomial of degree $2n + 2$. The theorem then follows from eq. (2.40) by simple power counting.

Let us denote by $\mathcal{F}^{(n\zeta)}$ the vector space of polynomials of degree n and parity ζ . In section 3 we will show that the polynomial deformation P belongs to such a space $\mathcal{F}^{(n\zeta)}$. The importance of the above theorem resides in having established that the correspondence

$$P \xrightarrow{\Gamma} \mathbf{VF}[\Xi P] \tag{2.42}$$

is a homomorphism of $\mathcal{F}^{(n\zeta)}$ into $\mathcal{F}^{(n\zeta)}$. Consequently, the bifurcation equation (2.26) may be regarded as an equation for the polynomial deformation P . This is a convenient point of view for the discussion of symmetry breaking, so let us write explicitly the bifurcation equation in this form. For the ellipsoidal equilibrium solutions, eqs. (2.3) to (2.15) imply that the potential at equilibrium is

$$4 \varphi(\mathbf{x}; J^2) = \alpha + \varepsilon_3 \alpha_3 [S(\mathbf{x}; J^2) - 1], \quad S(\mathbf{x}; J^2) \geq 0. \tag{2.43}$$

Equation (2.27) then becomes

$$B[P](J^2) = (\varepsilon_3 \alpha_3 P - \Gamma[P])(J^2), \tag{2.44}$$

where $\Gamma[P]$ is defined by eq. (2.42). [For polynomials of degree $n \leq 4$ this Γ -transform can be calculated explicitly from Chandrasekhar's identities [7].] On the other hand, for polynomial deformations eq. (2.26) is satisfied if and only if

$$B[P] = RS + \text{constant}, \tag{2.45}$$

$R(\mathbf{x})$ being an arbitrary polynomial. Solving the bifurcation equation in the form (2.45) amounts now to the identification of polynomial coefficients.

The term *bifurcation solution* will be used for a solution $P(x; J^2)$ of the eigenvalue equation (2.45) in the quotient space $\mathcal{F}_a/\mathcal{F}_t$, where \mathcal{F}_a and \mathcal{F}_t are, in the vector space of polynomials P , the subspaces of the admissible and trivial polynomials respectively.

3. Selection rules for symmetry breaking. — We now proceed to discuss the symmetry properties of the equilibrium and bifurcation equations and their non-trivial solutions, in order to determine the patterns of symmetry breaking that may appear at bifurcations. This will be done by using the powerful formalism of group theory, which only recently has been applied to bifurcation problems [23, 24]. We will briefly recall some basic notions [10], then apply them to our specific problem.

3.1 GROUP-THEORETICAL DESCRIPTION OF SYMMETRY BREAKING. — We consider an equation having the general form

$$\psi(u, \mu) = 0, \tag{3.1}$$

where $u(\mathbf{x})$ is the unknown function, and μ is a real parameter; ψ is a smooth map, otherwise arbitrary. Equation (3.1) is assumed to be covariant under a group G , in the following sense. Let g be an element of G , and $T(g)$ the 3×3 matrix which represents its action on the three-dimensional Euclidean space :

$$\mathbf{x} \xrightarrow{g} g \cdot \mathbf{x} = T(g) \mathbf{x}. \tag{3.2}$$

Then, its action on any function $u(\mathbf{x})$ is represented by a linear operator O_g defined by

$$u(\mathbf{x}) \xrightarrow{g} [O_g u](\mathbf{x}) = u[T^{-1}(g) \mathbf{x}]. \tag{3.3}$$

Equation (3.1) is said to be *covariant* under G if the action of this group commutes with the map ψ , i.e. if

$$O_g \psi(u, \mu) = \psi(O_g u, \mu). \tag{3.4}$$

An immediate consequence of covariance is that, if u is a solution of eq. (3.1) for some value of the parameter μ , then $O_g u$ is also a solution for the same μ . The set of all solutions obtained from u by the action of G is called the *orbit* of u [11].

The solutions of eq. (3.1) are, in general, not invariant under G ; this is the phenomenon of spontaneous symmetry breaking. Given a solution u , the elements of G which transform u into itself form a subgroup H of G called the *isotropy group*, or *little group*, of u . If u has the isotropy group H , then $u' = O_g u$ has the isotropy group $H' = gHg^{-1}$, i.e. H and H' are conjugated subgroups of G . Therefore, the isotropy group H of a given solution u characterizes, up to a conjugation in G , the whole orbit of u , which will be denoted by the symbol G/H . The foregoing discussion referred to a fixed μ , but now it can be extended to all values of μ . The set of all orbits of the same type (i.e. having the same little group, up to a conjugation) is called a *stratum*. A symmetry-changing bifurcation point at a certain critical value of the parameter μ corresponds to a critical orbit marking the boundary between two adjacent strata.

Let us now apply these considerations to the equi-

librium equation (2.9). This equation is covariant under the group $D_{\infty h}$, generated by the rotations about the direction of the angular momentum, the reflexion through a plane containing this direction, and the reflexion through a plane perpendicular to this direction at the centre of mass. An equilibrium solution $S(\mathbf{x}; J^2)$, invariant under a subgroup G of $D_{\infty h}$ generates, under the action of $D_{\infty h}$, an orbit $D_{\infty h}/G$; the whole stratum is obtained by varying the parameter J^2 . The classical ellipsoidal solutions generate two types of orbits, hence two strata. One stratum is the Maclaurin sequence, the corresponding isotropy group being $D_{\infty h}$ itself. The other stratum is the Jacobi sequence; in this case the isotropy group is the subgroup D_{2h} of $D_{\infty h}$, generated by the reflexions through the three symmetry planes of the triaxial ellipsoid. Calculating explicitly the other orbits of equilibrium solutions is a formidable task which we do not attempt here; instead, we will give a simple and systematic method for finding, in the two strata of ellipsoidal solutions, the critical orbits corresponding to bifurcations.

The bifurcation equation was obtained by linearization of the equilibrium equation near a given equilibrium solution $S(\mathbf{x}; J^2)$ having an isotropy group G or, more precisely, by linearization near an orbit $D_{\infty h}/G$ [12]. Hence, the bifurcation equation, the admissibility conditions for the polynomial deformations and the definition of trivial deformations are automatically G -covariant. To any bifurcation solution $P(\mathbf{x}; J^2)$, invariant under a subgroup H of G , will then be associated, by the action of G , an orbit G/H . The problem is to find all the possible isotropy groups H of the bifurcation solutions, i.e. the subgroups of G onto which the symmetry may be broken at a bifurcation.

A decisive step is made by remarking that the bifurcation equation, the admissibility conditions, and the definition of trivial deformations, in addition to being G -covariant, are *linear*. Therefore, any G/H orbit of bifurcation solutions generates a *vector space carrying a linear representation of the group G* . We shall make the specific assumption that, for any given value of J^2 , this representation is *irreducible*. (We call *accidental degeneracy* the appearance of a *reducible* representation. The meaning and validity of the assumption that there is no accidental degeneracy will be discussed in section 5. In section 4 we show, by explicit calculation of the lowest-degree bifurcations, that accidental degeneracy is indeed not present in this problem.) This assumption imposes restrictive conditions on the subgroups onto which the symmetry may be broken, *viz.* any H must be the isotropy group of a non-trivial vector of an irreducible representation appearing in the action of G on polynomials. The next step is therefore to study the irreducible representations of the groups $D_{\infty h}$ and D_{2h} , their isotropy groups, and their covariant polynomials.

3.2 IRREDUCIBLE REPRESENTATIONS OF THE GROUPS D_{2h} AND $D_{\infty h}$. — Let us consider first D_{2h} , the symmetry group of a triaxial ellipsoid. The principal axes of the ellipsoid will be labelled by an index $i = 1, 2, 3$. The group D_{2h} consists of the eight elements

$$E, r_1, r_2, r_3, \sigma_1, \sigma_2, \sigma_3, I; \quad (3.5)$$

here E is the identity, r_i denotes the rotation by an angle π about axis i , σ_i denotes the reflexion through the principal plane perpendicular to axis i , and I is the inversion through the centre. These elements form a group under the commutative composition laws

$$\begin{aligned} r_i^2 = \sigma_i^2 = I^2 = E, \quad r_i r_j = \sigma_i \sigma_j = r_k, \\ r_i \sigma_j = r_j \sigma_i = \sigma_k, \end{aligned} \quad (3.6)$$

where (ijk) represents a permutation of (123) .

D_{2h} is an Abelian group, hence its irreducible representations are all real and one-dimensional. The whole group being generated by the three reflexions σ_i ($i = 1, 2, 3$), the irreducible representations may be conveniently labelled by the eigenvalues $\zeta_i = \pm 1$ of these operators. To each irreducible representation corresponds only one isotropy group H of non-zero vectors, which coincides with the kernel of the representation (i.e. the set of elements of the group represented by the identity). The most general polynomial which transforms according to a given irreducible representation of D_{2h} is of the form [26]

$$P(\mathbf{x}) = \theta(x_1^2, x_2^2, x_3^2) \varphi(\mathbf{x}). \quad (3.7)$$

Here $\theta(x_1^2, x_2^2, x_3^2)$ is an arbitrary polynomial invariant under D_{2h} , and $\varphi(\mathbf{x})$ is a given polynomial, invariant under the isotropy group H corresponding to the irreducible representation.

The symmetry group of an axially-symmetric ellipsoid, denoted by $D_{\infty h}$, is an infinite group, generated by the rotations $r_3(\omega)$ by an angle ω about the third axis (the symmetry axis), and the reflexions σ_3 and σ_1 . It is customary to introduce the notation

$$\sigma_{v(\omega)} = r_3(\omega) \sigma_1, \quad \sigma_h = \sigma_3, \quad r_{h(\omega)} = \sigma_h \sigma_{v(\omega)}. \quad (3.8)$$

Here $\sigma_{v(\omega)}$ denotes the reflexion through the « vertical » symmetry plane of azimuth $\omega/2$, σ_h is the reflexion through the « horizontal » symmetry plane, and $r_{h(\omega)}$ denotes the rotation of angle π about the « horizontal » symmetry axis of azimuth $\omega/2$. The four sets

$$\begin{aligned} C_\infty &= \{ r_3(\omega) \}, & C_{\infty v} &= \{ r_3(\omega), \sigma_{v(\omega)} \}, \\ C_{\infty h} &= \{ r_3(\omega), \sigma_h \}, & D_\infty &= \{ r_3(\omega), r_{h(\omega)} \}, \end{aligned} \quad (3.9)$$

are all groups under the usual composition laws of rotations and reflexions [13]. Their union is $D_{\infty h}$, the symmetry group of an axially-symmetric ellipsoid.

Each of the groups $C_{\infty v}$, $C_{\infty h}$ and D_{∞} is a proper subgroup of $D_{\infty h}$; C_{∞} is a proper subgroup of each of them. All the other proper subgroups of $D_{\infty h}$ are finite.

The irreducible representations of $D_{\infty h}$ are all real, either two-dimensional or one-dimensional. In the two-dimensional ones the generating elements are represented as follows :

$$r_3(\omega) \rightarrow \begin{pmatrix} \cos m\omega & -\sin m\omega \\ \sin m\omega & \cos m\omega \end{pmatrix}, \tag{3.10}$$

$$\sigma_3 \rightarrow \zeta_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here m is a positive integer ($m = 1, 2, \dots$), and $\zeta_3 = \pm 1$. These representations are conveniently labelled by the doublet (m, ζ_3) . The two representations that would obtain from eq. (3.10) for $m = 0$ are reducible; they decompose into one-dimensional irreducible representations in which the generating elements are now represented as follows :

$$r_3(\omega) \rightarrow 1, \quad \sigma_3 \rightarrow \zeta_3, \quad \sigma_1 \rightarrow \zeta_1. \tag{3.11}$$

Here $\zeta_1 = \pm 1$ distinguishes between the two irreducible representations corresponding to $m = 0$ and the same ζ_3 . The one-dimensional irreducible representations will be labelled by the triplet $(m = 0, \zeta_3, \zeta_1)$.

To each one-dimensional irreducible representation of $D_{\infty h}$ corresponds only one isotropy group H , which coincides with the kernel of the representation; this is not true for the two-dimensional representations, for which H is determined only up to a conjugation, and is larger than the kernel. The most general polynomial which transforms according to a given irreducible representation of $D_{\infty h}$ is of the form

$$P(\mathbf{x}) = \sum_{i=1}^d \theta_i(x_1^2 + x_2^2, x_3^2) \varphi_i(\mathbf{x}). \tag{3.12}$$

Here $d = 1, 2$ is the dimension of the representation, $\theta_i(x_1^2 + x_2^2, x_3^2)$ are arbitrary polynomials invariant under $D_{\infty h}$, and $\varphi_i(\mathbf{x})$ are given polynomials, invariant under the corresponding isotropy group H .

In table I we list all the irreducible representations of the groups D_{2h} and $D_{\infty h}$. For both groups the head entries indicate the irreducible representations (labelled as described above), the corresponding isotropy groups H (denoted by their classical names [4]), and the explicit form of the invariant polynomials φ [26].

3.3 SELECTION RULES. — These results may be summarized in a set of *selection rules* for symmetry breaking at bifurcations, indicating which patterns of symmetry breaking are *forbidden* under the assumption that accidental degeneracy does not occur. The contents of these selection rules, which are presented schematically in table I, is discussed below.

Table I. — Irreducible representations of the D_{2h} and $D_{\infty h}$ groups.

$\zeta_1 \zeta_2 \zeta_3$	D_{2h} H	φ	$m \zeta_3 \zeta_1$	$D_{\infty h}$ H	φ_i
+++	D_{2h}	1	0++	$D_{\infty h}$	1
+++	$C_{2v}(1)$	x_1	0-+	$C_{\infty v}$	x_3
+ - +	$C_{2v}(2)$	x_2	0+-	$C_{\infty h}$	—
+ + -	$C_{2v}(3)$	x_3	0--	D_{∞}	—
+ --	$C_{2h}(1)$	$x_2 x_3$	m+	D_{mh}	$c_m(\mathbf{x})$
- + -	$C_{2h}(2)$	$x_3 x_1$			$s_m(\mathbf{x})$
- - +	$C_{2h}(3)$	$x_1 x_2$	m-	D_{md}	$x_3 c_m(\mathbf{x})$
- - -	D_2	$x_1 x_2 x_3$			$x_3 s_m(\mathbf{x})$

Notation : $c_m(\mathbf{x}) = \text{Re}(x_1 + ix_2)^m$,
 $s_m(\mathbf{x}) = \text{Im}(x_1 + ix_2)^m$.

1) The only subgroups of D_{2h} and $D_{\infty h}$ that are allowed as isotropy groups of bifurcation solutions are those listed in table I; any other subgroup is forbidden. This eliminates exactly half of the subgroups of D_{2h} , viz. $C_s(k)$, $C_2(k)$, S_2 and $C_1 = \{E\}$, ($k = 1, 2, 3$). In the case of $D_{\infty h}$ the forbidden subgroups are C_{∞} , C_{mv} , C_{mh} , S_{2m} and D_m ($m = 1, 2, \dots$). It is interesting to note that the forbidden isotropy groups are always *smaller* than the allowed ones. In other words, the absence of accidental degeneracy leads in a natural way to *minimal symmetry breaking*.

2) When only *polynomial deformations* are considered, the subgroups $C_{\infty h}$ and D_{∞} of $D_{\infty h}$ are also forbidden, because they do not appear in the action of the group on polynomials.

Symmetry considerations do not impose any restriction on the *degree* of the polynomial deformation $P(\mathbf{x})$; indeed, in eqs. (3.7) and (3.12) the θ 's are arbitrary polynomials. However, once the degree n has been fixed, new selection rules come into force :

3) All the isotropy groups for which the φ 's are of degree greater than n are forbidden. (Example : D_{mh} and $D_{m-1,d}$ are forbidden if $m > n$.)

4) All the irreducible representations of D_{2h} and $D_{\infty h}$ have definite parity $\zeta = \pm 1$ (the parity of the φ 's in table I). This forbids the isotropy groups with the wrong parity, i.e. with $\zeta \neq (-1)^n$, eliminating half of the subgroups in table I, both for D_{2h} and $D_{\infty h}$.

The selection rules represent a powerful instrument, which drastically reduces the list of possible types of symmetry breaking at bifurcations. Whether the *allowed* types of breaking actually occur is a dynamical question which can be answered only by explicitly solving the bifurcation equation. If there is no solution, the bifurcation, although allowed *kinematically* (i.e. from the point of view of symmetry breaking), is *dynamically* forbidden (i.e. is incompatible with the equation of hydrostatic equilibrium).

4. Lowest-degree polynomial bifurcations. — From the foregoing discussion the following practical procedure for the calculation of bifurcations has emerged. Given the isotropy group G of the original equilibrium

solution [12], table I lists all the possible isotropy groups H of the bifurcation solutions. Once the degree n of the polynomial deformation has been chosen, the most general form of an H-invariant admissible P is found from eqs. (3.7) or (3.12), table I, and eqs. (2.33) to (2.36). The Γ-transform of P can be calculated explicitly [7]; then, identification of the coefficients in the bifurcation equation (2.45) yields

a system of linear homogeneous algebraic equations for the set of coefficients { λ } which parametrize P. The bifurcation solution is found by solving this system.

This procedure is now illustrated by explicit computation of the bifurcations corresponding to polynomial deformations of the lowest degree compatible with each type of allowed symmetry breaking. For

Table II. — Second-degree bifurcations from the Maclaurin and Jacobi sequences.

G	H	P	Bifurcation equation		Bifurcation solution		
					J ²	e ²	η ²
D _{2h}	D _{2h}	$\lambda \left(\frac{x_1^2}{\varepsilon_1} - \frac{x_2^2}{\varepsilon_2} \right)$	(4.4)		No solution		
	C _{2h(k)}	$\lambda x_i x_j$ (ijk) permutation of (123)	(4.12)	k = 1, 2			
				k = 3	Equation satisfied identically (*)		
D _{∞h}	D _{∞h}	No non-trivial admissible P	—		—		
	D _{2h}	$\lambda(x_1^2 - x_2^2)$	(4.15)		0.384 436	0.660 433	0
	D _{1d}	$\lambda x_2 x_3$	(4.12) i = 2, j = 3; ε ₁ = ε ₂		No solution		

(*) This spurious solution does not correspond to a bifurcation; see the text.

Table III. — Third-degree bifurcations from the Maclaurin and Jacobi sequences.

G	H	P	Bifurcation equation		Bifurcation solution		
					J ²	e ²	η ²
D _{2h}	C _{2v(k)}	$x_k \left[\lambda_0^{(k)} - \sum_{i=1}^3 \frac{\lambda_i^{(k)}}{\varepsilon_i} x_i^2 \right]$ $5 \lambda_0^{(k)} = \sum_{i=1}^3 (1 + 2 \delta_{ki}) \lambda_i^{(k)}$	(4.23) and (4.29)	k = 1	0.632 243	0.880 928	0.813 175
	D ₂	$\lambda x_1 x_2 x_3$	(4.33)		No solution		
D _{∞h}	C _{∞v}	$x_3 \left[\lambda_0 - \frac{\lambda_1}{\varepsilon_1} (x_1^2 + x_2^2) - \frac{\lambda_3}{\varepsilon_3} x_3^2 \right]$ $5 \lambda_0 = 2 \lambda_1 + 3 \lambda_3$	(4.29) k = 3; ε ₁ = ε ₂				
	D _{1h}	$x_1 \left[\lambda_0 - \frac{\lambda_1}{\varepsilon_1} (x_1^2 + x_3^2) - \frac{\lambda_3}{\varepsilon_3} x_3^2 \right]$ $5 \lambda_0 = 4 \lambda_1 + \lambda_3$	(4.38)		1.346 350	0.939 685	0
	D _{3h}	$\lambda x_1 (x_1^2 - 3 x_2^2)$	(4.39)		0.662 777	0.808 664	0
	D _{2d}	$\lambda x_1 x_2 x_3$	(4.34) ε ₁ = ε ₂		No solution		

$n = 1$, no non-trivial admissible P exists. For $n = 2$ and $n = 3$ the situation is summarized in tables II and III, respectively. In both cases the head entries indicate the isotropy groups G of the original sequences ($D_{\infty h}$ or D_{2h}), the isotropy groups H of the bifurcation solutions (as given in table I), the admissible polynomial deformations P (parametrized in terms of a set of coefficients $\{\lambda\}$), the bifurcation equations (indicated by their number in the text), and the bifurcation solutions (the values of J^2 at which they occur, and the corresponding eccentricities squared e^2 and η^2).

The types of symmetry breaking listed in tables II and III are examined below; in each case we write explicitly the bifurcation equation and indicate its solution. The two infinite families of bifurcations from the Maclaurin sequence associated with the D_{mh} and D_{md} symmetry breakings [14] are discussed at the end of this section; only the first one is found to be dynamically allowed.

4.1 SECOND-DEGREE BIFURCATIONS FROM THE JACOBI SEQUENCE. — 1) When $H = D_{2h}$ there is no symmetry breaking. The most general admissible P is of the form

$$P = \lambda \left(\frac{x_1^2}{\varepsilon_1} - \frac{x_2^2}{\varepsilon_2} \right), \tag{4.1}$$

yielding

$$B[P] = \frac{\lambda}{2} \left(Q_0 - \sum_{i=1}^3 \frac{Q_i}{\varepsilon_i} x_i^2 \right), \tag{4.2}$$

where, in terms of the functions $\alpha_{i_1 \dots i_n}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ defined by eq. (A.4),

$$Q_0 = \tilde{\alpha}_1 - \tilde{\alpha}_2, \quad Q_1 = 3 \tilde{\alpha}_{11} - \tilde{\alpha}_{12} - 2 \tilde{\alpha}_3, \tag{4.3}$$

$$Q_2 = \tilde{\alpha}_{12} - 3 \tilde{\alpha}_{22} + 2 \tilde{\alpha}_3, \quad Q_3 = \tilde{\alpha}_{13} - \tilde{\alpha}_{23}.$$

Equation (2.45) then requires

$$Q_0 = Q_1 = Q_2 = Q_3. \tag{4.4}$$

A more convenient form of this system of equations is

$$Q_1 + Q_2 + Q_3 = 3 Q_0, \tag{4.5}$$

$$Q_1 + Q_2 = 2 Q_3, \tag{4.6}$$

$$Q_1 = Q_3. \tag{4.7}$$

Using the sum rules (A.7), we find that eq. (4.5) is an identity. Equation (4.6) may be cast in the form

$$\varepsilon_1 \beta_{13} = \varepsilon_2 \beta_{23}, \tag{4.8}$$

where the β 's are defined by eq. (A.5). Using the definition (A.3) and the integral representation (A.2), this can be transformed into

$$(\varepsilon_1 - \varepsilon_2) \varphi(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0, \tag{4.9}$$

where φ is a function that never vanishes. Therefore, eqs. (4.4) have no solution on the Jacobi sequence, where $\varepsilon_1 \neq \varepsilon_2$.

2) Next, we examine the breakings onto

$$H = C_{2h}(k), \quad (k = 1, 2, 3).$$

The corresponding second-degree polynomial deformations are

$$P = \lambda x_i x_j, \tag{4.10}$$

where (ijk) is a permutation of (123) . Then

$$B[P] = \lambda(\tilde{\alpha}_3 - \tilde{\alpha}_{ij}) x_i x_j, \tag{4.11}$$

and eq. (2.45) can be satisfied if and only if

$$\tilde{\alpha}_3 - \tilde{\alpha}_{ij} = 0. \tag{4.12}$$

For $k = 1, 2$ this equation has no solutions [15], this is seen immediately by writing it in the form

$$\beta_{i3} = 0, \quad (i = 1, 2), \tag{4.13}$$

and using the fact that $\beta_{i3} > 0$ (see Appendix). On the other hand, for $k = 3$ eq. (4.12) is satisfied identically all along the Jacobi sequence; compare with eq. (2.15). However, this solution is not related to a bifurcation. The deformation

$$S \rightarrow S + \lambda x_1 x_2 \tag{4.14}$$

represents in fact a *rigid rotation* of S about the direction of \mathbf{J} . The original and *deformed* surfaces lie on the same $D_{\infty h}/D_{2h}$ orbit and therefore describe the same physical state [11].

4.2 SECOND-DEGREE BIFURCATIONS FROM THE MACLAURIN SEQUENCE. — 3) There exists no non-zero second-degree admissible P corresponding to the case $H = D_{\infty h}$.

4) For $H = D_{2h}$, the bifurcation equation is obtained by taking $\varepsilon_1 = \varepsilon_2$ in eq. (4.7), whence

$$\tilde{\alpha}_3 - \tilde{\alpha}_{11} = 0. \tag{4.15}$$

This gives the Maclaurin-Jacobi bifurcation; compare with eq. (2.15).

5) When $H = D_{1d}$ [14] we have

$$P = \lambda x_2 x_3. \tag{4.16}$$

The bifurcation equation is obtained by taking $\varepsilon_1 = \varepsilon_2$ in eq. (4.12) for $i = 2, j = 3$. We have shown that there is no solution [15].

4.3 THIRD-DEGREE BIFURCATIONS FROM THE JACOBI SEQUENCE. — 6) The case $H = C_{2v}(k)$ ($k = 1, 2, 3$), is more involved and therefore more illustrative. The most general P is of the form

$$P = x_k \left(\lambda_0^{(k)} - \sum_{i=1}^3 \frac{\lambda_i^{(k)}}{\varepsilon_i} x_i^2 \right), \tag{4.17}$$

with the restriction

$$5 \lambda_0^{(k)} = \sum_{m=1}^3 (1 + 2 \delta_{km}) \lambda_m^{(k)} \quad (4.18)$$

imposed by the conservation of the centre of mass. We note that when

$$\lambda_1^{(k)} = \lambda_2^{(k)} = \lambda_3^{(k)} \quad (4.19)$$

equation (4.17) reduces to a *trivial* deformation. Calculating the Γ -transform [7] one obtains

$$B[P] = x_k \left(Q_0^{(k)} - \sum_{i=1}^3 \frac{Q_i^{(k)}}{\varepsilon_i} x_i^2 \right), \quad (4.20)$$

where $Q_i^{(k)}$, ($i = 0, 1, 2, 3$), are linear forms in the three independent parameters $\lambda_m^{(k)}$, ($m = 1, 2, 3$). Equation (2.45) requires that

$$Q_0^{(k)} = Q_1^{(k)} = Q_2^{(k)} = Q_3^{(k)}; \quad (4.21)$$

written in the form

$$Q_l^{(k)} - Q_0^{(k)} = 0, \quad (l = 1, 2, 3), \quad (4.22)$$

these relations form a system of three linear, homogeneous equations for the three parameters $\lambda_m^{(k)}$. Explicitly, they may be written as

$$\sum_{m=1}^3 C_{lm}^{(k)} \lambda_m^{(k)} = 0, \quad (l = 1, 2, 3), \quad (4.23)$$

where

$$C_{lm}^{(k)} = (1 + 2 \delta_{km}) \left(\frac{2}{3} \tilde{\alpha}_3 + \frac{3}{5} \tilde{\alpha}_k - \tilde{\alpha}_{kl} - \tilde{\alpha}_{km} + \tilde{\alpha}_{klm} \right) - 2 \delta_{lm} (\tilde{\alpha}_3 - \tilde{\alpha}_{klm}). \quad (4.24)$$

Equations (4.23) have a non-zero solution if and only if

$$\text{Det } C_{lm}^{(k)} = 0. \quad (4.25)$$

This condition turns out to be satisfied identically not only all along the Jacobi sequence, but also for any values of the ε 's. Indeed, using the sum rules (A.7) one finds that the sum of the three columns in the determinant in eq. (4.25) is zero :

$$\sum_{m=1}^3 C_{lm}^{(k)} = 0, \quad (l = 1, 2, 3). \quad (4.26)$$

The origin of this disease is traced down to the presence in our equations of the trivial deformation (4.19). To cure it, we may choose anyone of the λ 's equal to zero, which is equivalent to removing from eq. (4.17) a trivial deformation proportional to $x_k S$ [16]. For the remaining deformation to be non-trivial, all the second-order minors of the determinant in eq. (4.25) must then vanish. Omitting the proofs, which are elementary but tedious, let us state the main

results. All the second-order minors $D_{lm}^{(k)}$ are proportional :

$$D_{lm}^{(k)} = \gamma_{lm}^{(k)} \psi^{(k)}. \quad (4.27)$$

The γ 's are functions of the ε 's that never vanish. Equation (4.25) then factorizes into

$$\chi^{(k)} \psi^{(k)} = 0 \quad (4.28)$$

where, by the sum rules (A.7), $\chi^{(k)}$ is identically zero. Removing this spurious zero, the bifurcation is given by the equation

$$\psi^{(k)}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0. \quad (4.29)$$

In terms of the eigenvalues of the matrix $C_{lm}^{(k)}$, all this boils down to the fact that one eigenvalue is identically zero, whereas the zeros of the other two eigenvalues are given by eq. (4.29). Once a solution of this equation is known, the coefficients $\lambda_i^{(k)}$ (up to an arbitrary common additive constant) are found as the components of the corresponding eigenvector of $C_{lm}^{(k)}$. A numerical calculation shows that, on the Jacobi sequence, eq. (4.29) admits a solution only [15] for $k = 1$; the corresponding eigenvector is

$$\lambda_1^{(1)} = 0, \quad \lambda_2^{(1)} = 0.78375, \quad \lambda_3^{(1)} = 0.69553. \quad (4.30)$$

This is the bifurcation to the famous *pear-shaped* figure of Poincaré, invariant under $C_{2v}(1)$.

7) When $H = D_2$, we have

$$P = \lambda x_1 x_2 x_3, \quad (4.31)$$

whence

$$B[P] = \lambda (\tilde{\alpha}_3 - \tilde{\alpha}_{123}) x_1 x_2 x_3. \quad (4.32)$$

Equation (2.45) is satisfied only if

$$\tilde{\alpha}_3 - \tilde{\alpha}_{123} = 0, \quad (4.33)$$

or equivalently

$$\beta_{13} + \varepsilon_1 \beta_{123} = 0. \quad (4.34)$$

Using the positivity of the β 's, one concludes that the D_2 bifurcation does not occur [15].

4.4 THIRD-DEGREE BIFURCATIONS FROM THE MACLAURIN SEQUENCE. — 8) When $H = C_{\infty v}$, the third-degree admissible P is obtained by setting $\lambda_1 = \lambda_2$, $\varepsilon_1 = \varepsilon_2$ in eq. (4.17) for $k = 3$. It is sufficient, therefore, to look for solutions of eq. (4.29) in this particular case; there are none [15].

9) For $H = D_{1h}$ [14], the polynomial deformation is again a special case of eq. (4.17) : $\lambda_1 = \lambda_2$, $\varepsilon_1 = \varepsilon_2$, and $k = 1$. Then, eq. (4.29) factorizes into

$$\varphi_1(\varepsilon_1, \varepsilon_3) \varphi_3(\varepsilon_1, \varepsilon_3) = 0, \quad (4.35)$$

where

$$\varphi_1 = \frac{2}{3} \tilde{\alpha}_3 + \frac{3}{5} \tilde{\alpha}_1 - \tilde{\alpha}_{13} + \tilde{\alpha}_{11} - \tilde{\alpha}_{113}, \quad (4.36)$$

$$\varphi_3 = \tilde{\alpha}_3 - \tilde{\alpha}_{111}. \quad (4.37)$$

Each factor in eq. (4.35) yields one solution. The associated types of symmetry breaking are found by computing the corresponding eigenvectors. The solution

$$\varphi_1 = 0, \quad \lambda_1 = \lambda_2, \quad \lambda_3 = 0, \quad (4.38)$$

represents the breaking onto D_{1h} .

10) The bifurcation to $H = D_{3h}$ is obtained from the other solution of eq. (4.35) :

$$\varphi_3 = 0, \quad 3\lambda_1 + \lambda_2 = 0, \quad \lambda_3 = 0. \quad (4.39)$$

11) For $H = D_{2d}$, the bifurcation equation is (4.34), with $\varepsilon_1 = \varepsilon_2$. There exists no bifurcation of this type [15].

4.5 HIGHER-DEGREE BIFURCATIONS FROM THE MACLAURIN SEQUENCE. — Finally, let us examine the bifurcations from the Maclaurin sequence accompanied by symmetry breaking onto the subgroups D_{mh} or D_{md} ($m \geq 2$) [14]. As before, the discussion is restricted to the lowest-degree polynomial deformation compatible with the breaking.

12) For $H = D_{mh}$, taking

$$P = \lambda c_m, \quad (4.40)$$

where $c_m(\mathbf{x}) = \text{Re}(x_1 + ix_2)^m$, one obtains

$$B[P] = \lambda'_m(\varepsilon_1) [\tilde{\alpha}_3 - \tilde{\alpha}_1^{(m)}] c_m. \quad (4.41)$$

Here $\tilde{\alpha}_1^{(m)}$ stands for $\tilde{\alpha}_{1\dots 1}$ with m indices 1 and λ'_m is a function that never vanishes. To satisfy eq. (4.41) one must have

$$\tilde{\alpha}_3 - \tilde{\alpha}_1^{(m)} = 0. \quad (4.42)$$

The bifurcation equation (4.42) has one solution for each $m \geq 2$. The solution for the first few values of m are given in table IV. It seems that only the $m = 3$ and

Table IV. — *Lowest-degree bifurcations from the Maclaurin sequence corresponding to symmetry breaking onto D_{mh} groups.*

m	J^2	e^2
2	0.384 436	0.660 433
3	0.662 777	0.808 664
4	0.875 341	0.870 017
5	1.046 271	0.902 742
6	1.188 890	0.922 829
7	1.311 121	0.936 312
8	1.418 011	0.945 942
9	1.512 959	0.953 139
10	1.598 355	0.958 708

$m = 4$ bifurcations were known : see [19] chap. 6 ; see however [30] for a related work.

13) When $H = D_{md}$, the lowest-order polynomial deformation is

$$P = \lambda x_3 c_m, \quad (4.43)$$

yielding

$$B[P] = \lambda''_m(\varepsilon_1) [\tilde{\alpha}_3 - \tilde{\alpha}_{13}^{(m+1)}] x_3 c_m, \quad (4.44)$$

where $\tilde{\alpha}_{13}^{(m+1)}$ stands for $\tilde{\alpha}_{1\dots 13}$ with m indices 1 and one index 3. The bifurcation equation is

$$\tilde{\alpha}_3 - \tilde{\alpha}_{13}^{(m+1)} = 0. \quad (4.45)$$

In terms of the positive β 's defined by eq. (A.5), this may be written in the form

$$\sum_{k=1}^m \varepsilon_1^k \beta_{13}^{(k)} = 0, \quad (4.46)$$

showing that there are no D_{md} bifurcations [15].

5. Concluding remarks. — In the scheme presented here, the calculation of bifurcations is based on two general assumptions : the absence of accidental degeneracy, and the arbitrary choice of the degree of the polynomial deformation. We would like to add a few comments on the meaning and justification of these assumptions.

It has been shown [27] that the Maclaurin-Jacobi bifurcation is similar to second-order phase transitions, as described by the Landau theory [28]. We now extend this point of view to an arbitrary bifurcation. Let us consider the fluid in a configuration bounded by the surface $S(\mathbf{x}) = 0$, where $S(\mathbf{x})$ is a polynomial in \mathbf{x} . Its energy is a functional of S , depending also on the angular momentum squared ; we denote it by $E[S](J^2)$. For each value of J^2 , the equilibrium solutions are given by the minima of E , the lowest minimum corresponding to stable equilibrium. The covariance group of the equilibrium equation, $D_{\infty h}$, acts on the space \mathcal{F} of real polynomials by an orthogonal representation. Thus, \mathcal{F} is a real Hilbert space, and the energy becomes a real-valued function on \mathcal{F} ; we denote it by $E(J^2, \xi)$, where ξ stands for the vector coordinate of S in \mathcal{F} .

Let now $S(\mathbf{x}, J^2)$ be an equilibrium solution, and the subgroup G of $D_{\infty h}$ its isotropy group. We denote by $T(G)$ the representation of G on \mathcal{F} , which can always be decomposed into irreducible representations. Correspondingly, \mathcal{F} decomposes into a direct sum of subspaces $\mathcal{F}^{(\alpha n)}$; here α labels the factorial representations (direct sums of equivalent irreducible representations) appearing in the decomposition of $T(G)$, and n labels the irreducible representations inside each factorial representation. An expansion of the energy in the coordinates of \mathcal{F} , around the minimum corresponding to the equilibrium solution considered, is of the form

$$E(J^2, \xi) = E_0(J^2) + \sum_{\alpha n} E_2^{(\alpha n)}(J^2) \xi_{\alpha n}^2 + \dots \quad (5.1)$$

Here $\xi_{\alpha n}$ is the vector component of ξ in the subspace $\mathcal{F}^{(\alpha n)}$; for simplicity, the minimum of the energy has been chosen as origin. There are no linear terms in the expansion (5.1), and the quadratic terms may always be brought to normal form (a sum of squares) by an orthogonal transformation.

The point $\xi = 0$ will no longer be a minimum if one of the coefficients of the quadratic terms vanishes, i.e. if for a certain label (αn) one has

$$E_2^{(\alpha n)}(J^2) = 0. \quad (5.2)$$

This equation determines the subspace $\mathcal{F}^{(\alpha n)}$ in which the bifurcation occurs, and the corresponding critical value of the angular momentum. Of course, one cannot *a priori* rule out the possibility that eq. (5.2) hold simultaneously for two, or more, values of α . However, satisfying several different conditions with one value of the parameter J^2 would be a casual coincidence (*accidental degeneracy*) which, if it happened, would require a separate discussion. We have shown by explicit calculation that this is not the case in our problem.

The label n , which distinguishes between equivalent irreducible representations, is related to the degree of the arbitrary polynomials θ in eqs. (3.7) and (3.12); it is convenient to identify it with the degree of the polynomial deformation P . Intuitively, a large angular momentum is needed to sustain against gravity a configuration bounded by a complicated surface. Hence, for a given α , the critical values of J^2 , given by eq. (5.2), are expected to be increasing functions of n .

Assuming that an appropriate algorithm, is available one could go beyond the linearized equation (2.27), and determine not only the bifurcation points, but the complete sequences branching off. The whole game could then be played again, to find the bifurcation points on these new sequences, etc. In the absence of accidental degeneracy, minimal symmetry breaking is expected at each bifurcation. The procedure must eventually terminate when the isotropy group of the bifurcation solution reduces trivially to the identity. At this point all the symmetry of the problem has disappeared, and the method loses all its predictive power.

Actually, the method becomes inapplicable even before all the symmetry has gradually been lost by the mechanism of spontaneous symmetry breaking. The cause is the onset of dynamical instability, which leads to non-stationary phenomena [6]. (In our problem it would lead to the fragmentation of the fluid mass.) Then, dynamical details such as fluctuations, impurities, etc., become important, and the ensuing symmetry breaking is typically maximal. Unlike stationary bifurcations, dynamical instabilities are perhaps similar to first-order phase transitions.

In order to simplify the numerical calculations which illustrate the method, homogeneity, incompress-

sibility and rigid rotation have been assumed, thereby reducing the degrees of freedom of the fluid mass to the degrees of freedom of its surface. In order to build a more realistic model for a rotating star, these assumptions must be relaxed. New degrees of freedom can then be excited; such as changes of volume and internal motions. Nevertheless, the equations of motion remain $D_{\infty h}$ -covariant, and the foregoing discussion of symmetry breaking — in particular table I and the selection rules — remain unaffected.

The authors are grateful to one of the referees for his remarks and for pointing out other works [29], [30], [31] where group theory methods are applied to astrophysical problems.

Appendix

Parametrization of ellipsoidal solutions. — Let the fluid have mass m and, in the corotating system, be bounded by an ellipsoidal surface of semi-axes a_1, a_2, a_3 . We adopt a special system of units, in which the unit of length is $a = (a_1 a_2 a_3)^{1/3}$, and the unit of moment of inertia is $(2/5) ma^2$; angular momentum squared is measured in units $(12/25) Gm^3 a$, potential in units $3 Gm/a$, and energy in units $(3/5) Gm^2/a$, where G is the gravitational constant. Then, the energy of the fluid is

$$E = -\frac{\alpha}{2} + \frac{J^2}{\varepsilon_1 + \varepsilon_2}, \quad (A.1)$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the squares of the semi-axes, J^2 is the square of the angular momentum, and

$$\alpha(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \int_0^\infty \frac{du}{\sqrt{(u+\varepsilon_1)(u+\varepsilon_2)(u+\varepsilon_3)}}. \quad (A.2)$$

A convenient description of the fluid's properties at equilibrium is given in terms of the infinite set of parameters [17]

$$\alpha_{i_1 \dots i_n}(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \frac{(-2)^n}{\prod_{k=1}^3 (2n_k - 1)!!} \frac{\partial^n \alpha}{\partial \varepsilon_{i_1} \dots \partial \varepsilon_{i_n}}, \quad (A.3)$$

$$\sum_{k=1}^3 n_k = n, \quad (n = 1, 2, \dots),$$

where n_k is the number of times the index k appears in the string i_1, \dots, i_n . In particular, the Γ -transform of a polynomial is again a polynomial (see section 2), whose coefficients are simple functions of the α 's [7]. Equation (2.12), which gives the gravitational potential at an internal point of a fluid, is such an example.

To simplify certain calculations, it is useful to introduce the quantities

$$\tilde{\alpha}_{i_1 \dots i_n} = \varepsilon_{i_1} \dots \varepsilon_{i_n} \alpha_{i_1 \dots i_n}. \quad (A.4)$$

and

$$\beta_{i_1 \dots i_{n_j}} = \alpha_{i_1 \dots i_n} - \varepsilon_j \alpha_{i_1 \dots i_{n_j}}. \quad (A.5)$$

Combining the definition (A.3) and the integral representation (A.2), it is easy to show that the β 's are always *positive*, a property which is needed in certain proofs.

A set of useful identities satisfied by the α 's is obtained by observing that $\alpha_{i_1 \dots i_n}$ is a homogeneous function of degree $-(n + 1/2)$, i.e. that

$$\alpha_{i_1 \dots i_n}(\lambda \varepsilon_1, \lambda \varepsilon_2, \lambda \varepsilon_3) = \lambda^{-(n+1/2)} \alpha_{i_1 \dots i_n}(\varepsilon_1, \varepsilon_2, \varepsilon_3). \quad (A.6)$$

Applying Euler's theorem one obtains

$$\sum_{j=1}^3 (2n_j + 1) \varepsilon_j \alpha_{i_1 \dots i_{n_j}} = (2n + 1) \alpha_{i_1 \dots i_n}. \quad (A.7)$$

Equations (A.7) will be referred to as *sum rules*.

Finally, let us remark that the equilibrium equations (2.14) and (2.15) can be obtained by requiring that the energy (A.1) have a local extremum with respect to variations of the ε 's, subject to the additional conditions (2.11).

References and footnotes

- [1] See, for example [18].
- [2] This *spontaneous* symmetry breaking is to be distinguished from the more familiar *dynamical* symmetry breaking (symmetry breakdown in the equations of motion).
- [3] A detailed presentation of the problem, with references, is found in [19], [20]; these two references are almost complementary. From 1969 on a string of papers have been published on this subject, mostly in *The Astrophysical Journal*. They are not relevant to our point of view here, and are too numerous to be cited only for completeness.
- [4] For a description of the symmetry groups appearing in this paper see, for example [21].
- [5] Throughout this paper, all quantities are dimensionless. They represent the values of the corresponding dimensional quantities, measured in appropriate *natural* units, which are described in the Appendix.
- [6] See, for example, [19], [22], and references therein.
- [7] See [19], Ch. 3, Theorems 3, 14, 15 and 16.
- [8] See [19], Ch. 3, Lemma 8.
- [9] See [19], Ch. 3, Theorem 13.
- [10] For details see [24], [25].
- [11] Equation (3.1) must be regarded as the equation of motion of a physical system, depending upon the external parameter μ . Its solutions describe the dynamical states of the system. The covariance under G means that all solutions belonging to the same orbit represent the same physical state, seen from different reference frames. Only in presence of a perturbation destroying the G -covariance (e.g. impurities, defects, fluctuations, etc.) would one be able to distinguish physically between the solutions belonging to the same orbit.
- [12] We are interested in the bifurcations from the ellipsoidal equilibrium solutions, so G is either $D_{\infty h}$ or D_{2h} .
- [13] See [21]; in particular note that $r_3(\omega) \sigma_1 = \sigma_1 r_3(-\omega)$, whereas σ_3 commutes with any rotation $r_3(\omega)$.
- [14] For $m = 1$, the notations D_{mh} and D_{md} are not appropriate; the symmetry pattern is better described by the alternative notations $D_{1h} = C_{2v}(\omega)$ and $D_{1d} = C_{2h}(\omega)$, where ω denotes an arbitrary direction in the (x_1, x_2) plane. The case $m = 1$ is examined separately.
- [15] This verifies the theorem of reference [31]; the plane $x_3 = 0$ (orthogonal to the angular momentum) is always a symmetry plane of the figure of equilibrium.
- [16] A similar trick has probably been used, but not mentioned, in Chandrasekhar's calculation of this bifurcation ([19], p. 109). His eqs. (52) to (54) do not follow from eq. (51). The numerical result, however, is correct.
- [17] See [2], section 2.1.
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