SYMMETRY IN CONDENSED MATTHER PHYSICS

Louis MICHEL
Institut des Hautes Etudes Scientifiques
91440 Bures-sur-Yvette (France)

This is one of the oldest subject in mathematical physics. In 1829 the finite subgroups of O(3), the three dimensional orthogonal group, had been enumerated by Hessel. Completing and correcting the work of Frankenheim, Bravais determined the 14 (Bravais) classes of lattices in 1850. In 1879 Sohncke described the 65 classes of crystallographic groups, subgroups of the connected Euclidean group $E_o(3)$. Reflections had to be added. This was done independently by Fedorov and Schönflies. In 1891, by correspondence, they compared their results correcting minor errors on both sides, they reached an agreement on the listing of the $c_3 = 230$ crystallographic classes. This is a beautiful result of mathematical physics : indeed Schönflies, a mathematician, published in Mathematische Annalen while Fedorov, working for the Russian Geological Survey, published in the Proceedings of the Saint Petersburg Mineralogical Society ! Fedorov also described the $c_2 = 17$ crystallographic classes for dimension 2.

"Is the number c_n of crystallographic classes finite for any dimension n ?" became a mathematical problem, precisely Hilbert 18 th problem. The affirmative answer was given by Biebenbach : in 1910, he proved that γ_n , the number of isomorphism classes of crystallographic groups is finite for any n and in 1912 he established the theorem : isomorphic crystallographic groups are conjugated in Aff(n) , the n dimensional affine group (= inhomogeneous linear group).

The interatomic distances in a given crystal phase depends on temperature t , and pressure p , so its crystallographic group does depend on t and p , hence the natural equivalence for the classification of crystal symmetry groups is up to a conjugation by an element of Aff(n) the connected affine group. When two isomorphic crystallographic groups (conjugated in Aff(n) by Biebenbach theorem) are not conjugated by elements of $Aff_o(n)$ their two crystallographic classes are "enantiomorphic" For n = 3 there are eleven pairs of enantiomorphic classes so $\gamma_3 = 219 = c_3 - 11$.

The $c_4 = 4895$ ($\gamma_4 = 4783$) four dimensional crystallographic classes have been recently listed and described : Brown et. al. [1] . The c_n are unknown for n > 4 ; an asymptotic limit has been given by Schwarzenberger [2] :

$$\lim_{n \to \infty} \frac{1}{n^2} \, \text{Log } c_n = \text{Log } 2 \tag{1}$$

It is difficult for many mathematical physicists to study crystallography because most of its fondamental concepts were conceived when group theory was less developped and they are defined only implicitly in the literature. This situation is chan-

ging now, but one may find that a n-independent extension of the intuitive concept for n = 2 or 3 is far from obvious e.g. Neübusser et. al. [3] for the definition of crystal system. (Two of the three authors had each proposed earlier a definition ; they both failed for n = 7 !) Let me try to contribute here to an easier problem : the classification of crystallographic groups (and not of the crystal themselves).
I have first to recall some definitions and results. An action of a group G on a set M is defined by a map ϕ which satisfies :

$$G \times M \xrightarrow{\phi} M , \quad \phi(1,m) = m , \quad \phi(g_2, \phi(g_1,m)) = \phi(g_2 g_1, m) \quad (2)$$

I will often use g.m as a shorthand for $\phi(g,m)$. The <u>isotropy group</u> G_m (physicists also say the little group of m) is the G-subgroup which leaves m fixed :

$$G_m = \{g \in G , g.m = m\} \quad (3)$$

The G-<u>orbit</u> of m , G(m) , is the set of G-transforms of m . We easily compute

$$G_{g.m} = g G_m g^{-1} \quad (4)$$

The set of the isotropy groups of an orbit is a conjugation class of subgroups of G . There is a natural definition of equivalence of G-orbits : those with same isotropy groups. In a group action, a stratum is the union of all equivalent orbits e.g. the stratum S(m) of m is the set of all $m' \in M$ whose isotropy group $G_{m'} \in [G_m]$ the conjugation class of G_m . If $m' = g_2 \cdot m = g_1 \cdot m$, $g_2^{-1} g_1 \in G_m$, i.e. g_1 and g_2 belong to the same coset of G_m : $g_1 G_m = g_2 G_m$ and the orbit G(m) is equivalent to $[G:G_m]$ the set of left G-cosets of G_m with the G action by left translation : $g_2 \cdot g_1 G_m = (g_2 g_1) G_m$. When $G_m \triangleleft G$, i.e. G_m invariant subgroup of G ($\sim [G_m]$ has one element !), $[G:G_m]$ has also a natural group structure, that of the quotient group G/G_m . Finally we recall that $N_G(H)$ the normalizer of H is G is the largest G-subgroup which contains H as invariant subgroup. The points of the orbit G(m) with same isotropy group G_m form an orbit of $N_G(G_m)$.

<u>A n-dimensional crystallographic group</u> G (in crystallography one says a <u>space-group</u>) is a closed discrete subgroup of E(n) , the n-dimensional Euclidean group such that the orbit [E(n):G] be compact. So T_G , the translation subgroup of G is generated by n linearly independent vectors and is isomorphic to Z^n . The quotient $G/T_G = P$ is called a point group. It is a finite subgroup of O(n) . The conjugate class of P in O(n) is called a "<u>geometric class</u>". The macroscopic properties of a crystal depends only on its geometric class. The number of G·C. (Geometric class) and ICPG (isomorphic class of point groups) is given in table 1 for $n \leq 4$. They are respectively 32 and 18 for n = 3 .

The elements of T_G form a lattice in the vector space E_n . The set of lattice is an orbit [GL(n,R) : GL(n,Z)] (GL is for general linear). The strata of the action of O(n) on T (obtained by restriction of the GL(n,R) action) are one possi-

ble definition of <u>crystallographic systems</u> : there are 7 for n = 3 , namely triclinic,
monoclinic., rhombohedric ; tetragonal, trigonal, hexagonal, cubic . The corresponding
isotropy groups are called the <u>holohedries</u>, and denoted by P_H . An <u>arithmetic class</u>
is a free action of a point group P on a translation group T , i.e. it is an in-
jective homomorphism $P \xrightarrow{\Delta} \text{Aut } T = GL(n, Z)$. Therefore arithmetic classes corres-
pond to conjugation classes of finite subgroups of $GL(n, Z)$. It was a remarkable the-
orem by Jordan (1880) that the number of these classes is finite for any n : it is
73 for n = 3 . The <u>Bravais classes</u> of lattices correspond to the arithmetic classes
of the holohedries. There are 14 for n = 3 . As extensions of P by T with ac-
tion Δ , the space groups of an arithmetic class correspond to the elements of the
cohomology group $H^2_\Delta(P,T)$. Isomorphic extensions are the orbits of the normalizer
$N_{GL(n,Z)}(\Delta(P))$ in its natural action on $H^2_\Delta(P,T)$. The semi direct product $T_\Delta P$
corresponding to $0 \in H^2_\Delta(P,T)$ is called a symmorphic space group.

We give in Table 1 some crystallography statistics for $1 \leq n \leq 4$

Table 1	Crystallography statistics for $1 \leq n \leq 4$			
n =	1	2	3	4
Crystallographic Systems	1	4	7	33(7)
Geometric classes	2	10	32	227(44)
Bravais classes	1	5	14	64(10)
Arithmetic classes	2	13	73	710(70)
Isomorphic classes of spaces groups	2	17	219(11)	4783(112)

The numbers between bracket give the number of splitting into
enantiomorphic pairs

It is only for eight space groups that no crystal of that structure has been yet found [4]
Of course enantiomorphism classes always appear together in the same piece of material
(twining or macle). Do not believe that crystals of the pure chemical elements corres-
pond to symmorphic groups of holohedries. This is true only if there is one atom per
fundamental cell. Diamond, the metastable phase of carbon, has two, its point group,
0_h , is a holohedry , its space group, Fd3m   , is not symmorphic. In general mo-
lecular crystal space groups are expected far from symmorphic.

Since the development of solid state physics in the thirties, the use of the
unirreps (unitary irreducible linear representations) of the space group became more
and more necessary : labelling and structure of electronic levels and energy bands,
phonon dispersion relations, localized states, study of selection rules in any tran-
sition between quantum states ; see e.g. for applications Birman [5], Cracknell
[6] Bir and Pincus [7] ; these books contain a study of the unirreps. They are
also a necessary tool for the study of symmetry changes predicted by Landau theory
of second order phase transitions (see below).

In 1939 [8] , [9] began the study of these unirreps. One considers a
character k of the translation group T_G i.e. $k \in T^*_G$ the dual group of T_G ; in

physicist langage this is a wave vector of the Brillouin zone. Since $T_G^* = U(1)^3$, k
can be specified by three real parameters k_1 , k_2 , k_3 defined modulo 1 ("The wave
vector k is defined up to an element of the dual lattice of the translations"). Let
G_k and $P_k = G_k/T_G$ the isotropy groups of k in the actions of G and P on T_G^*.
(Remark that T_G acts trivially and also that G_k is a space group). The unirreps
of G coresponding to the orbit G(k) (in physics books one says "the star of k")
are obtained as induced representations $\Gamma_{G_k}^{(\alpha)} \uparrow G$ from the unirreps $\Gamma_{G_k}^{(\alpha)}$ of G_k .
The dimension of an unirrep satisfies

$$\dim \Gamma_{G_k}^{(\alpha)} \uparrow G = (\#G(k)) \cdot (\dim \Gamma_{G_k}^{(\alpha)}) \quad . \tag{5}$$

It is a divisor of 48 for three dimensional space groups. The image of $\Gamma_{G_k}^{(\alpha)} \uparrow G$ is a
finite group when the three k_i's are rational. Since there is an infinity of G-
orbits in T_G , every space group has an infinity of inequivalent unirreps. This has
not discouraged physicists to tabulate them ! [10] to [15] .

Most tables treat only a few thousands unirreps, those corresponding to "higher
symmetry k's ", i.e. those k's in the zero dimensional strata for the action of
P_H on T_G^* . These unirreps are the most useful for applications ; they have finite
image. These tables leave also the induction $\Gamma_{G_k}^{(\alpha)} \uparrow G$ to the user and give only the
unirreps Γ_{G_k} .

One could use for years tables of trigonometric functions without knowing
their geometrical meaning. This seems to be the situation in solid state physics for
the use of the nearly 4000 unirreps of the G_k whose k is of "higher symmetry".
The structure of these unirreps is interesting and they have only 37 unequivalent ima-
ges whose dimensions are :

dimension	1	2	3	4	6	
Number of images	7	20	6	3	1	(6)

To show it, Mozrzymas and I [16] noted that these images are homomorphic images
of the "extended little group $\tilde{P}_k$ " (already considered by some physicists, [17],
[18]). It is a central extension $\tilde{P}_k/\text{Im } k = P_k$. Remark that Im k and P_k depends
only on k and the geometrical class P . The corresponding $\tilde{P}_k$ of a "little space
group" G_k can be obtained inside each arithmetic class by the functorial map.

$$H_\Lambda^2(P_k,T) \xrightarrow{\tilde{k}} H_0^2(P_k,\text{Im}_k) \quad .$$

These P_k are direct products of an (eventually trivial) Abelian group by one of a
family of 26 finite groups whose order are :

order	6	8	12	16	24	32	48	96	
non Abelian fac-tors of $\tilde{P}_k$'s	1	2	2	6	4	5	4	2	(7)

Only the "allowed" images (i.e. those faithful for Im k) appears as the correspon-
ding images of the G_k .

      If a physical system S has a symmetry group H , all physical properties
of S should be described by invariants of H . If, in the physical theory, the ac-
tion of the symmetry group appears through a linear representation, the group inva-
riants depend only on the representation image. This shows the interest of the clas-
sification of the images of the space groups unirreps. As we have seen, these unirreps
are of finite dimension m . When the image is infinite, it is not closed in the group
U(m) . The physically relevant invariants are those of the closure Q of the image :
it is a compact subgroup of U(m) . There has been recent progress for computing the
algebra of invariants of compact or finite groups and this is very useful to physicists
[e.g. 19 to 22]. For a m dimensional compact image Q , G. Schwarz proved [23]
that invariant smooth functions are smooth functions of invariants polynomials. Since
Hilbert we know that the ring of invariant polynomials is finitely generated. It is
now known that it is a free module over a ring of polynomials i.e. every K-invariant
polynomial is of the form

$$p(x_1, x_2, \ldots, x_m) = \sum_{\alpha=0}^{\nu} \phi_\alpha \, Q_\alpha(\theta_1, \theta_2, \ldots, \theta_{m'}) \qquad m' \leq m \qquad (8)$$

where ϕ_α , θ_k are homogeneous polynomials of degree δ_α and d_i respectively ;
the θ_i are algebraically independent, the Q_α are arbitrary polynomials in m'
variables (m' = m in the particular case where K is finite) $\phi_0 = 1$, for each
$\alpha > 0$ there is a smallest integer $\nu_\alpha > 1$ such that $\phi_\alpha^{\nu_\alpha}$ is a polynomial in the
θ_k's and for $1 < n < \nu_\alpha$, ϕ_α^n is another ϕ_α . So the dimension μ_n of the vector
space of K-invariant homogeneous polynomials is given by the generating function

$$M(t) = \sum_{n=0}^{\infty} \mu_n \, t^n = \frac{1 + \sum_{\alpha=1}^{\nu} t^{\delta_\alpha}}{\prod_{i>1}^{m'} (1-t^{d_i})} \qquad (9)$$

When K is finite, M(t) was calculated by Molien [24]

$$M(t) = \frac{1}{|K|} \sum_{\Delta(g) \in K} \det(1 + t\Delta(g))^{-1} \qquad . \qquad (10)$$

For compact images, replace $\frac{1}{|K|} \Sigma$ by the Haar integral.

      There have been many generalizations of the crystallographic groups : magnetic
groups, black and white and color groups, matacrystallographic groups, etc. There is
no time to review them ; for a recent survey of references see [25] . I prefer to
mention with some details the most active fields in the last five years, where symme-
try has played and important role in condensed matter physics.

1. Modulated and incommensurate crystal structures

      Most crystals have defects (see 4, below) ; these must be neglected for defi-
ning a perfect crystal state and its symmetry group. However the crystal symmetry may
be more subtle in what is now called a modulated structure. This may occur in the most

common material, such as the historical example of Na_2CO_3 discovered in 1969 [26]. It was established in 1976 that, at ordinary temperature, this mineral has a incommensurate structure [27] ; so its symmetry cannot be described by a 3-dimensional space group.

Consider a crystal with a periodic deformation depending on a parameter λ , so that the point $\vec{n} = \sum_{i=1}^{3} n_i \vec{t}_i$ of the crystal lattice Λ is displaced into $\vec{r}_n(\lambda)$

$$\vec{n} \in \Lambda \quad , \quad \vec{r}_{\vec{n}}(\lambda) = \vec{n} + \vec{f}(\lambda) \sin(2\pi\vec{q}\cdot\vec{n}) \tag{11}$$

where $\vec{f}(\lambda)$ is continuous in λ with $\vec{f}(0) = 0$. If $\vec{q} \in \Lambda^*$ (the Brillouin zone) has rational coordinates $0 \leq q_i < 1$, $i = 1,2,3$, the function $\vec{r}_n(\lambda)$ is again invariant by a discrete translation group $(\sim Z^3)$, which is a subgroup of that of Λ . The smaller is $|\vec{q}|$, the larger is the modulation in the dual space direction. When λ diverges from zero (for Na_2CO_3, at temperature $-138°C$, (then $\vec{q} = (\frac{1}{6}, 0, \frac{1}{3})$) around the X-ray diffraction peaks $\vec{K}$ in the Bragg reflections, weak satellites appear at $\vec{K} + g\cdot\vec{q}$ where $g \in P$, the point group. Generally $\vec{q}$ is also a continuous function of temperature so the generic values of its coordinates are irrational, the deformed lattice is then called "incommensurate" and, stricto sensu, it has only a translation group $\sim Z^\nu$ of invariance, where ν is the number of (fixed) zero coordinates.

There are other types of incommensurate structures, for instance the nonstochiometric ones. They can be decomposed into two periodical subsystems whose ratio of characteristic lengths is a temperature function : e.g. $Fe_{1-\delta}S$, $Hg_{3-\delta}$ As F_6 where δ has a small value function of T . These composite structures may also be modulated (as is the case of the last compound). Now that the existence of incommensurate structures is known, physicists are looking for them and they are founding many.

What is the best way to describe them from the symmetry point of view ? The answer has been given by Janner and Jensen [28]. I can explain it by the following remark : it is true that the function $a \sin(2\pi px+r) + b \sin(2\pi qx+s)$ has no period when p/q is irrational, but its general behaviour is known when we note that it is the restriction on the line $y = x$ of a doubly periodic function on the two plane x , y , i.e. $a \sin(2\pi px+r) + b \sin(2\pi qy+s)$. Any function $\rho(\vec{r})$ describing a physical property of an incommensurate structure can be written with the Fourier decomposition :

$$\rho(\vec{r}) = \sum_{\vec{k}\in\Lambda^*} \sum_{\vec{q}\in\Lambda^d} \hat{\rho}(\vec{k},\vec{q}) e^{2\pi i(\vec{k}+\vec{q})\cdot\vec{r}} \tag{12}$$

It is invariant by a 3+d dimensional crystallographic group $(d \leq 3)$ acting separately on the $\vec{k}$'s and the $\vec{q}$'s and transforming $\Lambda^* \sim Z^3$ into itself. Equation (12) can indeed be interpreted as the projection of a 3+d dimensional crystal structure on the dual space ; this means that the incommensurate crystal can be considered as the intersection, by our 3 dimensional space, of a "super-crystal" in 3+d dimensions. Physicists are really interested by the classification of crystallographic groups in

4,5,6 dimensions, which are reducible on a 3 dimensional subspace.

2. Landau theory of second order phase transitions

Consider equation (11) again when $\vec{q}$ has rational coordinates in Λ^* and λ
diverges from zero. Then the crystal symmetry decreases from the space group G to
a subgroup H ; discontinuities in some physical properties of the crystal appear but
the volume, for instance, is a continuous function of λ . This is an example of a
second order phase transition. In 1937 Landau [29][30] gave a mathematical model for
them.

As we have noted, every physical function describing the crystal, for instance
F , the Gibbs free energy, is an invariant of its space group G . The thermodynamic
equilibrium is described by the lowest minimum of t when the temperature T , the
pressure p (and any external field) are fixed. In general this minimum is G-invariant,
but when the parameters change there can appear a set of critical values T_c, p_c
(generally a curve in the T,p plane) for which the minimum splits into an orbit
[G : H] of minima. Assume F to be a smooth function : by Morse Lemma and Schwarz
theorem [23],this will appear when a coefficient of the positive G-invariant quadra-
tic form vanishes over the space E of a real-irreducible representation of G . For
the splitting of the minimum to occur, the expansion of F on E must not contain
a third degree G-invariant polynomial ; this is the Landau criterion for second order
phase transition. (See below what happens when it is violated). If the subgroup H
is still a space group, there is a largest translation subgroup K of H which is an
invariant subgroup of G ; this subgroup K is subgroup of all isotropy groups of the
orbit of minima ; by linear completion of this orbit and from the irreducibility (on
the real) of the representation on E , one sees that K is the kernel of the re-
presentation ; its image Q = G/K is therefore finite. A more detailed analysis from
Lifschitz [31] for transitions from crystal is crystal imposes the rule : the antisym-
metric part of the tensor square of the unirrep must not contain the vector represen-
tation of G (i.e. the natural representation of the image P = G/T).

Both Landau and Lifschitz rules are in pretty good agreement with the known
corpus of experimental data. The Landau model is also able to describe transitions
to incommensurate structures ; in those cases the Lifschitz rule is always violated.
Most first order transitions occur by local collapses of the crystal structure and
formation of a new crystal (e.g. the well known tin pest which may occur in very cold
climate) ; for such transitions there is no a priori relations between the space groups
G_o and G_1 of the two phases. Other first order transitions are called "quasi second
order" because they can be described by the same Landau model with the Landau crite-
rion violated : at the critical T , p values, the minimum of F (which has been
raised and is no longer the lowest minimum) becomes a saddle point, through a zero
frequency mode , due to the presence of a small third degree term ; then the sys-

tem "falls" into the (not far) minimum essentially governed by the third degree term.

If the Landau model and its extensions have a good - but not complete - success for predicting second order - or quasi second order - transitions, it completely fails to predict the correct critical exponents. As suggested [32 to 36] this requires the use of renormalization group. This technique has been applied to the Landau model considered as a mean field theory. The absence of stable, non G-invariant fixed points explains in some cases why some expected second order transitions are instead of first order. [37] has made good predictions on the nature of the phase transition when some external constraint is applied. However it is still difficult to draw definitive conclusions on this topic.

3. Symmetry of mesomorphic phases

This subject is nearly one hundred years old since Lehmann discovered the first liquid crystal in 1885 and one of the basic papers was written by G. Friedel in 1923. These are states of condensed matter intermediary between liquid and crystals ; they might be just liquid with oriented molecules (nematics), they might be just so tiny micro-crystals that they behave like the molecules of a liquid (smectics B ?). There should be an a priori classification of their symmetry group as there were for crystals. Some partial attempts were made in [38 to 41].

As is known to the majority of this audience, G-invariant states of a C^* algebra can be decomposed into extremal states which are not in general G-invariant. This is one of the mathematical scheme for spontaneous symmetry breaking. Following Kastler et. al.[40] this decomposition can be ergodic or transitive. For the latter case this decomposition is into an integral over an orbit [G:H] carrying a finite G-invariant measure. For the Euclidean group E(3) they are the compact orbits. What are the corresponding closed subgroups H_α ? Let $T_\alpha = H_\alpha \cap R^3$ where R^3 is the translation subgroup of E(Z). Let $Q_\alpha = N_{E(3)}(T_\alpha) / R^3$, the quotient of the normalizer of T_α by the translations. The list of possible T_α is R^3, $R^2 \times Z$, $R \times Z^2$, Z^3 and exceptionnally R^2. The corresponding Q_α are easy to compute and the possible H_α are obtained as solutions of $H_\alpha \cap R^3 = T_\alpha < H_\alpha <_\square R^3_\square Q_\alpha$ ($\square$ = semi direct product). Let us denote by H_o the connected component of the identity of H and by

$$P_H = H/(T \cap H) \quad .$$

Table 2 gives the general classification of the mesomorphic phases. (See also [42],[43]).

TABLE 2   Mesomorphic phases ; their symmetry group H < E(3) .

α	$T_H = H \cap R^3$	$P_H = H/T_H$	H_0	Name	nb of group classes (cjg in Aff(3))
1	R^3	$D_{\infty h}$	$R^3_\square U(1)$	Nematics	"
"		F	R^3	e.g. biaxial nematics	∞ (= Finite < O(3))
2	$R^2 \times Z$	D_∞	$R^2_\square R$	Cholesterics (cheral)	"
"		$D_{\infty h}$	$R^2_\square U(1)$	Smectics A	"
"		F	R^2	Smectics C	∞ ⎫
"		F	R^2	Chiral Smectics C	∞ ⎬ to be classified
5	R^2	$Z_\square F$			⎭
3	$R \times Z^2$	F	R	Lyotrops Rod Lattices Discotics	80 Belov and Tarakhova
4	Z^3	F	1	Crystals	230 Fedorov, Schönflies

For the last 25 years the study of liquid crystal has become a very active topic of physics. The Landau theory of second order phase transitions can be applied to many cases ; e.g. the transition from smectics A to smectics C ; in the mono or bi-molecular layers of the smectics, the orientation of the molecules, instead to be orthogonal to the layers becomes tilted. For the last two years liquid crystals with new symmetry types have been looked for and found, especially triaxial nematics and discotics. There are many new phases, mainly among smectics, whose symmetry is not yet determined.

4. Topological classification of symmetry defects

Symmetry defects, mainly dislocations which are defects of the translation symmetry, play an important role in the properties of actual crystals. They have been very much studied for the last thirty years. Dislocations are classified by the Burger's vector. This is really a classification by a homotopy group [44] . Toulouse and Kléman [45] introduced a general scheme for a topological classification of defects in an ordered phase and, as an application, predicted that the line defects in the super-fluid phase A of ^{3}He would annihilate by pairs, just as they do in nematics. When the symmetry G (G is a gauge group, or E(3) for mesomorphic phases) is spontaneously broken into the subgroup H , topologically stable point defects, line defects, wall defects are classified by elements of the homotopy groups $\pi_n([G:H])$, n = 2,1,0 , respectively. [46] These results were also found by Volovik and Mineev [47]. This approach is very similar to the topological classification of t'Hooft-Polyakov monopoles by $\pi_2([G:H])$ where H is the stabilizer in the action of the gauge group on the asymptotic orientations of the Higgs field [48][49]. Instantons are characterized by an element of $\pi_3(G)$. Similarly there can exist defectless "knotted" configurations in nematics or cholesterics, classified by elements of $\pi_3([G:H])$ (see e.g. [50]).

The classification of defects for the mesomorphic phases is given in [42][43][47] . For general reviews of topological classification of symmetry defects, see e.g. [51] [52][53]. These topological methods have a validity for all domains of physics. For condensed matter physics they have been very illuminating. Of course, topology alone is not enough. It has to be blended now with energetic and dynamical studies. Continuous density of defects are now studied. It is not unrelated to group theory since gauge theory approach can be introduced. This might be the subject of a report at the next conference !

Conclusion

I hope that this fast survey has shown you the great variety of symmetries which appear in condensed matter physics. The study of these symmetries, their change by phase transition, their defects is partly an old topic and mainly a new lively field of research. There are some interesting problems in the different statistical mechanics models of strange materials (e.g. spin glasses [54]). Let me single out an amusing and a bit paradoxical approach for the study of amorphous materials. Are they like frozen liquids or do they have a lot of unseen regularities ? Kléman and Sadoc [55] have proposed a tentative description of the crystallography of amorphous solids. They considered them as regular crystals in 3 dimensional constant negative curvature (Lobatchewski) space with a density of defects due to their embedding in our Euclidean space !

R E F E R E N C E S

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