Table V*c*. — *Three-dimensional point groups* (G). *Tetrahedral groups* (T_h, T_d, T) : *Zeros of a general vector field* $\mathbf{v} = \Sigma_\sigma q_\sigma(\theta)\,\mathbf{e}_\sigma$ *at each stratum* Σ[L]. *The basic fields and relations determining the strata are given in tables* V*a* *and* V*b respectively. A dash in the last column indicates that* $\mathbf{v} = 0$ *reduces to an identity.*

G	Σ[L]	Zeros of $\mathbf{v} = \Sigma_\sigma q_\sigma(\theta)\,\mathbf{e}_\sigma$
T_h	$\Sigma[T_h]$	———
	$\Sigma[C_{2v}]$	$q_1 + \theta_1 q_2 + \theta_1^2 q_3 = 0$
	$\Sigma[C_3]$	$9 q_1 + 3\theta_1 q_2 + \theta_1^2 q_3 = 0$
	$\Sigma[C_s']$	$\theta_1 q_1 + \theta_2 q_2 + \theta_3 q_3 + \varphi_1 q_5 = 0$
		$2\varphi_1 q_2 + 2\theta_1 \varphi_1 q_3 + 2(\theta_1^3 - 4\theta_1\theta_2 + 3\theta_3) q_4 +$ $+ (\theta_1^4 - 4\theta_1^2\theta_2 + 2\theta_1\theta_3 + \theta_2^2) q_5 = 0$
	$\Sigma[C_1]$	$\theta_1 q_1 + \theta_2 q_2 + \theta_3 q_3 + \varphi_1 q_5 = 0$
		$6\varphi_1 q_2 + 6\theta_1\varphi_1 q_3 + 6(\theta_1^3 - 4\theta_1\theta_2 + 3\theta_3) q_4 +$ $+ 3(\theta_1^4 - 4\theta_1^2\theta_2 + 2\theta_1\theta_3 + \theta_2^2) q_5 + (3\theta_2 - \theta_1^2)(\theta_1^3 - 3\theta_1\theta_2 + 2\theta_3) q_6 = 0$
		$6\theta_2 q_1 + 6\theta_3 q_2 + (\theta_1^4 - 6\theta_1^2\theta_2 + 8\theta_1\theta_3 + 3\theta_2^2) q_3 - 6\varphi_1 q_4 = 0$
T_d	$\Sigma[T_d]$	———
	$\Sigma[C_{3v}]$	$3\theta_1 q_1 + 9\theta_2 q_2 + \theta_1^2 q_3 = 0$
	$\Sigma[C_{2v}]$	$q_1 + \theta_1 q_3 = 0$
	$\Sigma[C_s]$	$\theta_1 q_1 + 3\theta_2 q_2 + \theta_3 q_3 = 0$
		$6\theta_2 q_1 + (\theta_1^2 - \theta_3) q_2 + 2\theta_1\theta_2 q_3 = 0$
	$\Sigma[C_1]$	$q_1 = q_2 = q_3 = 0$
T	Σ[T]	———
	$\Sigma[C_3]$	$3\theta_1 q_1 + 9\theta_2 q_2 + \theta_1^2 q_3 = 0$
	$\Sigma[C_2]$	$q_1 + \theta_1 q_3 = 0$
	$\Sigma[C_1]$	$\theta_1 q_1 + 3\theta_2 q_2 + \theta_3 q_3 - \varphi_1 q_6 = 0$
		$6\theta_2 q_1 + (\theta_1^2 - \theta_3) q_2 + 2\theta_1\theta_2 q_3 + 2\varphi_1 q_5 = 0$
		$2\varphi_1 q_3 - (\theta_1^3 - \theta_1\theta_3 - 18\theta_2^2) q_4 + 2\theta_2(3\theta_3 - \theta_1^2) q_5 +$ $+ (\theta_1^2\theta_3 - 6\theta_1\theta_2^2 - \theta_3^2) q_6 = 0$

Solutions on $\Sigma[C_s']$ are found from the equation of $\Sigma[C_s']$, cf. table IV*b*,

$$\theta_1^3 - 3\theta_1\theta_2 + 2\theta_3 = 0 \qquad (56)$$

and the equations (cf. Table IV*c*)

$$q_2 + \theta_1 q_3 = 0 \qquad (57)$$

$$2 q_1 + (\theta_2 - \theta_1^2) q_3 = 0\,. \qquad (58)$$

One first solves (57) for θ_2 (as a quadratic polynomial in θ_1). Substituting this θ_2 into (56) one finds θ_3 (as a cubic polynomial in θ_1). Finally, substituting these θ_2 and θ_3 into (58) a single cubic equation for θ_1 is obtained. Solutions on $\Sigma[C_{4v}]$, $\Sigma[C_{3v}]$ and $\Sigma[C_{2v}]$ are, similarly, found from a third degree equation in θ_1, whereas equations for $\Sigma[C_s]$ are more complicated.

5. Conclusion.

We hope that this paper will help both to acquaint a broader group of physicists with new, sophisticated, and powerful group theoretical tools developed over the last few decades by the mathematicians, and to shed some light on an acute problem in physics, namely, that of solving systems of non-linear equations. We have completed treatment of all two- and three-dimensional point groups. However, we also presented mathematical background which should be sufficient for further applications. For example, it is both challenging and useful to apply the method presented here to : four- and five-dimensional irreducible representations of icosahedral groups Y and Y_h (this would complete calculation for all irreducible representations of all finite three-dimensional point groups), to $l > 3$

Table VI*a*. — *Three-dimensional finite, reducible point groups* ($G = D_{nh}, D_{nd}, D_n, n \geqslant 2$) : *Denominator* ($\theta_i$) *and numerator* ($\varphi_\beta$) *invariants and basic vector fields* ($\mathbf{e}_\sigma$). *Cartesian coordinates x, y and z are so chosen that the origin is a fixed point and the x and z axes are two and n-fold axes respectively.* $\hat{i}$, $\hat{j}$ *and* $\hat{k}$ *denote unit vectors in the x, y and z directions respectively. The radius vector in the xy plane is* $\boldsymbol{\rho} = x\hat{i} + y\hat{j}$, $\rho^2 = x^2 + y^2$. *Other quantities entering the table are* : $\gamma_n = \mathrm{Re}\,(x + iy)^n$; $\sigma_n = \mathrm{Im}\,(x + iy)^n$; $J\boldsymbol{\rho} = -y\hat{i} + x\hat{j}$. *Note that* $\frac{1}{n}\nabla\gamma_n = \gamma_{n-1}\,\hat{i} - \sigma_{n-1}\,\hat{j}$ *and* $\frac{1}{n}\nabla\sigma_n = \sigma_{n-1}\,\hat{i} + \gamma_{n-1}\,\hat{j}$.

G	θ_i	φ_β	$\mathbf{e}_\sigma$
D_{nh} $n \geqslant 2$	$\theta_1 = z^2$ $\theta_2 = \rho^2$ $\theta_3 = \gamma_n$	$\varphi_0 = 1$	$\mathbf{e}_1 = z\hat{k}$ $\mathbf{e}_2 = \boldsymbol{\rho}$ $\mathbf{e}_3 = \frac{1}{n}\nabla\gamma_n$
D_{nd} $n \geqslant 2$	$\theta_1 = z^2$ $\theta_2 = \rho^2$ $\theta_3 = \gamma_{2n}$	$\varphi_0 = 1$ $\varphi_1 = z\sigma_n$	$\mathbf{e}_1 = z\hat{k}$ $\mathbf{e}_2 = \boldsymbol{\rho}$ $\mathbf{e}_3 = \frac{1}{2n}\nabla\gamma_{2n}$ $\mathbf{e}_4 = \sigma_n\,\hat{k}$ $\mathbf{e}_5 = z\sigma_n\,\boldsymbol{\rho}$ $\mathbf{e}_6 = \frac{1}{n}z\nabla\sigma_n$
D_n $n \geqslant 2$	$\theta_1 = z^2$ $\theta_2 = \rho^2$ $\theta_3 = \gamma_n$	$\varphi_0 = 1$ $\varphi_1 = z\sigma_n$	$\mathbf{e}_1 = z\hat{k}$ $\mathbf{e}_2 = \boldsymbol{\rho}$ $\mathbf{e}_3 = \frac{1}{n}\nabla\gamma_n$ $\mathbf{e}_4 = \mathbf{e}_3 \times \mathbf{e}_2$ $\mathbf{e}_5 = \mathbf{e}_1 \times \mathbf{e}_2$ $\mathbf{e}_6 = \mathbf{e}_1 \times \mathbf{e}_3$

irreducible representations of O(3), to crystallographic space groups etc. Finally, the most challenging would be an extension of our results to compact groups.

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Table VI*b*. — *Three-dimensional finite reducible point groups* ($G = D_{nh}, D_{nd}, D_n, n \geqslant 2$) : *Classes of little groups* ([L]), *relations determining corresponding stratum* (Σ[L]) *and the global basis on* E_3^{L} ($\mathbf{e}'_\sigma$). *The relations are given in terms of invariants ; the relations necessary in orbit space, but redundant in the carrier space, are enclosed in curly brackets. The last rows for groups* D_{nd} *and* D_n *give algebraic relations satisfied on each stratum by the non-trivial numerator invariants. See table* VI*a for definitions of the invariants and the vector fields.*

G	[L]	Σ[L]	$\mathbf{e}'_\sigma$
D_{nh} $n \geqslant 2$	$[D_{nh}]$	$\theta_1 = \theta_2 = 0, \{\theta_3 = 0\}$	0
	$[C_{nv}]$	$\theta_1 > 0, \theta_2 = 0, \{\theta_3 = 0\}$	$\mathbf{e}_1$
	$[C_{2v}^{(\pm)}]$ ([a])	$\theta_1 = 0, \theta_2 > 0, \theta_3^2 = \theta_2^n$	$\mathbf{e}_2$
	$[C_s^{(\pm)}]$ ([a])	$\theta_1 > 0, \theta_2 > 0, \theta_3^2 = \theta_2^n$	$\mathbf{e}_1, \mathbf{e}_2$
	$[C_{1h}]$	$\theta_1 = 0, \theta_2 > 0, \theta_3^2 < \theta_2^n$	$\mathbf{e}_2, \mathbf{e}_3$
	$[C_1]$	$\theta_1 > 0, \theta_2 > 0, \theta_3^2 < \theta_2^n$	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
D_{nd} $n \geqslant 2$	$[D_{nd}]$	$\theta_1 = \theta_2 = 0, \{\theta_3 = 0\}$	0
	$[C_{nv}]$	$\theta_1 > 0, \theta_2 = 0, \{\theta_3 = 0\}$	$\mathbf{e}_1$
	$[C_2]$	$\theta_1 = 0, \theta_2 > 0, \theta_3 = \theta_2^n$	$\mathbf{e}_2$
	$[C_s]$	$\theta_2 > 0, \theta_3 = -\theta_2^n, \{\theta_1 \geqslant 0\}$	$\mathbf{e}_4, \mathbf{e}_2$
	$[C_1]$	$\theta_2 > 0, \theta_3 > -\theta_2^n, \theta_1 + \theta_2^n - \theta_3 > 0, \{\theta_1 \geqslant 0\}$	$\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_4 - \mathbf{e}_6, \mathbf{e}_3$
	$\varphi_1^2 = \frac{1}{2}\,\theta_1(\theta_2^n - \theta_3)\,\{\geqslant 0\}$		
D_n $n \geqslant 2$	$[D_n]$	$\theta_1 = 0, \theta_2 = 0, \{\theta_3 = 0\}$	0
	$[C_n]$	$\theta_1 > 0, \theta_2 = 0, \{\theta_3 = 0\}$	$\mathbf{e}_1$
	$[C_2^{(\pm)}]$ ([a])	$\theta_1 = 0, \theta_2 > 0, \theta_3^2 = \theta_2^n$	$\mathbf{e}_2$
	$[C_1]$	$\theta_2 > 0, \theta_1 + \theta_2^n - \theta_3^2 > 0, \{\theta_1 \geqslant 0\}$	$\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_4 - \mathbf{e}_6, \mathbf{e}_3$
	$\varphi_1^2 = \theta_1(\theta_2^n - \theta_3^2)\,\{\geqslant 0\}$		

([a]) For n odd the two classes, (+) and (−), coincide ; for n even they are disjoint and correspond to $\theta_3 = \pm\,\theta_2^{n/2}$.

Table VI*c*. — *Three-dimensional finite reducible point groups* (G = D_{nh}, D_{nd}, D_n, $n \geqslant 2$) : *Zeros of a general vector field* $\mathbf{v} = \Sigma_\sigma q_\sigma(\theta)\, \mathbf{e}_\sigma$ *at each stratum* $\Sigma[L]$. *The basic fields and relations determining the strata are given in tables* VI*a and* VI*b respectively. A dash in the last column indicates that* $\mathbf{v} = 0$ *reduces to an identity.*

G	$\Sigma[L]$	Zeros of $\mathbf{v} = \Sigma_\sigma q_\sigma(\theta)\, \mathbf{e}_\sigma$
D_{nh}	$\Sigma[D_{nh}]$	—
$n \geqslant 2$	$\Sigma[C_{nv}]$	$q_1 = 0$
	$\Sigma[C_{2v}^{(\pm)}]$	$\theta_2 q_2 + \theta_3 q_3 = 0$
	$\Sigma[C_s^{(\pm)}]$	$q_1 = 0, \theta_2 q_2 + \theta_3 q_3 = 0$
	$\Sigma[C_{1h}]$	$q_2 = q_3 = 0$
	$\Sigma[C_1]$	$q_1 = q_2 = q_3 = 0$
D_{nd}	$\Sigma[D_{nd}]$	—
$n \geqslant 2$	$\Sigma[C_{nv}]$	$q_1 = 0$
	$\Sigma[C_2]$	$q_2 + \theta_2^{n-1} q_3 = 0$
	$\Sigma[C_s]$	$\varphi_1 q_1 + \theta_2^n q_4 = 0$
		$\theta_2 q_2 - \theta_2^n q_3 + \varphi_1(\theta_2 q_5 + q_6) = 0$
	$\Sigma[C_1]$	$\theta_1 q_1 + \theta_2 q_2 + \theta_3 q_3 + \varphi_1(q_4 + \theta_2 q_5 + q_6) = 0$
		$\varphi_1(q_1 - q_2 + \theta_2^{n-1} q_3) + \frac{1}{2}(\theta_2^n - \theta_3)(q_4 - \theta_1 q_5) - \theta_1 \theta_2^{n-1} q_6 = 0$
		$\theta_3 q_2 + \theta_2^{2n-1} q_3 + \varphi_1(\theta_3 q_5 - \theta_2^{n-1} q_6) = 0$
D_n	$\Sigma[D_n]$	—
$n \geqslant 2$	$\Sigma[C_n]$	$q_1 = 0$
	$\Sigma[C_2^{(\pm)}]$	$\theta_2 q_2 + \theta_3 q_3 = 0$
	$\Sigma[C_1]$	$\theta_1 q_1 + \theta_2 q_2 + \theta_3 q_3 + \varphi_1(q_4 + q_6) = 0$
		$\varphi_1(q_1 - q_2) + (\theta_2^n - \theta_3^2) q_4 - \theta_1(\theta_3 q_5 + \theta_2^{n-1} q_6) = 0$
		$\theta_3 q_2 + \theta_2^{n-1} q_3 - \varphi_1 q_5 = 0$

Table VII*a*. — *Three-dimensional finite reducible point groups* (G = C_{nv}, C_{nh}, C_n, $n \geqslant 2$) : *Denominator* (θ_i) *and numerator* (φ_β) *invariants and basic vector fields* ($\mathbf{e}_\sigma$). *Cartesian coordinates x, y and z are so chosen that the origin is a fixed point and the z axis is the n-fold axis which together with the x axis forms a vertical reflection plane.* $\hat{i}$, $\hat{j}$ *and* $\hat{k}$ *denote unit vectors in the x, y and z directions respectively. The radius vector in the xy plane is* $\boldsymbol{\rho} = x\hat{i} + y\hat{j}$, $\rho^2 = x^2 + y^2$. *Other quantities entering the table are* : $\gamma_n = \mathrm{Re}(x + iy)^n$; $\sigma_n = \mathrm{Im}(x + iy)^n$; $J\boldsymbol{\rho} = -y\hat{i} + x\hat{j}$.

G	θ_i	φ_β	$\mathbf{e}_\sigma$
C_{nv}, $n \geqslant 2$	$\theta_1 = z$	$\varphi_0 = 1$	$\mathbf{e}_1 = \hat{k}$
	$\theta_2 = \rho^2$		$\mathbf{e}_2 = \boldsymbol{\rho}$
	$\theta_3 = \gamma_n$		$\mathbf{e}_3 = \frac{1}{n} \nabla \gamma_n$
C_{nh}, $n \geqslant 2$	$\theta_1 = z^2$	$\varphi_0 = 1$	$\mathbf{e}_1 = z\hat{k}$
	$\theta_2 = \rho^2$	$\varphi_1 = \sigma_n$	$\mathbf{e}_2 = \boldsymbol{\rho}$
	$\theta_3 = \gamma_n$		$\mathbf{e}_3 = \frac{1}{n} \nabla \gamma_n$
			$\mathbf{e}_4 = z\sigma_n \hat{k}$
			$\mathbf{e}_5 = J\boldsymbol{\rho}$
			$\mathbf{e}_6 = \frac{1}{n} \nabla \sigma_n$
C_n, $n \geqslant 2$	$\theta_1 = z$	$\varphi_0 = 1$	$\mathbf{e}_1 = \hat{k}$
	$\theta_2 = \rho^2$	$\varphi_1 = \sigma_n$	$\mathbf{e}_2 = \boldsymbol{\rho}$
	$\theta_3 = \gamma_n$		$\mathbf{e}_3 = \frac{1}{n} \nabla \gamma_n$
			$\mathbf{e}_4 = \mathbf{e}_3 \times \mathbf{e}_2$
			$\mathbf{e}_5 = \mathbf{e}_1 \times \mathbf{e}_2$
			$\mathbf{e}_6 = \mathbf{e}_1 \times \mathbf{e}_3$

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Table VII*b*. — *Three-dimensional finite reducible point groups* ($G = C_{nv}, C_{nh}, C_n, n \geqslant 2$) : *Classes of little groups* ([L]), *relations determining corresponding stratum* (Σ[L]) *and the global basis on* $E_3^{\perp}$ (e'_σ). *The relations are given in terms of invariants; the relations necessary in orbit space, but redundant in the carrier space, are enclosed in curly brackets. The last rows for groups* C_{nh} *and* C_n *give algebraic relations satisfied on each stratum by the non-trivial numerator invariants. See table* VII*a for definitions of the invariants and the vector fields.*

G	[L]	Σ[L]	e'_σ
C_{nv} $n \geqslant 2$	$[C_{nv}]$	$\theta_2 = 0, \{\theta_3 = 0\}$	e_1
	$[C_s^{(\pm)}]$ ([a])	$\theta_2 > 0, \theta_3^2 = \theta_2^n$	e_1, e_2
	$[C_1]$	$\theta_3^2 < \theta_2^n, \{\theta_2 > 0\}$	e_1, e_2, e_3
C_{nh} $n \geqslant 2$	$[C_{nh}]$	$\theta_1 = 0, \theta_2 = 0, \{\theta_3 = 0\}$	0
	$[C_n]$	$\theta_1 > 0, \theta_2 = 0, \{\theta_3 = 0\}$	e_1
	$[C_s]$	$\theta_1 = 0, \theta_2 > 0$	e_2, e_5
	$[C_1]$	$\theta_1 > 0, \theta_2 > 0$	e_1, e_2, e_5
	$\varphi_1^2 = \theta_2^n - \theta_3^2 \{\geqslant 0\}$		
C_n $n \geqslant 2$	$[C_n]$	$\theta_2 = 0, \{\theta_3 = 0\}$	e_1
	$[C_1]$	$\theta_2 > 0$	e_1, e_2, e_5
	$\varphi_1^2 = \theta_2^n - \theta_3^2 \{\geqslant 0\}$		

([a]) For n odd the two classes, (+) and (−), coincide; for n even they are disjoint and correspond to $\theta_3 = \pm \theta_2^{n/2}$.

Table VII*c*. — *Three-dimensional finite reducible point groups* ($G = C_{nv}, C_{nh}, C_n, n \geqslant 2$) : *Zeros of a general vector field* $\mathbf{v} = \Sigma_\sigma q_\sigma(\theta)\, \mathbf{e}_\sigma$ *at each stratum* Σ[L]. *The basic fields and relations determining the strata are given in tables* VII*a and* VII*b, respectively. A dash in the last column indicates that* $\mathbf{v} = 0$ *reduces to an identity.*

G	Σ[L]	Zeros of $\mathbf{v} = \Sigma_\sigma q_\sigma(\theta)\, \mathbf{e}_\sigma$
C_{nv} $n \geqslant 2$	$\Sigma[C_{nv}]$	$q_1 = 0$
	$\Sigma[C_s^{(\pm)}]$	$q_1 = 0, \theta_2 q_2 + \theta_3 q_3 = 0$
	$\Sigma[C_1]$	$q_1 = q_2 = q_3 = 0$
C_{nh} $n \geqslant 2$	$\Sigma[C_{nh}]$	———
	$\Sigma[C_n]$	$q_1 = 0$
	$\Sigma[C_s]$	$\theta_2 q_2 + \theta_3 q_3 + \varphi_1 q_6 = 0$ $\varphi_1 q_3 - \theta_2 q_5 - \theta_3 q_6 = 0$
	$\Sigma[C_1]$	$q_1 + \varphi_1 q_4 = 0$ $\theta_2 q_2 + \theta_3 q_3 + \varphi_1 q_6 = 0$ $\varphi_1 q_3 - \theta_2 q_5 - \theta_3 q_6 = 0$
C_n $n \geqslant 2$	$\Sigma[C_n]$	$q_1 = 0$
	$\Sigma[C_1]$	$q_1 + \varphi_1 q_4 = 0$ $\theta_2 q_2 + \theta_3 q_3 + \varphi_1 q_6 = 0$ $\varphi_1 q_3 - \theta_2 q_5 - \theta_3 q_6 = 0$

Table VIII*a*. — *Three-dimensional finite reducible point groups* ($G = S_{2n}, n \geqslant 1, C_s, C_1$) : *Denominator* ($\theta_i$) *and numerator* ($\varphi_\beta$) *invariants and basic vector fields* ($\mathbf{e}_\sigma$). *Cartesian coordinates x, y and z are chosen such that the origin is a fixed point and the z axis is the n-fold axis perpendicular to the reflection plane.* $\hat{i}, \hat{j}$ *and* $\hat{k}$ *denote unit vectors in the x, y and z directions respectively. The radius vector in the xy plane is* $\boldsymbol{\rho} = x\hat{i} + y\hat{j}$, $\rho^2 = x^2 + y^2$. *Other quantities entering the table are* : $\gamma_n = \mathrm{Re}\,(x + iy)^n$; $\sigma_n = \mathrm{Im}\,(x + iy)^n$; $J\boldsymbol{\rho} = -y\hat{i} + x\hat{j}$; $\frac{1}{n}\nabla\gamma_n = \gamma_{n-1}\,\hat{i} - \sigma_{n-1}\,\hat{j}$; $\frac{1}{n}\nabla\sigma_n = \sigma_{n-1}\,\hat{i} + \gamma_{n-1}\,\hat{j}$.

G	θ_i	φ_β	$\mathbf{e}_\sigma$
S_{2n} ([a]) $n \geqslant 1$	$\theta_1 = z^2$ $\theta_2 = \rho^2$ $\theta_3 = \gamma_{2n}$	$\varphi_0 = 1$ $\varphi_1 = z\gamma_n$ $\varphi_2 = z\sigma_n$ $\varphi_3 = \sigma_{2n}$	$\mathbf{e}_1 = z\hat{k}$ $\mathbf{e}_2 = \boldsymbol{\rho}$ $\mathbf{e}_3 = \frac{1}{2n}\nabla\gamma_{2n}$ $\mathbf{e}_4 = \gamma_n\,\hat{k}$ $\mathbf{e}_5 = z\gamma_n\,\boldsymbol{\rho}$ $\mathbf{e}_6 = \frac{1}{n} z\,\nabla\gamma_n$ $\mathbf{e}_7 = \sigma_n\,\hat{k}$ $\mathbf{e}_8 = z\sigma_n\,\boldsymbol{\rho}$ $\mathbf{e}_9 = \frac{1}{n} z\,\nabla\sigma_n$ $\mathbf{e}_{10} = z\sigma_{2n}\,\hat{k}$ $\mathbf{e}_{11} = J\boldsymbol{\rho}$ $\mathbf{e}_{12} = \frac{1}{2n}\nabla\sigma_{2n}$
C_s	$\theta_1 = z^2$ $\theta_2 = x$ $\theta_3 = y$	$\varphi_0 = 1$	$\mathbf{e}_1 = z\hat{k}$ $\mathbf{e}_2 = \hat{i}$ $\mathbf{e}_3 = \hat{j}$
C_1	$\theta_1 = x$ $\theta_2 = y$ $\theta_3 = z$	$\varphi_0 = 1$	$\mathbf{e}_1 = \hat{i}$ $\mathbf{e}_2 = \hat{j}$ $\mathbf{e}_3 = \hat{k}$

([a]) $S_2 = C_i$.

Appendix A.

GLOBAL BASIS OF VECTOR FIELDS ON A STRATUM. — We have proven, theorem 3.1, that at any point $\mathbf{r}$ of a stratum, the values $\mathbf{v}_\alpha(\mathbf{r})$ of the equivariant vector fields span the tangent plane to the stratum. If for a stratum of dimension m, there exist m covariant vector fields whose values are linearly independent at every point of the stratum, and therefore form a basis for each tangent plane to the stratum, we shall say that there

Table VIII*b*. — *Three-dimensional finite reducible point groups* ($G = S_{2n}$, $n \geqslant 2$, C_i, C_s, C_1) : *Classes of little groups* ([L]), *relations determining corresponding stratum* (Σ[L]) *and the global basis on* E_3^L (e'_σ). *The relations are given in terms of invariants; the relations necessary in orbit space, but redundant in the carrier space, are enclosed in curly brackets. The last three rows for* S_{2n} ($n \geqslant 2$) *and* C_i *give algebraic relations satisfied on each stratum by the non-trivial numerator invariants. See table* VIII*a, for definitions of the invariants and the vector fields.*

G	[L]	Σ[L]	e'_σ
S_{2n} $n \geqslant 2$	$[S_{2n}]$ $[C_n]$ $[C_1]$	$\theta_1 = 0, \theta_2 = 0, \{ \theta_3 = 0 \}$ $\theta_1 > 0, \theta_2 = 0, \{ \theta_3 = 0 \}$ $\theta_2 > 0, \{ \theta_1 \geqslant 0, \theta_2^{2n} \geqslant \theta_3^2 \}$	0 e_1 $e_1 + e_2, e_{11}, e_4 - e_6, e_7 - e_9$ ([a])
	$\varphi_1^2 = \frac{1}{2}\theta_1(\theta_2^n + \theta_3)$ $\varphi_2^2 = \frac{1}{2}\theta_1(\theta_2^n - \theta_3)$ $\varphi_3^2 = \theta_2^{2n} - \theta_3^2$		
C_i	$[C_i]$ $[C_1]$	$\theta_1 = \theta_2 = 0, \{ \theta_3 = 0 \}$ $\theta_1 + \theta_2 > 0, \{ \theta_1 \geqslant 0, \theta_2 \geqslant 0, \theta_3^2 \leqslant \theta_2^2 \}$	0 $e_1 + e_2, e_{11}, e_4 - e_6, e_7 - e_9$ ([a])
	$\varphi_1^2 = \frac{1}{2}\theta_1(\theta_2 + \theta_3)$ $\varphi_2^2 = \frac{1}{2}\theta_1(\theta_2 - \theta_3)$ $\varphi_3^2 = \theta_2^2 - \theta_3^2$		
C_s	$[C_s]$ $[C_1]$	$\theta_1 = 0$ $\theta_1 > 0$	e_2, e_3 e_1, e_2, e_3
C_1	$[C_1]$	The whole space	e_1, e_2, e_3

([a]) A minimal set of basic vector-fields which spans $\Sigma[C_1]$, which is *three* dimensional, consists of *four* basic fields, manifesting non-existence of a global basis on this stratum (see also Appendix B).

exists a *global basis* of vector fields for this stratum. We can give two such examples :

i) One dimensional stratum (which does not contain the origin); the vector field **r** is equivariant and forms global basis.

ii) Generic stratum of a pseudo-reflection group acting on an n-dimensional space E_n. In that case the basic covariant vector fields are the gradients of the n algebraically independent basic invariants. By theorem 3.1 the values of these n gradients must form a basis for E_n at any point of the n dimensional generic stratum.

These two examples are sufficient to prove the lemma :

Lemma A.1 For two dimensional real representations of finite groups there exists a global basis of covariant vector fields for any stratum.

Indeed for reflection groups this is true by ii) on the generic stratum and by i) on the nongeneric — and therefore one dimensional strata. The non reflection groups are the C_n's and the Grammian of the basic vector fields $\boldsymbol{\rho}$ and $J\boldsymbol{\rho}$ is ρ^4 which does not vanish outside the origin $\left(J \text{ is the matrix } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)$. For all groups we consider, acting on three dimensional space, we will find only one family of groups with no global invariant vector field basis for their strata : These are S_{2n}; see table VIII*c*.

We now prove the following :

Theorem A.1 There is a global covariant vector field basis for all strata of the reflection groups in two dimensions and for A_n, B_n, H_3.

Lemma A.1 proves it for 2-dimensional reflection groups. For the others the proof is based on an idea

Table VIII*c*. — *Three-dimensional finite reducible point groups* ($G = S_{2n}, n \geqslant 2, C_i, C_s, C_1$) : *Zeros of a general vector field* $\mathbf{v} = \Sigma_\sigma q_\sigma(\theta)\, \mathbf{e}_\sigma$ *at each stratum* $\Sigma[L]$. *The basic fields and relations determining the strata are given in tables* VIII*a and* VIII*b*, *respectively. A dash in the last column indicates that* $\mathbf{v} = 0$ *reduces to an indentity.*

G	$\Sigma[L]$	Zeros of $\mathbf{v} = \Sigma_\sigma q_\sigma(\theta)\, \mathbf{e}_\sigma$
S_{2n}	$\Sigma[S_{2n}]$	—
$n \geqslant 2$	$\Sigma[C_n]$	$q_1 = 0$
	$\Sigma[C_1]$	$\theta_1 q_1 + \theta_2 q_2 + \theta_3 q_3 + \varphi_1(q_4 + \theta_2 q_5 + q_6) + \varphi_2(q_7 + \theta_2 q_8 + q_9) + \varphi_3(\theta_1 q_{10} + q_{12}) = 0$ $-\varphi_3 q_3 - \varphi_2 q_6 + \varphi_1 q_9 + \theta_2 q_{11} + \theta_3 q_{12} = 0$ [$\varphi_1(q_1 - q_2 - \theta_2^{n-1} q_3) + \frac{1}{2}(\theta_2^n + \theta_3)(q_4 - \theta_1 q_5) - \theta_1 \theta_2^{n-1} q_6 + \frac{1}{2}\varphi_3(q_7 - \theta_1 q_8) + \varphi_2[(\theta_2^n + \theta_3) q_{10} + q_{11} - \theta_2^{n-1} q_{12}] = 0$ $\varphi_2(q_1 - q_2 + \theta_2^{n-1} q_3) + \frac{1}{2}\varphi_3(q_4 - \theta_1 q_5) + \frac{1}{2}(\theta_2^n - \theta_3)(q_7 - \theta_1 q_8) - \theta_1 \theta_2^{n-1} q_9 + \varphi_1[(\theta_2^n - \theta_3) q_{10} - q_{11} - \theta_2^{n-1} q_{12}] = 0$] (^{a})
C_i	$\Sigma[C_i]$	—
	$\Sigma[C_1]$	$\theta_1 q_1 + \theta_2 q_2 + \theta_3 q_3 + \varphi_1(q_4 + \theta_2 q_5 + q_6) + \varphi_2(q_7 + \theta_2 q_8 + q_9) + \varphi_3(\theta_1 q_{10} + q_{12}) = 0$ [$-\varphi_3 q_3 - \varphi_2 q_6 + \varphi_1 q_9 + \theta_2 q_{11} + \theta_3 q_{12} = 0$ $\varphi_1(q_1 - q_2 - q_3) + \frac{1}{2}(\theta_2 + \theta_3)(q_4 - \theta_1 q_5) - \theta_1 q_6 + \frac{1}{2}\varphi_3(q_7 - \theta_1 q_8) + \varphi_2[(\theta_2 + \theta_3) q_{10} + q_{11} - q_{12}] = 0$ $\varphi_2(q_1 - q_2 + q_3) + \frac{1}{2}\varphi_3(q_4 - \theta_1 q_5) + \frac{1}{2}(\theta_2 - \theta_3)(q_7 - \theta_1 q_8) - \theta_1 q_9 + \varphi_1[(\theta_2 - \theta_3) q_{10} - q_{11} - q_{12}] = 0$] (^{b})
C_s	$\Sigma[C_s]$	$q_2 = q_3 = 0$
	$\Sigma[C_1]$	$q_1 = q_2 = q_3 = 0$
C_1	$\Sigma[C_1]$	$q_1 = q_2 = q_3 = 0$

(^{a}) On the generic stratum, the equations outside the brackets must be supplement by one of the equations within the brackets : either equation can be used outside the two families of planes, $\theta_3 = \theta_2^n$ and $\theta_3 = -\theta_2^n$, whereas the first equation must be used at the first family of planes and the second at the second family of planes, respectively. This manifests non-existence of a global basis for S_{2n} at $\Sigma[C_1]$. (See also Appendix B.)

(^{b}) On the generic stratum, the single equation outside the brackets must be supplemented by two of the equations within the brackets. Outside the three planes $\theta_3 = \theta_2, \theta_3 = -\theta_2, \theta_1 = 0$ (the three coordinate planes) any two of the equations can be used. At the first plane ($y = 0$) the first and the second equations can be used and at the second plane ($x = 0$) the first and the third equations can be used whereas at the intersection of these two planes (z axis) the second and the third equations must be used. This manifests non-existence of a global basis for C_i at $\Sigma[C_i]$. (See also Appendix B.)

introduced in § 6 of [14]. Given a homogeneous polynomials $p(x)$ of degree d, by « polarization » one can obtain a multilinear form in d variables ; indeed define

$$D_{\mathbf{y}}\, p(\mathbf{x}) = \lim_{\lambda \to 0} (p(\mathbf{x} + \lambda \mathbf{y}) - p(\mathbf{x}))\, \lambda^{-1} = (\mathbf{y}, \nabla p(\mathbf{x})) \quad (\text{A.1})$$

and note that

$$D_{\mathbf{y}}\, D_{\mathbf{z}}\, p(\mathbf{x}) = D_{\mathbf{z}}\, D_{\mathbf{y}}\, p(\mathbf{x}) . \quad (\text{A.2})$$

The multilinear form $\tilde{p}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_d)$ is obtained by

$$\tilde{p}(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_d) = (1/d\,!)\, D_{\mathbf{x}_1} D_{\mathbf{x}_2} \cdots D_{\mathbf{x}_d}\, p(\mathbf{x}) . \quad (\text{A.3})$$

It is completely symmetrical in the d variables and it satisfies :

$$\tilde{p}(\mathbf{x}, \mathbf{x}, \mathbf{x}, ..., \mathbf{x}) = p(\mathbf{x}) . \quad (\text{A.4})$$

For a fixed $\mathbf{x}$, the bilinear form in $\mathbf{y}$ and $\mathbf{z}$, $\tilde{p}(\mathbf{x}, \mathbf{x}, ..., \mathbf{x}, \mathbf{y}, \mathbf{z})$ defines a $n \times n$ matrix $T(\mathbf{x})$, homogeneous of degree $d - 2$:

$$\tilde{p}(\mathbf{x}, \mathbf{x}, ..., \mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{y}, T(\mathbf{x})\, \mathbf{z}) . \quad (\text{A.5})$$

This matrix is symmetrical since $(\mathbf{y}, \mathrm{T}(\mathbf{x})\, \mathbf{z}) = (\mathrm{T}(\mathbf{x})\, \mathbf{y}, \mathbf{z})$. Consider now a set of basic invariants for one of the groups we consider and order them in order of increasing degree. Let $\theta_1 = (\mathbf{x}, \mathbf{x}) = \Sigma x_i^2$ and $\theta_2(\mathbf{x}) = (\mathbf{x}, \mathrm{T}(\mathbf{x})\, \mathbf{x})$. Then $(\mathbf{x}, \mathrm{T}^{k-1}\, \mathbf{x}) = \theta_k, k = 1, ..., n$ form a set of basic invariants. This can be verified by direct calculation for θ_3 of H_3 (which is Coxeter's notation for the group $\mathrm{Y_h}$, see Appendix D).

The group B_n is the symmetry group of the hypercube in n dimensions ($\mathrm{B}_3 = \mathrm{O_h}$). Its basic invariants can be chosen as $\theta_k = \sum_i x_i^{2k}$. Then $\mathrm{T}(\mathbf{x})_{ij} = x_i^2\, \delta_{ij}$ and we do verify that $\theta_k = (\mathbf{x}, \mathrm{T}^{k-1}(\mathbf{x})\, \mathbf{x})$. Similarly, A_n is the symmetry group of the regular simplex in n dimension ($\mathrm{A}_3 = \mathrm{T}_d$). Consider this n dimensional space as the hyperplane $\sum_{i=1}^{n+1} x_i = 0$ in the $(n + 1)$ dimensional space and in it the regular simplex whose $(n + 1)$ vertices have coordinates are

$$(\xi_\alpha)_i = \delta_{i_\alpha} - \frac{1}{n+1}. \qquad (\mathrm{A}.6)$$

Then $\mathrm{A}_n \sim \mathrm{S}_{n+1}$, the group of permutations of $(n + 1)$ objects, is represented by the $(n + 1)$ by $(n + 1)$ permutation matrices (zero everywhere except one by line and column). A basis of invariants can be chosen such that

$$0 \leqslant k \leqslant n, \quad \theta_k = \sum_{i=1}^{n+1} x_i^{k+1} \quad \text{with} \quad \theta_0 = 0. \quad (\mathrm{A}.7)$$

Then

$$\mathrm{T}(\mathbf{x})_{ij} = x_i\, \delta_{ij}, \quad \theta_k = (\mathbf{x}, \mathrm{T}(\mathbf{x})^{k-1}\, \mathbf{x}) \quad 1 \leqslant k \leqslant n. \qquad (\mathrm{A}.8)$$

For the reflection groups which we consider, the basic vector fields can be chosen as

$$\mathbf{e}_k(\mathbf{x}) = \mathrm{T}(\mathbf{x})^{k-1}\, \mathbf{x} = \frac{1}{c_k} \nabla\theta_k(\mathbf{x}) \qquad (\mathrm{A}.9)$$

where $c_k = [(k-1)\, d + 2]$, $d =$ degree of $\mathrm{T}(\mathbf{x})$. Assume that for a given $l < n$, at a point $\mathbf{x}$,

$$\mathbf{e}_l(\mathbf{x}) = \mathrm{T}^{l-1}(\mathbf{x})\, \mathbf{x} = \sum_{s=1}^{l-1} \alpha_s\, \mathrm{T}(\mathbf{x})^{s-1}\, \mathbf{x} = \sum_{s=1}^{l-1} \alpha_s\, \mathbf{e}_s(\mathbf{x}) \qquad (\mathrm{A}.10)$$

i.e. $\mathbf{e}_l(\mathbf{x})$ is a linear combination of the $(l - 1)$ $\mathbf{e}_s(\mathbf{x})$, $1 \leqslant s \leqslant (l - 1)$. This is also true for the $\mathbf{e}_k(\mathbf{x})$, $l \leqslant k \leqslant n$. Indeed, applying the operator $\mathrm{T}(\mathbf{x})^{k-l}$ to the two sides of equation A.10 one obtains

$$\mathbf{e}_k(\mathbf{x}) = \sum_{t=k-l+1}^{k-1} \alpha_{t+l+k}\, \mathbf{e}_t(\mathbf{x}) \qquad (\mathrm{A}.11)$$

and for each value of t, $l \leqslant t \leqslant (k - 1)$, $\mathbf{e}_t(\mathbf{x})$ can be expanded as a linear combination of the $\mathbf{e}_s(\mathbf{x})$, $1 \leqslant s \leqslant (l - 1)$, by repeated use of the equation A.10.

If the point $\mathbf{x}$ belongs to a stratum of dimension l, equation A.10 cannot hold, because the set of n vectors $\mathbf{e}_k(\mathbf{x})$ would be of rank $(l - 1)$ and theorem 3.1 would not hold. Hence the vectors $\mathbf{e}_k(\mathbf{x})$; $1 \leqslant k \leqslant l$ are linearly independent; this is true for any point x of the l-dimensional stratum. This concludes the proof of the theorem.

We make a remark, which applies in $n = 3$ dimensions only, for subgroups H of reflections groups R which have the same set of rotation axes i.e. same set of points Σ_1 of one-dimensional strata. If $\mathbf{e}_\alpha$, $\alpha = 1, 2, 3$ are basic vector fields of R, by theorem A.1, $\mathbf{e}_1(\mathbf{x})$ and $\mathbf{e}_2(\mathbf{x})$ are not collinear where $0 \neq \mathbf{x} \notin \Sigma_1$ and therefore $(\mathbf{e}_1(\mathbf{x}) \times \mathbf{e}_2(\mathbf{x}), \mathbf{e}_1(\mathbf{x}) \times \mathbf{e}_2(\mathbf{x})) > 0$, i.e. $\mathbf{e}_1(\mathbf{x}), \mathbf{e}_2(\mathbf{x}), \mathbf{e}_1(\mathbf{x}) \times \mathbf{e}_2(\mathbf{x})$ form a global basic outside $\Sigma_1 \cup \{ 0 \}$. So we have proven :

Lemma A.2 In three dimensional space, if $\mathbf{e}_\alpha$, $\alpha = 1, 2, 3$ form a basis of covariant vector fields for the finite reflection group R and if $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_1 \times \mathbf{e}_2$ are covariant vector fields for the R-subgroup H possessing the same rotation axes, then $\mathbf{e}_1(\mathbf{x})$, $\mathbf{e}_2(\mathbf{x})$, $\mathbf{e}_1(\mathbf{x}) \times \mathbf{e}_2(\mathbf{x})$ form a global basis for all H-strata of dimension $l > 1$.

This lemma applies to the unimodular subgroups of R, thus to the pairs $\mathrm{Y} < \mathrm{Y}_h$, $\mathrm{O} < \mathrm{O}_h$, $\mathrm{T} < \mathrm{T}_d$, $\mathrm{D}_n < \mathrm{D}_{nh}$, $\mathrm{C}_n < \mathrm{C}_{nv}$. We are left to study as particular cases the groups T_h, (Appendix C) D_{nd}, C_{nh}, S_{2n} (Appendix B). As we will see there are no global bases for the S_{2n} groups.

Appendix B.

STRATA, INVARIANTS, COVARIANT VECTOR FIELDS, GLOBAL BASES OF VECTOR FIELDS FOR THE GROUPS C_n, S_{2n}, C_{nv}, C_{nh}, D_n, D_{nd}, D_{nh}. — In this appendix, we give some information for the construction of tables I, VI, VII, VIII. Let us first define the groups. In the three dimensional space C_n is the group of rotations around the axis Oz by angles $2\,\pi k/n$, $0 \leqslant k < n$; it is a cyclic group of n elements. C_{nv} is generated by C_n and a reflection through the plane xOz i.e. $y = 0$. Let $l_\alpha = 0$, $\alpha = 1$ to n be the equations of the n reflection planes of C_{nv}. The product of the l_α's is

$$\sigma_n = \prod_{\alpha=1}^{n} l_\alpha = \mathrm{Im}\,(x + iy)^n. \qquad (\mathrm{B}.1)$$

C_{nv} is generated by reflections. Both C_n and C_{nv} act trivially on the z axis. They are the finite groups acting on the two dimensional space of coordinates x, y. The basic invariants of C_{nv} for this action are

$$\theta_1 = \rho^2 = x^2 + y^2, \quad \theta_2 = \gamma_n = \mathrm{Re}\,(x + iy)^n. \qquad (\mathrm{B}.2)$$

Since C_n is the unimodular subgroup of the reflection group C_{nv}, its numerator invariant is [See Eqs. 31 and 32]

$$\varphi_1 = \frac{D(\rho^2, \gamma_n)}{D(x, y)} = \sigma_n. \qquad (\mathrm{B}.3)$$

Remark that

$$\rho^{2n} = \gamma_n^2 + \sigma_n^2 . \quad (B.4)$$

Basic vector fields of C_{nv} are the gradients $\boldsymbol{\rho} = \frac{1}{2}\nabla\rho^2 = \begin{pmatrix} x \\ y \end{pmatrix}$, $\frac{1}{n}\nabla\gamma_n = \begin{pmatrix} \gamma_{n-1} \\ -\sigma_{n-1} \end{pmatrix}$.

Since the real two dimensional representation of C_n is irreducible on the real but reducible on the complex, it leaves invariant the antisymmetric bilinear form

$$x_1 y_2 - x_2 y_1 = (x_2, y_2)\, J \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (B.5)$$

Therefore, $J\boldsymbol{\rho}$ and $\frac{1}{n} J \nabla\gamma_n = \frac{1}{n}\nabla\sigma_n$ complete the basis of vector fields for C_n. We have seen in appendix A that these groups have a global basis of vector fields for each stratum.

For the representation in three dimensional space of the same groups, one adds the invariant z, the gradient field $\hat{k}$ (unit vector of the axis Oz) and for C_n, the cross-products of the C_{nv} covariant vector fields. The other finite, reducible, groups acting on the three dimensional space are discussed below.

i) D_{nh} is generated by C_{nv} and the reflection through the plane xOy, i.e. $z = 0$. It is a group generated by reflections and it is straight forward to construct the tables for it. As explained in [3], [12] and [4], the Stanley method yields the invariants for the subgroups D_n, C_{nh}, as well as for the subgroups D_{nd} and S_{2n} of D_{2nh}. The choice of covariant basic vector fields is less obvious and should be guided by the computation of the Poincaré (Molien) functions for vector fields (Tables VI*a*, *b*, *c*).

ii) D_n is the unimodular subgroup of D_{nh}. If $\mathbf{e}_\alpha$, $\alpha = 1, 2, 3$ are the basic vector fields of D_{nh}, we can add $\mathbf{e}_1 \times \mathbf{e}_2$, $\mathbf{e}_2 \times \mathbf{e}_3$, $\mathbf{e}_3 \times \mathbf{e}_1$ to complete the basis of vector fields of D_n. Then, as we have seen in appendix A, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \times \mathbf{e}_2$ form a global basis on the generic stratum. Note that the « horizontal » rotation axes, for the rotation by π, have for equation $z = 0$ *and* $\sigma_n = 0$. (See also Tables VI*a*, *b*, *c*.)

iii) D_{nd} contains D_n and the n plane reflections whose planes are defined by $\gamma_n = 0$. It is a subgroup of index 2 of D_{2nh}. It is possible to find a global basis for the two dimensional and the generic strata given in tables VI*a*, *b*, *c*.

iv) C_{nh} is the direct product of C_n and the plane reflection through the « horizontal » plane xOy. It is an index two subgroup of D_{nh} and Stanley method applies. With the help of the Molien function, it is easy to find the three other basic vector fields and the global basis for the two dimensional and the generic stratum, tables VII*a*, *b*, *c*.

v) S_{2n}, is a $2\,n$ element cyclic group generated by the product of the space inversion $-$ I and a rotation around Oz by an angle $2\,\pi/n$. It is sometimes denoted C_{ni} (and for $n = 1$ C_i is more often used than S_2). Outside the generic stratum, for $n > 1$, there is a one dimensional stratum, the z axis. S_{2n} is an index-four subgroup of D_{2nh}. The Stanley method yields the three φ_i invariants. There are twelve basic vector fields and no global basis for the generic stratum. However it is possible to find four covariant vector fields (see Table VIII*a*)

$$\mathbf{r} = \mathbf{e}_1 + \mathbf{e}_2 , \quad \mathbf{a} = \mathbf{e}_{11} , \quad \mathbf{b} = \mathbf{e}_4 - \mathbf{e}_6 , \quad \mathbf{c} = \mathbf{e}_7 - \mathbf{e}_9 \quad (B.6)$$

such that

$$\text{Gram}\,(\mathbf{r}, \mathbf{a}, \mathbf{b}) + \text{Gram}\,(\mathbf{r}, \mathbf{a}, \mathbf{c}) = (\theta_1 + \theta_2)^2\, \theta_2^n \quad (B.7)$$

is positive everywhere on $\Sigma[C_1]$ for $n \geqslant 2$; Gram is a shorthand for the Grammian of a set of vectors :

$$\text{Gram}\,(\mathbf{r}_1, \mathbf{r}_2, \ldots) = \det\,(\mathbf{r}_\alpha, \mathbf{r}_\beta)\,, \quad (B.8)$$

which reduces to $(\mathbf{r}_1, \mathbf{r}_2 \times \mathbf{r}_3)^2$ for a set of three vectors in the three dimensional space. For $n = 1$ the above expression vanishes in $\Sigma[C_1]$ along the z-axis ($\theta_2 = 0$). However, in that case

$$\text{Gram}\,(\mathbf{r}, \mathbf{a}, \mathbf{b}) + \text{Gram}\,(\mathbf{r}, \mathbf{a}, \mathbf{c}) + \text{Gram}\,(\mathbf{r}, \mathbf{b}, \mathbf{c}) = (\theta_1 + \theta_2)^3 \quad (B.9)$$

which is positive everywhere on $\Sigma[C_1]$. Consequently, an arbitrary S_{2n}-covariant vector field, $\mathbf{v} = \sum_{\alpha=1}^{12} q_\alpha\, \mathbf{e}_\alpha$ is completely determined everywhere on the generic stratum by its four projections : $(\mathbf{v}, \mathbf{r})$, $(\mathbf{v}, \mathbf{a})$, $(\mathbf{v}, \mathbf{b})$ and $(\mathbf{v}, \mathbf{c})$. However, no three of the above projections are (linearly) independent everywhere on $\Sigma[C_1]$ and there is no global basis on $\Sigma[C_1]$. For example, if we determine $\mathbf{v}$ by the first two projections, $(\mathbf{v}, \mathbf{r})$ and $(\mathbf{v}, \mathbf{a})$, then the projections $(\mathbf{v}, \mathbf{b})$ and $(\mathbf{v}, \mathbf{c})$ are equivalent everywhere on $\Sigma[C_1]$ *except* on the two families of planes $\theta_3 = \theta_2^n$ and $\theta_3 = -\theta_2^n$ where Gram $(\mathbf{r}, \mathbf{a}, \mathbf{b}) = 0$ and Gram $(\mathbf{r}, \mathbf{a}, \mathbf{c}) = 0$, respectively, and where $(\mathbf{v}, \mathbf{c})$, respectively, $(\mathbf{v}, \mathbf{b})$ projections must be used. In particular, for $n = 1$ the intersection of the above planes (z-axis) also belongs to $\Sigma[C_1]$ and both projections, $(\mathbf{v}, \mathbf{b})$ and $(\mathbf{v}, \mathbf{c})$, together with $(\mathbf{v}, \mathbf{r})$ determine $\mathbf{v}$ at this intersection.

Although the Molien functions for vector fields were necessary for our work, we do not give them in this paper since they can be deduce immediately from our tables. To build these tables, we need also to compute the Grammian matrix of the basic covariant vector fields and express it in terms of θ_i and φ_α. We leave this as an exercise to an interested reader. We present a detailed calculation for cubic groups in the following appendix.

Appendix C.

COMPUTATIONS FOR THE CUBIC GROUPS O_h, O, T_h, T_d, T. — These groups are subgroups of O_h, the symmetry group of the cube. We choose the origin of coordinates at the center of the cube and the coordinate axes normal to the cube faces. The cube symmetry group is generated by nine plane reflections. We denote by $l_\alpha = 0$, $\alpha = 1, 2, \dots, 9$ the equations of these reflection planes. By the action of O_h on the planes of the 3-dimensional space, these nine reflection planes fall into two orbits : one is made of the coordinate planes and yields

$$\prod_{\alpha=1}^{3} l_\alpha = I_3 = xyz . \qquad \text{(C.1)}$$

The other orbit, of six planes, yields

$$\prod_{\alpha=4}^{9} l_\alpha = J_6 = (x^2 - y^2)(y^2 - z^2)(z^2 - x^2) . \qquad \text{(C.2)}$$

We also introduce the notation :

$$\prod_{\alpha=1}^{9} l_\alpha = J_9 = I_3 J_6 . \qquad \text{(C.3)}$$

The group T_d is generated by the six reflections through planes (C.2). The T_h group has three plane reflections, those of (C.1). Rotation axes are at the intersection of the symmetry planes. The incidence relations are given in table C.1. This table shows that the rotation axes fall into three O_h orbits.

Table C.1. — *Reflection planes and rotation axes of the cubic groups. Each of the three A-axes is at the intersection of four reflection planes; each of the four B-axes is at the intersection of three reflection planes of the 6-plane orbit. Each of the six C-axes is at the intersection of two reflection planes. Only the groups O_h and O have C-axes of rotation.*

	Planes \ Axes	*A*			*B*				*C*					
O_h, T_h	$x = 0$		×	×					×	×				
	$y = 0$	×		×							×	×		
	$z = 0$	×	×										×	×
O_h, T_d	$y - z = 0$	×			×	×			×					
	$y + z = 0$	×					×	×		×				
	$z - x = 0$		×		×		×				×			
	$z + x = 0$		×			×		×				×		
	$x - y = 0$			×	×			×					×	
	$x + y = 0$			×		×	×							×
Order of rotation axis	O_h, O	4			3				2					
	T_d, T_h, T	2			3									

The reflection through the coordinate planes changes the sign of the variables x, y, z so O_h and T_h invariant polynomials depend only on x^2, y^2, z^2 and the homogeneous ones have even degree. From equations 13 and 33, we know that the degrees d_1, d_2, d_3 of the three basic invariants of O_h satisfy :

$$d_1 = 2 , \quad d_1 d_2 d_3 = 48$$

$$d_1 + d_2 + d_3 = 9 + 3 \quad \text{so} \quad d_2 = 4 , \; d_3 = 6 \qquad \text{(C.4)}$$

(48 is the order of O_h, 9 is the number of its reflection planes). The rotation around the *A* and *B* axes induce respectively odd and even permutations of the coordinate axes, so the three basic O_h invariants are invariant by the permutations of x^2, y^2, z^2. A possible choice is :

$$I_2 = x^2 + y^2 + z^2 \quad I_4 = x^4 + y^4 + z^4$$
$$I_6 = x^6 + y^6 + z^6 . \qquad \text{(C.5)}$$

The O_h basic vector fields can be chosen proportional to the gradients of the basic invariants

$$\mathbf{V}_1 = (x, y, z) \quad \mathbf{V}_3 = (x^3, y^3, z^3) \quad \mathbf{V}_5 = (x^5, y^5, z^5) . \qquad \text{(C.6)}$$

They satisfy the relations (see equations 31 and 32)

$$(\mathbf{V}_1, \mathbf{V}_3 \times \mathbf{V}_5) = \frac{1}{48} \frac{D(I_2, I_4, I_6)}{D(x, y, z)} = J_9 . \qquad \text{(C.7)}$$

The six reflection planes of (C.2), which generate T_d, permute the coordinate axes but do not change separately the sign of x, y, z (they change it by pairs). So the fundamental invariants of T_d are I_2, I_3, I_4. With the notation

$$\nabla I_3 = \mathbf{V}_2 = (yz, zx, xy) \qquad \text{(C.8)}$$

equations 31 and 32 applied to T_d invariants yield :

$$(\mathbf{V}_1, \mathbf{V}_2 \times \mathbf{V}_3) = \frac{1}{8} \frac{D(I_2, I_3, I_4)}{D(x, y, z)} = J_6 . \quad (C.9)$$

The unimodular subgroups O of O_h and T of T_d have respectively the same denominator invariants θ_i, $i = 1, 2, 3$. Their numerator invariant φ_1 is J_9 for O and J_6 for T. The general Stanley method (see Eq. 30) applies to the index-2 subgroup T_h of O_h. Its θ_i's are those of O_h. Its φ_1 is the product of the l_α of O_h plane reflections which do not belong to T_h; this is J_6 again.

We now pass to the choice of basic vector fields.

For the reflection groups O_h and T_d we have seen that $\mathbf{e}_\alpha$, $\alpha = 1, 2, 3$, can be taken proportional to the gradients of the denominator invariants θ_i, $i = 1, 2, 3$. For the three other groups, the Molien function indicates the number nk, and the degrees of the basic vector fields. For the unimodular groups O and T we can choose $\mathbf{e}_1 \times \mathbf{e}_2$, $\mathbf{e}_1 \times \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3$ for $\mathbf{e}_4$, $\mathbf{e}_5$, $\mathbf{e}_6$. This does not apply to T_h. $\mathbf{V}_1$, $\mathbf{V}_3$, $\mathbf{V}_5$ are the basic vector fields $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ for T_h (as for O_h). Since $\mathbf{V}_2$ is a pseudo-vector field for T_h one can take $\mathbf{e}_4 = \mathbf{V}_1 \times \mathbf{V}_2$, $\mathbf{e}_5 = \mathbf{V}_3 \times \mathbf{V}_2$. Although $\mathbf{V}_2 \times \mathbf{V}_5$ has the right degree indicated by the Molien function, it is not independent from $\mathbf{e}_4$ and $\mathbf{e}_5$. Indeed

$$\mathbf{V}_2 \times \mathbf{V}_5 + I_2 \mathbf{V}_3 \times \mathbf{V}_2 - \tfrac{1}{2}(I_2^2 - I_4) \mathbf{V}_1 \times \mathbf{V}_2 = 0 . \quad (C.10)$$

Noting that I_3 is a pseudo-scalar of T_h; one checks that a possible choice for $\mathbf{e}_6$ is $I_3 \mathbf{V}_1 \times \mathbf{V}_3$.

We have studied in appendix A the choice of global basis of covariant vector fields for a given stratum. For computing the equations for zeros of covariant vector fields on a given stratum, we need the scalar product of all basic vector fields with those of the global basis for each stratum. These scalar products are given in table C.2. With this table we can also compute the Grammian (i.e. the determinant of the matrix of scalar products) of the set of vectors $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ and $\mathbf{V}_1$, $\mathbf{V}_3$, $\mathbf{V}_5$. This is respectively J_3^2 and J_6^2 which are O_h invariants :

$$6 J_3^2 = I_2^3 - 3 I_2 I_4 + 2 I_6 , \quad (C.11)$$

$$6 J_6^2 = - 18 I_6^2 + 36 I_2 I_4 I_6 - 8 I_2^3 I_6 + 3 I_4^2 - 21 I_2^2 I_4^2 + 9 I_2^4 I_4 - I_2^6 . \quad (C.12)$$

Table C.2. — *Scalar products of various basic vector fields with vector fields of the global bases chosen for different strata.*

	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_5$	$\mathbf{V}_1 \times \mathbf{V}_3$	$\mathbf{V}_1 \times \mathbf{V}_2$
$\mathbf{V}_1$	I_2	$3 I_3$	I_4	I_6	0	0
$\mathbf{V}_2$	$3 I_3$	A_4	$I_2 I_3$	$I_3 I_4$	J_6	0
$\mathbf{V}_3$	I_4	$I_2 I_3$	I_6	L_8	0	$- J_6$
$\mathbf{V}_5$	I_6	$I_3 I_4$	L_8	L_{10}	J_9	$- I_2 J_6$
$\mathbf{V}_1 \times \mathbf{V}_3$	0	J_6	0	J_9	$I_2 I_6 - I_4^2$	$- I_3 B_4$
$\mathbf{V}_1 \times \mathbf{V}_5$	0	$I_2 J_6$	$- J_9$	0	$I_2 L_8 - I_4 I_6$	$I_3(I_2 I_4 - 3 I_6)$
$\mathbf{V}_3 \times \mathbf{V}_5$	J_9	$- A_4 J_6$	0	0	$I_4 L_8 - I_6^2$	$I_3(I_4^2 - I_2 I_6)$
$\mathbf{V}_1 \times \mathbf{V}_2$	0	0	$- J_6$	$- I_2 J_6$	$- I_3 B_4$	$I_2 A_4 - 9 I_3^2$
$\mathbf{V}_2 \times \mathbf{V}_3$	$- J_6$	0	0	$- A_4 J_6$	$I_3(3 I_6 - I_2 I_4)$	$3 I_2 I_3^2 - I_4 A_4$

$A_4 = \frac{1}{2}(I_2^2 - I_4)$ $B_4 = 3 I_4 - I_2^2$; $L_8 = \frac{1}{6}(8 I_2 I_6 + 3 I_4^2 - 6 I_2^2 I_4 + I_2^4)$; $L_{10} = \frac{1}{6}(5 I_4 I_6 + 5 I_6 I_2^2 - 5 I_4 I_2^3 + I_2^5)$.
Use $(\mathbf{V}_a \times \mathbf{V}_b, \mathbf{V}_c \times \mathbf{V}_d) = (\mathbf{V}_a, \mathbf{V}_c)(\mathbf{V}_b, \mathbf{V}_d) - (\mathbf{V}_a, \mathbf{V}_d)(\mathbf{V}_b, \mathbf{V}_c)$ = minor of Grammian for lines a, b, column c, d.

Appendix D.

CALCULATIONS FOR THE ICOSAHEDRAL GROUPS. — We follow Coxeter [21] in orienting the icosahedron so that one pair of opposite edges is parallel to each coordinate axis; it is centred at the origin and has edges of length 2, and its vertices are at $(\pm \tau, \pm 1, 0)$, $(0, \pm \tau, \pm 1)$, $(\pm 1, 0, \pm \tau)$, where $\tau = (1 + \sqrt{5})/2$. The associated dodecahedron has its vertices at the midpoints of the faces of the icosahedron. Meyer [5] has given explicit expressions for the basic Y-invariants r^2, I_6', I_{10} and J_{15} (J_{15} is a pseudoinvariant of Y_h). They are reproduced in the caption of our table III*a* (we correct a misprint in Meyer's I_{10}). $r^2 = x^2 + y^2 + z^2$ is the O(3) invariant; I_6' is the product of the six planes through the origin orthogonal to lines joining pairs of opposite vertices; I_{10} is the product of the ten planes through the origin parallel to pairs of opposite faces; J_{15} is the product of the fifteen planes through the origin orthogonal to lines joining the midpoints of pairs of opposite edges; they are the symmetry planes, so J_{15} is proportional to the Jacobian $D(r^2, I_6', I_{10})/D(x, y, z)$.

The maxima, $(2\tau + 1)/27$, of I_6'/r^6 occur in the directions of the vertices of the dodecahedron; I_6' vanishes along lines (planes) joining midpoints of adjacent edges of the icosahedron.

The absolute maximum, $5(2\tau - 1)/81$, of I_{10}/r^{10} occurs in the directions of the vertices of the dodecahedron; local maxima $(2\tau - 1)/125$ occur in the direction of the vertices of the icosahedron; I_{10} vanishes along lines (planes) joining midpoints of alternate edges meeting at a vertex of the icosahedron. The absolute minima, $-(2\tau - 1)/250$, occur in the directions of two points of each edge of the icosahedron at distance τ^{-1} from each vertex.

J_{15} vanishes along the edges of the icosahedron and along those of the dodecahedron.

The six basic covariant vector fields for Y are $\mathbf{e}_1 = \mathbf{r}$, $\mathbf{e}_2 = \frac{1}{2}\nabla I_6'$, $\mathbf{e}_3 = \frac{1}{2}\nabla I_{10}$, $\mathbf{e}_4 = \mathbf{e}_1 \times \mathbf{e}_2$, $\mathbf{e}_5 = \mathbf{e}_1 \times \mathbf{e}_3$ and $\mathbf{e}_6 = \mathbf{e}_2 \times \mathbf{e}_3$; $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are also the basic covariant vector fields for Y_h. Their scalar products are computed straightforwardly. For instance, since $(\mathbf{e}_2, \mathbf{e}_2)$ is an invariant of degree 10, we must have $(\mathbf{e}_2, \mathbf{e}_2) = Ar^{10} + Br^4 I_6' + CI_{10}$, where A, B and C are constants to be determined by comparing coefficients of powers of x, y, z, say of x^{10}, $x^8 y^2$ and $x^6 y^4$; additional coefficients are compared as a check. In table D, we list the products of $(\mathbf{e}_i, \mathbf{e}_j)$ for $i = 1, 2, 3$ and $j = 1, ..., 6$. The remaining scalar products can be found by using elementary formula $(\mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d}) = (\mathbf{a}, \mathbf{c})(\mathbf{b}, \mathbf{d}) - (\mathbf{a}, \mathbf{d})(\mathbf{b}, \mathbf{c})$. $J_{15}^2 = (\mathbf{e}_1, \mathbf{e}_2 \times \mathbf{e}_3)^2$ is evaluated similarly as a Grammian determinant,

$$J_{15}^2 = \mathrm{Gram}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \equiv \det [(\mathbf{e}_i, \mathbf{e}_j)]_{i,j=1}^3 . \quad (\mathrm{D}.1)$$

The result can be found in table III*b*.

We turn to the derivation of the equations of the strata of Y and Y_h. The stratum $[C_5]$ of Y, or $[C_{5v}]$ of Y_h, consists of lines through the origin in the directions of the vertices of the icosahedron, but excluding the origin itself. Since I_6'/r^6 takes its absolute minimum, $-(2\tau + 1)/5$, in these directions, we can specify the stratum by the conditions

$$5 I_6' + (2\tau + 1) r^6 = 0, \quad r^2 > 0. \quad (\mathrm{D}.2)$$

The equation $A_6 \equiv 5 I_6' + (2\tau + 1) r^6 = 0$ implies $125 I_{10} - (2\tau - 1) r^{10} = 0$, but not the other way around, because I_{10}/r^{10} has only a local, not an absolute, maximum in the directions of the icosahedron vertices.

The stratum $[C_3]$ of Y, or $[C_{3v}]$ of Y_h, consists of lines in the directions of the vertices of the dodecahedron. In those directions I_6'/r^6 has an absolute maximum $(2\tau + 1)/27$ and I_{10}/r^{10} has an absolute maximum $5(2\tau - 1)/81$. Hence the stratum is specified by the equation $B_6 \equiv (2\tau + 1) r^6 - 27 I_6' = 0$ or by $5(2\tau - 1) r^{10} - 81 I_{10} = 0$, together with $r^2 > 0$.

The stratum $[C_2]$ of Y, or $[C_{2v}]$ of Y_h, consists of lines in the directions from the origin to midpoints of the edges of the icosahedron, or dodecahedron. Those points are specified by the conditions $I_6' = 0$, $I_{10} = 0$ or $r^8 I_6'^2 + I_{10}^2 = 0$, together with $r^2 > 0$.

The stratum $[C_s]$ of Y_h consists of the reflection planes, on which $J_{15} = 0$, with the exception of the lines belonging to the one-dimensional strata and the origin. It is specified by the invariant conditions $J_{15}^2 = 0$, $A_6 B_6 [I_{10}^2 + r^8 I_6'^2] > 0$; the inequality excludes the lower dimensional strata. Finally, the generic stratum is given by $A_6 B_6 (I_{10}^2 + r^8 I_6'^2) > 0$ for Y (adding $J_{15}^2 > 0$ for Y_h); the inequalities exclude all lower dimensional strata.

We now mention the Poincaré functions for invariants, given, e.g., in [9]. Those Poincaré functions are valid for the generic stratum; on lower dimensional strata, because of identities satisfied there by the invariants, they assume a reduced, or collapsed, form. Thus, for Y_h the generic Poincaré function is $[(1 - t^2)(1 - t^6)(1 - t^{10})]^{-1}$ and for Y it is $(1 + t^{15})[(1 - t^2)(1 - t^6)(1 - t^{10})]^{-1}$. On all one-dimensional strata, $J_{15} = 0$ and I_6' and I_{10} are numerical multiples of r^6 and r^{10} respectively. Hence the reduced Poincaré functions are $(1 - t^2)^{-1}$. [Similarly, the basic vectors $\mathbf{e}_i$ all vanish or are multiples of powers of r^2 and $\mathbf{e}_1 = \mathbf{r}$; the generating function for covariant vector fields becomes $t(1 - t^2)^{-1}$.] On the two dimensional stratum of Y_h the only invariant condition additional to those for the generic stratum is $J_{15}^2 \equiv 0$. Hence the reduced Poincaré function for invariants is

$$(1 - t^{30})[(1 - t^2)(1 - t^6)(1 - t^{10})]^{-1} = (1 + t^{10} + t^{20})[(1 - t^2)(1 - t^6)]^{-1} . \quad (\mathrm{D}.3)$$

Only r^2 and I_6' behave as denominator invariants. I_{10} and I_{10}^2 are numerator invariants; I_{10}^3 can be expressed as a linear combination of r^{15} and $(I_6')^5$.

The global basis on a stratum has been discussed in appendix A. On the one-dimensional strata it is given by $\mathbf{e}_1 = \mathbf{r}$. On the two-dimensional stratum $[C_s]$ of Y_h, it is given by $\mathbf{e}_1$ and $\mathbf{e}_2$. For the generic stratum of Y_h the global basis is $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. However, since the generic stratum of Y includes the planes $J_{15} = (\mathbf{e}_1, \mathbf{e}_2 \times \mathbf{e}_3) = 0$, we take as its global basis $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_4 = \mathbf{e}_1 \times \mathbf{e}_2$.

Table D. — *Scalar products* $(\mathbf{e}_i, \mathbf{e}_j)$ *for the basic covariant vector fields of* Y ($i = 1, 2, 3$ *and* $j = 1, ..., 6$).

$$D_{10} = -(7/4)(2\tau + 1) r^4 I_6' + (1/4)(18\tau + 11) I_{10};$$
$$D_{14} = (1/4)(1 - 2\tau) r^8 I_6' + (3/4)(2\tau + 1) r^4 I_{10} + 2(4\tau - 7) r^2 (I_6')^2;$$
$$D_{18} = (3/4)(3 - 2\tau) r^{12} I_6' + (5/4)(2\tau - 1) r^8 I_{10} + 12(8\tau - 13) r^6 (I_6')^2 + 12(7 - 4\tau) r^2 I_6' I_{10} + 48(55 - 34\tau)(I_6')^3.$$

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$
$\mathbf{e}_1$	r^2	$3 I_6'$	$5 I_{10}$	0	0	J_{15}
$\mathbf{e}_2$	$3 I_6'$	D_{10}	D_{14}	0	$-J_{15}$	0
$\mathbf{e}_3$	$5 I_{10}$	D_{14}	D_{18}	J_{15}	0	0

Appendix E.

ON DISCREPANCIES. — A careful reader may notice some discrepancies between our results and previously published ones. Therefore, we must warn the reader of some errors in previous publications.

For the icosahedral group Y we use essentially the same integrity basis as given in [5] except that we found an error in I_{10} of [5] : the factor $(x^4 + y^4 + z^4)$ should read

$$(x + y + z)(x - y - z)(y - z - x)(z - x - y) = \\ = x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 .$$

With respect to [9] we have to make two remarks. First, we find for the octahedral group O that the basic covariant $E^{(8)}(\Gamma_4, \Gamma_4)$, in the notation of [9], is not independent of the other five basic covariants

$$E^{(8)}(\Gamma_4, \Gamma_4) = \tfrac{1}{2}\{ I^{(4)}(\Gamma_4) - [I^{(2)}(\Gamma_4)]^2 \} E^{(4)}(\Gamma_4, \Gamma_4) + \\ + I^{(2)}(\Gamma_4) E^{(6)}(\Gamma_4, \Gamma_4) . \quad \text{(E.1)}$$

An appropriate, eight degree, independent covariant is, for example

$$E^{(8)}(\Gamma_4, \Gamma_4) = E^{(3)}(\Gamma_4, \Gamma_4) \times E^{(5)}(\Gamma_4, \Gamma_4) \quad \text{(E.2)}$$

which is the choice in our table IV*a*. Second, we observe that equation 55 of [9] is not sufficient to obtain all the basic covariants of an improper group which is a direct product of the inversion and a proper group. For example, basic fields of the form

$$E^{(2k+1)}(\Gamma_l^0, \Gamma_m^0) = I^{(2p+1)}(\Gamma_m) E^{(2q)}(\Gamma_l, \Gamma_m) , \quad \text{(E.3)}$$

must be considered. This can be illustrated for T_h in which case one could not otherwise obtain the basic covariant

$$E^{(7)}(\Gamma_4^0, \Gamma_4^0) = I^{(3)}(\Gamma_4) E^{(4)}(\Gamma_4, \Gamma_4) . \quad \text{(E.4)}$$

In [11] we found that the $\Gamma_4^{(1)-}$ basic covariants for $\Gamma_4^{(1)-}(T_h)$ were incomplete, missing a seventh degree invariant [e.g. Eq. E.4].

Finally, in [13] equations for the strata of O_h were given. In the first and third row of table 3, [13], α should read α^2. In the second row of the same table the three-fold axes ($x = y = z$, etc.) should be excluded from the equations of [C_s] stratum. This can be achieved, for example, by adding the inequality $\theta_2 > \frac{1}{3}\theta_1^2$ to the relations already given.

Should some errors be detected in the present work, we would appreciate being notified about them.

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