Classification of the symmetries of ordinary differential equations.

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1 Introduction.

How can we make such a classification? One has to choose a group G acting on the set of ODE (ordinary differential equations). The stabilizer $G_{\mathcal{E}}$ of the equation $\mathcal{E} = 0$ is its symmetry group. The equations of the orbit $G.\mathcal{E}$ have "same symmetry": their stabilizers are conjugated to $G_{\mathcal{E}}$. More generally, the union of the orbits with stabilizers conjugated to $G_{\mathcal{E}}$ form the *stratum* of ODE with symmetry $G_{\mathcal{E}}$ (up to a conjugation). Such a classification of ODE symmetries was implicitly carried by S. Lie at the end of the last century. The number of strata is infinite; but we shall explain how they can be obtained, summarizing Lie's work (only a small part of it is preserved in text books) and adding some recent contributions.

In the XIX^{th} century, analytic transformations:

$$x \mapsto X(x,y), \quad y \mapsto Y(x,y), \tag{1(1)}$$

were applied to ODE in order to bring them to a simpler form: when possible to a form whose solutions are known. The order of ODE is preserved by transformations 1(1). More general transformations were also considered, for instance to decrease the order of equations. Soon groups of transformations were considered: for instance LODE (linear ODE) are transformed into themselves by the subgroup G' depending on three functions f, h, s of x:

$$X = f(x), \quad Y = h(x)(y - s(x)). \tag{1(2)}$$

It was also well-known that LODE of order 1,2 form respectively a unique orbit of G': any such equation can be transformed into $y' = 0$, $y'' = 0$ respectively; however the corresponding transformations are *built from solutions* of the starting equations: indeed, for the second order inhomogeneous equation, choose one solution for $s(x)$ and $h(x) = u(x)^{-1}$, $f(x) = v(x)u(x)^{-1}$ where u, v are linearly independent solutions of the homogeneous equation.

For LODE's of order $n > 2$, from their coefficients, Laguerre, Brioschi, Halphen, Forsyth and others built G'-invariants (labelling the group orbits for small n's). LODE can be written with the coefficient c_n of the term $y^{(n)}$ equal to 1. It was well known that the coefficient c_{n-1} of the term $y^{(n-1)}$ can be made zero by the transformation $X = x, Y = y\exp(\frac{1}{n}\int_{x_0}^{x} c_{n-1}(t)dt)$. So any order n linear ordinary differential equation can be written:

$$\mathcal{E}_n \equiv y^{(n)} + \sum_{k=0}^{n-2} c_k(x)y^{(k)} = 0. \tag{1(3)}$$

In 1879, as a by-product of his study of invariants, Laguerre [1] showed the existence of a diffeomorphism, which does not require the knowledge of the solutions of 1(3), and transforms to 0 the coefficient c_{n-2} in equation 1(3). Indeed, in 1(2) make $s(x) = 0$ and define $f(x)$, $h(x)$ from the function $\theta(x)$, a solution of the equation:

$$\binom{n+1}{3}\theta'' + c_{n-2}\theta = 0, \; f' = \theta^{-2}, \; h = \theta^{1-n}. \tag{1(4)}$$

Beware that the set of LODE with *constant* coefficients is not stable under such a transformation.

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2 Lie's work on the symmetry of differential equations.

Soon after, Lie published a series of four papers classifying all ODE's with a given symmetry group [2]. His aim was to extend to differential equations the Galois theory of polynomial equations. This extension cannot be completely carried through but this is still presently a topic in mathematics. However Lie obtained many results, thanks to his powerful differential method using Lie algebras!

Transformations 1(1) form $\mathrm{Diff}_2(x,y)$, the group [1] of analytic diffeomorphisms acting on the 2-dimension space of the complex variables x, y. This group action can be extended to the functions $y(x)$ and their derivatives. The Lie algebra $\mathcal{D}\mathrm{iff}_n$ of Diff_n is the algebra of vector fields $\sum \alpha^i(x^1,\ldots,x^n)\,\partial_i$, ($\partial_i$ is a short for $\frac{\partial}{\partial x^i}$) whose Lie bracket is:

$$[\sum_i \alpha^i\,\partial_i, \sum_j \beta^j\,\partial_j] = \sum_{ij}\left(\alpha^i(\partial_i\beta^j) - \beta^i(\partial_i\alpha^j)\right)\partial_j\,. \qquad 2(1)$$

The action of the Lie algebra $\mathcal{D}\mathrm{iff}_2(x,y)$ on the successive derivatives $y', y'', \ldots, y^{(n)}$ of the function $y(x)$ is given by the Lie algebra homomorphisms

$$\mathcal{D}\mathrm{iff}_2(x,y) \overset{pr_n}{\to} \mathcal{D}\mathrm{iff}_{n+2}(x,y,y_1,y_2,\ldots,y_n) \qquad 2(2)$$

where $y_1, y_2, y_3, \ldots, y_k, \ldots$ are the derivative of $y(x) \equiv y_0$ considered as independent variables. Given a vector field

$$\bar{v} = \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y \qquad 2(3)$$

$\mathrm{pr}_n(\bar{v})$ is called the n^{th} prolongation of $\bar{v}$.

Consider a basis $\{\bar{v}_\alpha\}$, $1 \le \alpha \le m$ of a m-dimensional subalgebra $\mathcal{G} \subset \mathcal{D}\mathrm{iff}_2(x,y)$. This algebra $\mathcal{G}$ is a symmetry algebra of the order n equation

$$\mathcal{E}(x,y,y_1,y_2,\ldots,y_n) = 0 \qquad 2(4)$$

if the $n+2$ variable function $\mathcal{E}$ satisfies the system of m linear partial differential equations [2]:

$$\mathrm{pr}_n(\bar{v}_\alpha)\mathcal{E} = \lambda(x,y,y_1,\ldots,y_n)\mathcal{E} = 0. \qquad 2(5)$$

Since the $\mathrm{pr}_n(\bar{v}_\alpha)$'s form a basis of a Lie subagebra of $\mathcal{D}\mathrm{iff}_{n+2}(x,y,y_1,\ldots,y_n)$ of dimension n this system is integrable. We will give more details in the next section. Lie extended his theory to partial differential equations and considered also the symmetry of differential equations under infinitesimal contact transformations:

$$\mathcal{D}\mathrm{iff}_3(x,y,y_1) \overset{c_n}{\to} Diff_{n+2}(x,y,y_1,y_2,\ldots,y_n) \qquad 2(6)$$

so useful in classical mechanics and used for years before in their integral form. Lie theory is explained in many textbooks: recent ones are [3] [4] [5]. They deal more with partial differential equations and, mainly [5], with generalisations of the symmetry. Except for the beginning of the next section, none of the material of this lecture can be found in the text books [3]. About

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1 In his contribution to these proceedings, A. Kirillov explains the existence of different such groups, with different topologies. Moreover one needs that the functions $X(x,y), Y(x,y)$ be defined only locally, around a regular point of the coefficients of the ODE. So one should consider pseudo-groups or, simply, local Lie groups. For lack of time, these technical points (ignored by Lie) cannot be discussed here.

2 We consider $\lambda(x,y,y_1,\ldots,y_n)\mathcal{E} = 0$ and $\mathcal{E} = 0$ as the same equation; using 2(4) we can write the second members of the equations 2(5) as 0.

3 and also in the lectures given in previous ICGTMP by Hamermesh, Olver, Winternitz.

ten years later Lie proved [6] that the dimension d of the symmetry algebra of an ordinary differential equation of order n satisfies:

$$d \leq n+4 \text{ for } n > 2, \quad d \leq 8 \text{ for } n = 2 \tag{2(7)}$$

(d is infinite for $n = 1$). For the order n, the maximal dimension of the symmetry algebra is reached by the equation $y^{(n)} = 0$. During the same period Lie published [7] the list $\mathcal{L}$ of equivalence classes (under the adjoint action of Diff_2) of the finite dimensional Lie subalgebras of $\mathcal{D}\mathrm{iff}_2$. Implicitly this yields the classification of symmetries of ODE since we have the list $\mathcal{L}$ of strata and, as explained in the next section, for each finite dimensional subalgebra $\mathcal{G} \subset \mathcal{D}\mathrm{iff}_2$ one can find all ODE invariant by $\mathcal{G}$.

3 ODE with a given symmetry algebra.

Equation 2(5) gives the system of m linear partial differential equations that must satisfy an order n ordinary differential equation $\mathcal{E} = 0$ which has the m dimensional subalgebra $\mathcal{G} \subset \mathcal{D}\mathrm{iff}_2$ as symmetry algebra. We need now to give the explicit form of the n^{th} prolongation of a vector field $\bar{v}$ (see [2]-X) or any text book on the subject:

$$pr_n(\bar{v}) = \xi\,\partial_x + \sum_{k=0}^{n} \eta^{[k]} \partial_{y_k}, \text{ with } \eta^{[k]} = \left(\frac{d}{dx}\right)^k (\eta - y_1\xi) + y_{(k+1)}\xi, \tag{3(1)}$$

where $\frac{d}{dx}$ is the total derivative: for a function $\varphi(x, y)$

$$\frac{d}{dx}\varphi = \varphi_x + y_1\varphi_y. \tag{3(2)}$$

One verifies that the η's satisfy the recursion relation:

$$\eta^{[k]} = \frac{d}{dx}\eta^{[k-1]} - y_k \frac{d}{dx}\xi. \tag{3(3)}$$

We give here the first two η's:

$$\eta^{[1]} = \eta_x + y_1(\eta_y - \xi_x) - y_1^2\xi_y, \tag{3(3)}$$

$$\eta^{[2]} = \eta_{xx} + y_1(2\eta_{xy} - \xi_{xx}) + y_1^2(\eta_{yy} - 2\xi_{xy}) - y_1^3\xi_{yy} + y_2(\eta_y - 2\xi_x) - 3y_1y_2\xi_y. \tag{3(4)}$$

Equation 3(1) shows that $\eta^{[k]}$ is a polynomial in $y_i, 1 \leq i \leq k$; indeed the terms in y_{k+1} cancel. The number of terms of $\eta^{[k]}$ increases very fast with k (17,29,47 for $k = 3, 4, 5$) and there are computer programs for determining them. However some families of their coefficient can be given in a compact form. Writing the polynomial as:

$$\eta^{[k]} = P_0^k + \sum_{p_1,p_2,\ldots,p_\ell} P^k_{1^{p_1}2^{p_2}\ldots\ell^{p_\ell}} y_1^{p_1} y_2^{p_2} \ldots y_\ell^{p_\ell} \text{ with } \ell \leq k, \ \prod_j jp_j \leq k+1, \tag{3(5)}$$

we have found for instance :

$$2 \leq \ell \leq k > 3, \ P^k_{1^p\ell} = \binom{\ell+p}{p}\binom{k}{\ell+p-1}\left(\frac{k+1-\ell-p}{\ell+p}\,\eta_{x^{k-\ell-p}\,y^{p+1}} - \xi_{x^{k+1-\ell-p}\,y^p}\right). \tag{3(6)}$$

(see also [8] equation (10) for other general terms).

For a Lie subalgebra of $\mathcal{D}\mathrm{iff}_n$ one has to distinguish between its usual dimension (linear rank of its elements for linear combinations with constant coefficients) and its functional dimension (linear rank of its elements for linear combinations with function coefficients), which is its dimension as a Lie algebra over the ring of functions. From the theory of systems of linear partial differential equations one knows that the most general solution of 2(5) depends of $n + 2 - m'$

arbitrary functions, where m' is the functional dimension of the Lie algebra $\mathrm{pr}_n(\mathcal{G})$. Assume $m' = m = n+2$; then a $\mathcal{G}$-invariant differential equation is given by the Lie determinant ([2]-X,p. 245):

$$\Delta \equiv \begin{pmatrix} \xi_1\,\partial_x\mathcal{E} & \eta_1\,\partial_y\mathcal{E} & \eta_1^{[1]}\,\partial_{y_1}\mathcal{E} & \dots & \eta_1^{[n-1]}\,\partial_{y_{n-1}}\mathcal{E} & \eta_1^{[n]}\,\partial_{y_n}\mathcal{E} \\ \xi_2\,\partial_x\mathcal{E} & \eta_2\,\partial_y\mathcal{E} & \eta_2^{[1]}\,\partial_{y_1}\mathcal{E} & \dots & \eta_2^{[n-1]}\,\partial_{y_{n-1}}\mathcal{E} & \eta_2^{[n]}\,\partial_{y_n}\mathcal{E} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \xi_{n+1}\,\partial_x\mathcal{E} & \eta_{n+1}\,\partial_y\mathcal{E} & \eta_{n+1}^{[1]}\,\partial_{y_1}\mathcal{E} & \dots & \eta_{n+1}^{[n-1]}\,\partial_{y_{n-1}}\mathcal{E} & \eta_{n+1}^{[n]}\,\partial_{y_n}\mathcal{E} \\ \xi_{n+2}\,\partial_x\mathcal{E} & \eta_{n+2}\,\partial_y\mathcal{E} & \eta_{n+2}^{[1]}\,\partial_{y_1}\mathcal{E} & \dots & \eta_{n+2}^{[n-1]}\,\partial_{y_{n-1}}\mathcal{E} & \eta_{n+2}^{[n]}\,\partial_{y_n}\mathcal{E} \end{pmatrix} = 0. \qquad 3(7)$$

For $n = m' - 2 + k,\ k > 1$ Lie denotes each solution which appears for increasing values of n, by $\varphi_1(x, y, y_1, \dots, y_{m'-1}),\ \varphi_2(x, y, y_1, \dots, y_{m'}), \dots,\ \varphi_k(x, y, y_1, \dots, y_{m'-2+k})$.

From the recursion relation 3(3) one can prove [5], eq.(2.92):

$$\frac{d}{dx}(pr_n(v)\phi) = \big(\frac{d\xi}{dx} + pr_{n+1}(v)\big)\frac{d\phi}{dx} \qquad 3(8)$$

So, when $\frac{d\xi}{dx} = 0$ one can choose $\varphi_2 = \frac{d}{dx}\varphi_1, \dots, \varphi_k = \frac{d}{dx}\varphi_{k-1}$. In the general case [4]:

$$\varphi_3 = \frac{d\varphi_2}{dx}(\frac{d\varphi_1}{dx})^{-1}, \dots, \varphi_k = \frac{d\varphi_{k-1}}{dx}(\frac{d\varphi_1}{dx})^{-1}. \qquad 3(9)$$

This means that with the solutions of the system 2(5) for $k = 2$, one can build up to an arbitrary order the general form of the ODE with a given symmetry algebra. Lie gave many examples; we give here three of them [5] .

$\mathcal{G} = \{\partial_x, x\,\partial_x, x^2\,\partial_x, \partial_y, y\,\partial_y, y^2\,\partial_y\} \sim SL_2 \times SL_2$; then $m' = m = 6$ and the Lie determinant gives the third order invariant equation:

$$\mathcal{E} \equiv y_1 y_3 - \frac{3}{2}y_2^2 = 0, \qquad 3(10)$$

Beside this equation, all $\mathcal{G}$-invariant ODE of order $n \geq 5$ are of the form $\Omega_n = 0$, with Ω_n an arbitrary functions of $\varphi_k,\ 1 \leq k \leq n-4$; with the notation $'$ for the total derivative [6]:

$$s = (y_1)^{-1}\mathcal{E}, \quad \varphi_1 = (4ss'' - 5s'^2)s^{-3}, \quad \varphi_2 = (4s^2s''' - 18ss's'' + 15s'^3)s^{-9/2}; \qquad 3(11)$$

the other φ_k's are computed according to 3(9).

$\mathcal{G} = \{\partial_x, \partial_y, x\,\partial_x, y\,\partial_y, y\,\partial_x, x\,\partial_y, xD, yD\}$ with $D = x\,\partial_x + y\,\partial_y$; so $\mathcal{G} \sim SL_3$. Then

$$\Delta \equiv -2y_2u^2 = 0 \text{ with } u = 9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 \qquad 3(12)$$

yields two invariant equations: $y'' = 0$ and a fifth order one: $u = 0$. The expressions of φ_i, $i = 1, 2$ given by Lie would take one page [7] ! So we skip them.

$\mathcal{G} = \{u_\alpha\,\partial_y\}, 1 \leq \alpha \leq m > 2$, with the u_α linearly independent, i.e. the Wronskian $w(u_\alpha) \neq 0$. So $\mathcal{G}$ is isomorphic to a m-dimensional Abelian Lie algebra. Then $m' = m-1$, which implies that the Lie determinant vanish identically. The u_α's are solutions of the linear differential equation

$$w_0 = 0, \qquad 3(13)$$

[4] LIE [2]-X,p. 247 obtained heuristically relation 3(9): in his very short proof, one step is incorrect.

[5] Their choice will become clear in the two next sections.

[6] This example is in [2]I,section 2.12, but with the Lie's implicit remark in [2]II-1.9 that the expressions he introduced are successsive total derivatives of m.

[7] Using total derivatives, probably they could be greatly simplified.

where:

$$w_i = \begin{pmatrix} u_1 & u_1' & u_1'' & \dots & u_1^{(m-1)} & u_1^{(m+i)} \\ u_2 & u_2' & u_2'' & \dots & u_2^{(m-1)} & u_2^{(m+i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_m & u_m' & u_m'' & \dots & u_m^{(m-1)} & u_m^{(m+i)} \\ y & y' & y'' & \dots & y^{(m-1)} & y^{(m+i)} \end{pmatrix}. \qquad 3(14)$$

So the ODE of order $m+k$, invariant by $\mathcal{G}$, are of the form [8]

$$\Omega_{m+k}(x, w_0, w_1, \dots, w_k) = 0. \qquad 3(15)$$

As a trivial application of this powerful method, for the one dimensional symmetry algebras generated by ∂_x (respectively, ∂_y, $x\,\partial_y$) we obtain the obvious result: the invariant ODE $\mathcal{E} = 0$ do not depend explicitly on x(resp. y, y'), i.e. $\partial_x \mathcal{E} = 0$, (resp. $\partial_y \mathcal{E} = 0$, $\partial_{y'} \mathcal{E} = 0$. Since all one dimensional subalgebras of vector fields are equivalent, any ODE with a non trivial symmetry algebra can be put under one of these forms.

4 Symmetry algebra of an ODE; examples of LODE.

The previous section summarizes Lie's paper [2]I; the three other papers were essentially devoted to the (eventually partial) integration of ODE with given symmetry algebra. The paper [2]-I also gives implicitly a method for finding *the* symmetry algebra [9] $\mathcal{G}_\mathcal{E}$ of a differential equation $\mathcal{E} = 0$. This technique is explained in detail in text books and powerful (interactive) computer programs have been developed for solving this problem, although an explicit expression for $\mathcal{G}_\mathcal{E}$ often requires solving the equation! Lie knew that the symmetry algebras of the LODE $y^{(n)} = 0$ are respectively, for $n = 1$, $\mathcal{G} = \{\xi(x,y)\,\partial_x + \eta(y)\,\partial_y\}$, for $n = 2$, $\mathcal{G} = \mathcal{S}L_3$ given [10] in section 3, for $n > 2$, $\mathcal{G} = \mathcal{A}_n\big(1, x, x^2, \dots, x^{(n-1)}\big) \rtimes \mathcal{G}L_2$, the center $\lambda y\,\partial_y$ of the general linear algebra acts on $\mathcal{A}_n$ by dilatation while its simple subalgebra $\mathcal{S}L_2(\partial_x, 2x\,\partial_x + (n-1)y\,\partial_y, -x^2\,\partial_x - (n-1)xy\,\partial_y)$ acts through its (unique up to equivalence) n-dimensional irreducible representation. It seems that a general study of the symmetry of order $n > 2$ LODE has been done only recently [8] [9]. Up to an isomorphism the subalgebra $\mathcal{A}_n \rtimes \mathcal{A}_1(y\,\partial_y)$ belongs to the symmetry algebra of all LODE. This is easy to understand: given an order n linear differential equation $\mathcal{E}_n = 0$, we know that its solutions form a n-dimensional vector space, so the set of solutions $\{u\}$ (and therefore the equation) is invariant under the translations by the solutions: infinitesimally they are represented by the n-dimensional Abelian Lie algebra of vector fields $\mathcal{A}_n = \{u\,\partial_y\}$. Since the equation is homogeneous in y, the symmetry algebra contains also $\mathcal{A}_1(y\,\partial_y)$. We will now characterize the LODE with a larger symmetry algebra.

Without loss of generality we write the order n linear differential equation in the form 1(3). To belong to its symmetry algebra, a vector field $\bar{v}$ must satisfy the equation:

$$\mathrm{pr}(\bar{v})\mathcal{E}_n \equiv \eta^{[n]} + \sum_{k=0}^{n-2}(c_k \eta^{[k]} + c_k' \xi y^{(k)}) = 0. \qquad 4(1)$$

As we have seen, this is a polynomial in the independent variables $y_k = y^{(k)}$, $1 \le k \le n$; to be zero, all coefficients of its independent monomials must vanish. For instance, from the terms in

8 Note that for all vector fields $\xi = 0$ and that up to terms in w_0, the successive derivatives of w_0 are the w_i's.

9 That is the stabilizer in $\mathcal{D}\mathrm{iff}_2$ of the equation.

10 Indeed $\mathcal{S}L_3$ is the projective linear algebra acting on the two dimensional linear manifold of solutions; here it acts locally. We recalled in the introduction that all order 2 LODE are on the same orbit of Diff_2.

$y_2 y_{n-1}$ and $y_1 y_{n-1}$ we obtain $\xi_y = 0$ and $\eta_{yy} - n\xi_{xy} = 0$, so $\eta_{yy} = 0$. With these results we can write explicitly [11] the non vanishing terms of $\eta^{[k]}$:

$$0 < k,\ \xi_y = 0 = \eta_{yy} \Rightarrow \eta^{[k]} = \eta_{x^k} + \sum_{\ell=1}^{k} y_l \Big(\binom{k}{\ell} \eta_{yx^{k-\ell}} - \binom{k}{\ell-1} \xi_{x^{k+1-\ell}} \Big). \qquad 4(2)$$

Moreover we know a general form for ξ, η:

$$\xi = f(x), \quad \eta = g(x)y + u(x). \qquad 4(3)$$

Using the expression 4(2) for $\eta^{[k]}$, replacing y_n by its value in $\mathcal{E}_n = 0$, interchanging the two sums on k, ℓ and separating the term in y_{n-1}, equation 4(1) becomes:

$$n y_{n-1}(\eta_{xy} - \tfrac{n-1}{2} \xi_{xx}) + \sum_{\ell=0}^{n-2} y_\ell(-c_\ell(\eta_y - \xi_x) + c'_\ell \xi +$$

$$+ \sum_{\ell=1}^{n-2} y_\ell \sum_{k=\ell}^{n} c_k \Big(\binom{k}{\ell} \eta_{yx^{k-\ell}} - \binom{k}{\ell-1} \xi_{x^{k+1-\ell}} \Big) + \sum_{\ell=0}^{n} c_k \eta_{x^k} = 0. \qquad 4(4)$$

The term in y_{n-1} yields:

$$g' = \tfrac{n-1}{2} f'', \text{ i.e. } g = \tfrac{n-1}{2} f' + K. \qquad 4(5)$$

So the functions f, u in 4(3) and the constant K in 4(5) are the only unknowns. The equations they satisfy are obtained directly from 4(4):

$$\ell = n-2: \ \binom{n+1}{3} f''' + 4c_{n-2} f' + 2c'_{n-2} f = 0 \qquad 4(6)$$

$$1 \le \ell \le n-3: \ (n-1-\ell)\binom{n+1}{\ell} f^{(n+1-\ell)} + \sum_{k=\ell+1}^{n-2} c_k\Big((n-1)\binom{k}{\ell} - \binom{k}{\ell-1}\Big) f^{(k+1-\ell)} +$$

$$+ 2c_\ell(n-\ell) f' + 2c'_\ell f = 0 \qquad 4(7)$$

$$\sum_{k=1}^{n} (n-1) c_k f^{(k+1)} + 2nc_0 f' + 2c'_0 f = 0, \quad \mathcal{E}(u) = 0. \qquad 4(8-9)$$

Equation 4(9) corresponds to the Abelian algebra $\mathcal{A}_n(\{u\})$ and no condition is imposed on the constant K: this corresponds to the algebra $\mathcal{A}_1(y\,\partial_y)$. The symmetry algebra of the linear equation $\mathcal{E}_n = 0$ is strictly larger iff (=if and only if) the $n-1$ equations 4(6-7-8) have at least one common solution. We verify that this is the case when the equation $\mathcal{E} = 0$ has constant coefficients: then $f =$ constant is a solution. To study the general case we can assume that the Laguerre transformation 1(4) has been performed. Then equation 4(6) reduces to $f''' = 0$, so the equations 4(7-8) have a common solution with it iff the following simpler system of $n-1$ equations

$$f''' = 0,\ 0 \le \ell \le n-3,\ (l+1)(n-1-\ell) c_{\ell+1} f'' + 2(n-\ell) c_\ell f' + 2c'_\ell f = 0 \qquad 4(10)$$

has non trivial solutions. The first equation is of order three, that with $\ell = n-3$ is simply:

$$3c_{n-3} f' + c'_{n-3} f = 0, \qquad 4(11)$$

11 Use 3(1), begin by a partial derivative ∂_y on η, and continue with the condition to never use ∂_y on ξ or again on η.

and the other are at most of order two (depending on the values of their coefficients). So the system has three linearly independent solutions (the set of polynomials in x of degree ≤ 2), iff all c_ℓ vanish:

Theorem *The symmetry algebra of an order $n > 2$ linear differential equation has dimension $n+4$ iff by the Laguerre transformation the equation reduces to $y^{(n)} = 0$. Then the symmetry algebra is $\mathcal{A}_n \rtimes \mathcal{G}L_2$.*

This theorem has been stated in [9] in an equivalent form. In [8] we obtained a similar theorem with the condition that the order n equation be iterative, that is of the form:

$$\mathcal{E} = L^n[y] = L^{n-1}\left[L[y]\right] = 0 \text{ with } L[y] \equiv r(x)y' + q(x)y. \qquad 4(12)$$

Reference [9] gives a similar structure for the equation, but in the form 1(3):

$$n \text{ odd}: \ \frac{d}{dx}\Big(\prod_{k=1}^{(n-1)/2}\Big(\binom{n+1}{3}\frac{d^2}{dx^2} + (2k)^2 c_{n-2}\Big)\Big)y = 0 \qquad 4(13')$$

$$n \text{ even}: \ \Big(\prod_{k=1}^{n/2}\Big(\binom{n+1}{3}\frac{d^2}{dx^2} + (2k-1)^2 c_{n-2}\Big)\Big)y = 0. \qquad 4(13")$$

Iterative equations 4(12) have a set of linear independent solutions of the form $u^{n-1-k}v^k, 0 \leq k \leq n-1$; this is a well known form for building the irreducible representations of SL_2 by tensor symmetric power from the two dimensional one with basis u, v.

If after the Laguerre transformation the coefficients are not all zero, let $k \leq n-3$ be the largest index of the non-vanishing ones; 4(10) shows that the equation for $\ell = k$ is a first order equation in f whose solutions satisfy $f = L_k c_k^{1/(k-n)}$ where L_k is a constant and, moreover, f a polynomial of degree ≤ 2. This shows that the dimension of the symmetry algebra of an order n LODE is either $n+4$ or $n+2$ or $n+1$, but [12] it cannot be $n+3$. It is $n+2$ iff, for the equation in the Laguerre form, its coefficients satisfy the non trivial equations of 4(10); explicitly:

$$f = Ax^2 + Bx + C, \quad k < \ell \leq n-3, c_\ell = 0, \quad c_k = K_k f^{k-n},$$

$$0 \leq \ell < k, \ c_\ell = f^{\ell-n}\Big(K_\ell + A(\ell+1)(n-\ell-1)\int_0^x c_{\ell+1}(t) f^{n-\ell-1}(t)dt\Big). \qquad 4(14)$$

We can prove here a simpler

Theorem *If the symmetry algebra of an order n linear ordinary differential equation has dimension n+2, this equation can be transformed into a linear differential equation with constant coefficients.*

We have already proven (see 4(3-4)) that the symmetry algebra has the structure: $\mathcal{A}_n(u(x)\,\partial_y) \rtimes \mathcal{A}_2(y\,\partial_y, v)$ with $v = f(x)\,\partial_x + \frac{n-1}{2} f' y\,\partial_y$ which normalizes the subalgebra $\mathcal{A}_n(u(x)\,\partial_y)$ where the u's are the solutions of the equation. The linear ordinary differential equation 1(3) with this symmetry algebra is transformed by the diffeomorphism of type 1(2):

$$X = \int_0^x f(t)^{-1}dt, \ Y = yr(x), \text{ with } r = f^{-(n-1)/2} \qquad 4(15)$$

into another one with symmetry algebra $\mathcal{G} = \mathcal{A}_n\big(u(\varphi(X))r(\varphi(X))\,\partial_Y\big) \rtimes \mathcal{A}_2(Y\,\partial_Y, \partial_X)$ where $x = \varphi(X)$ is the inverse function of $X(x)$ in 4(15). In the symmetry algebra the term in ∂_X shows

[12] This result is given in [9]. Although the contrary has sometimes be stated, a non linearizable differential equation of order n may have a symmetry algebra of dimension $n+3$: this is the case for the equation $u = 0$ with $n = 5$ in 3(12).

that the new linear equation has constant coefficients. In $\mathcal{D}\mathrm{iff}_2$ there are many non isomorphic subalgebras $\mathcal{G}$ as defined above: by a suitable basis in $\mathcal{A}_n$ the linear map induced on it by ∂_X can be put in the form of a Jordan matrix whose trace can be removed by a suitable combination with the multiple of the identity operator $Y\,\partial_Y$. So the generic family has $n-1$ parameters of isomorphic classes.

To summarize: there is an infinity of strata for the LODE of order $n > 2$, and equivalently an infinity of Lie subalgebras of $\mathcal{D}\mathrm{iff}_2$ containing a $\mathcal{A}_n$ algebra. For the Lie algebras $\mathcal{A}_n$ and $\mathcal{A}_n \rtimes \mathcal{A}_1$, this last factor acting by dilation on the first one, the equivalent classes are labelled by $n-2$ arbitrary functions, and this is also true of the strata of the generic LODE: the functions are their coefficients in the Laguerre form. There are $n-1$ parameter families of order n LODE strata with a $n+2$ dimensional symmetry algebra. Finally, as we have seen, the order n LODE with $n+4$ dimensional symmetry algebras form a unique orbit (and stratum) of Diff_2; their symmetry algebra class: $\mathcal{W}_n = \mathcal{A}_n \rtimes \mathcal{G}L_2$ are *maximal finite dimensional subalgebras* of $\mathcal{D}\mathrm{iff}_2$.

5 The equivalence classes of finite dimensional algebras of $\mathcal{D}\mathrm{iff}_2$.

As we explained at the end of section 2, the publication by Lie [7] of the list $\mathcal{L}$ of equivalence classes (under the adjoint action of Diff_2) of the finite dimensional Lie subalgebras of $\mathcal{D}\mathrm{iff}_2$ completed implicitly the problem of the classification of symmetries of ODE since we have the list $\mathcal{L}$ of strata and, as explained in section 3, for each finite dimensional subalgebra $\mathcal{G} \subset \mathcal{D}\mathrm{iff}_2$ one can find all $\mathcal{G}$-invariant ODE. Classification of finite dimensional Lie algebras was just beginning, but Lie used very cleverly the concept of primitive and imprimitive actions (the latter transform a given family of curves into themselves) of equivalence classes of finite subgroups of Diff_2 on the plane x, y. Indeed these concepts are very relevant to the problem. As we have already seen from some examples in the preceding section, this list $\mathcal{L}$ is infinite. Remark that $\mathcal{L}$ is a partially ordered set (by inclusion of subalgebras up to conjugation in Diff_2). Of course, with the results we now know on the structure and classification of finite dimensional Lie algebras, this list can be obtained faster [13]. This will be done in the companion summer school (at Rachov, Ukraine) and will be published in its proceedings. Here we just give the essential results; most of them are given in tables 1,2 and diagram 3.

ad d	$\begin{pmatrix}1 & 0\\ 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$
type	$\mathcal{S}^{\lambda}_{2,1}$	$\mathcal{S}^{J}_{2,1}$	$\mathcal{S}^{0}_{2,1}$	$\mathcal{N}_{2,1}$	$\mathcal{A}_3$
$[a \wedge b] = 0$	$[d \wedge a] = a$	$[d \wedge a] = a$	$[d \wedge a] = a$	$[d \wedge a] = 0$	$[d \wedge a] = 0$
	$[d \wedge b] = \lambda b$	$[d \wedge b] = a + b$	$[d \wedge b] = 0$	$[d \wedge b] = a$	$[d \wedge b] = 0$
$\dim \mathcal{G}'$	2	2	1	1	0
$\dim \mathcal{C}(\mathcal{G})$	0	0	1	1	3

Table 1 Types (=Isomorphism classes) of non simple 3-dimensional Lie algebras.

The symbols $\mathcal{S}, \mathcal{N}$ means respectively solvable, nilpotent Lie algebras; their first index is the dimension of the maximal Abelian ideal and the second is the dimension of the corresponding quotient. There is an infinity of types $\mathcal{S}^{\lambda}_{2,1} \sim \mathcal{S}^{1/\lambda}_{2,1}$ so we assume here $0 < |\lambda| \leq 1$. $\mathcal{N}_{2,1}$ is the Heisenberg algebra of center a.

[13] Also the results can be given with greater precision. Instead of giving the list $\mathcal{L}$, Lie gives examples of subalgebras, he is sometimes redundant and it is not always obvious how to construct the partial ordering of $\mathcal{L}$.

There are (29 + three one parameter families) equivalence classes of finite dimensional $\mathcal{D}\mathrm{iff}_2$ subalgebras which do not contain a three dimensional Abelian algebras; these classes are those of the subalgebras of $SL_2 \times SL_2$ and SL_3. The two corresponding classes $[SL_2 \times SL_2]$ and $[SL_3]$ are maximal in $\mathcal{L}$. The partially ordered set of these equivalence classes is given in diagram 3. The notations are explained in the previous tables.

$[\mathcal{G}]$	$[\mathcal{A}_2]^+$	$[\mathcal{A}_2]^-$	$[\mathcal{S}_{1,1}]^+$	$[\mathcal{S}_{1,1}]^-$	b, h	c, d, e_+	centralizer $\mathcal{C}$	normalizer $\mathcal{N}$
$[\mathcal{A}_3]^{+\phi}$	∞				$x\,\partial_y$	$\phi(x)\,\partial_y,\ \phi''(x) \neq 0$	$\mathcal{A}_\infty(\eta(x)\,\partial_y)$	variable
$[\mathcal{S}^{\lambda}_{2,1}]^+$	1			∞	$x\,\partial_y$	$-D + \lambda x\,\partial_x$	$\mathcal{A}_1(x^{(1-\lambda)^{-1}}\,\partial_y)$	$\mathcal{A}_3 \rtimes \mathcal{A}_2^-$
$[\mathcal{S}^{0}_{2,1}]^+$	1	∞		∞	$x\,\partial_y$	$-D$	$\mathcal{A}_1(x\,\partial_y)^c$	$[\mathcal{S}^-_{1,1} \oplus \mathcal{S}^-_{1,1}]$
$[\mathcal{S}^{1}_{2,1}]^+$	1		∞		$x\,\partial_y$	$-y\,\partial_y$	0	$[\mathcal{B}_3]$
$[\mathcal{S}^{J}_{2,1}]^+$	1			∞	$x\,\partial_y$	$\partial_x - y\,\partial_y$	$\mathcal{A}_1(e^{-x}\,\partial_y)$	$\mathcal{A}_3 \rtimes \mathcal{A}_2^-$
$[\mathcal{S}^{\lambda}_{2,1}]^-$		1		∞	∂_x	$-(\lambda x\,\partial_x + y\,\partial_y)$	0	$[\mathcal{S}^+_{1,1} \oplus \mathcal{S}^+_{1,1}]$
$[\mathcal{S}^{0}_{2,1}]^-$		∞	∞		∂_x	$-y\,\partial_y$	$\mathcal{A}_1(\partial_x)^c$	$[\mathcal{S}^+_{1,1} \oplus \mathcal{S}^+_{1,1}]$
$[\mathcal{S}^{1}_{2,1}]^-$		1		∞	∂_x	$-(x\,\partial_x + y\,\partial_y)$	0	$[\mathcal{A}\mathrm{ff}_2]^-$
$[\mathcal{S}^{J}_{2,1}]^-$		1		∞	∂_x	$-(x\,\partial_x + (x+y)\,\partial_y)$	0	$[\mathcal{S}^1_{2,1,1}]$
$[\mathcal{N}_{2,1}]$	1	∞			$y\,\partial_x$	∂_x	$\mathcal{A}_1(\partial_x)^c$	$\mathcal{A}_3 \rtimes (\mathcal{S}^0_{2,1})^-$
$[SL_2]^+$			∞		$2y\,\partial_y$	$-y^2\,\partial_y$	$\mathcal{D}\mathrm{iff}_1(x)$	$\mathcal{C} \oplus \mathcal{G}$
$[SL_2]^-$				∞	$2D$	$-y(2x\,\partial_x + y\,\partial_y)$	$\mathcal{A}_1(x\,\partial_x)$	$\mathcal{C} \oplus \mathcal{G}$
$[SL_2]^0$				∞	$2D$	$-2xy\,\partial_x - (x^2 + y^2)\,\partial_y$	0	$\mathcal{G}$

Table 2. Equivalence classes of two and three dimensional subalgebras of $Diff_2(x, y)$.

The first column lists the conjugation classes of the three dimensional subalgebras. There is an infinity of equivalence classes for algebras $\mathcal{A}_3$; they are labelled by $+$ and the function ϕ. In lines 2 and 6 there is an infinity of isomorphism types; they are labelled by the parameter $\lambda \neq 1$, $0 < |\lambda| \leq 1$ and they are defined in table 1. The columns 2,3,4,5 give the incidence of the four equivalence classes of two dimensional subalgebras into those of dimension three. A typical representative subalgebra is given by a basis of three vector fields: a or $e_- = \partial_y$; the two other generators are given in columns 6,7: D is a short for $x\,\partial_x + y\,\partial_y$. The last two columns give the centralizer and the normalizer in $\mathcal{D}\mathrm{iff}(x, y)$ of each subalgebra (the upper index ^{c} indicates that the centralizer is the center).

One can be astonished that Lie did not invent the simple concepts of centralizer and normalizer of a subalgebra. For example, isomorphic subalgebras of $\mathcal{D}\mathrm{iff}_2$ cannot be equivalent if their centralizers and normalizers (respectively denoted by $\mathcal{C}_{\mathcal{D}\mathrm{iff}_2}(\mathcal{G})$ and $\mathcal{N}_{\mathcal{D}\mathrm{iff}_2}(\mathcal{G})$) are not isomorphic. The centralizers are easy to compute and, in general, it is not difficult to compute the normalizers (recall that $\mathcal{G}$ and $\mathcal{C}_{\mathcal{D}\mathrm{iff}_2}(\mathcal{G})$ are ideals of $\mathcal{N}_{\mathcal{D}\mathrm{iff}_2}(\mathcal{G})$). For the subalgebras of $\mathcal{S}L_2 \times \mathcal{S}L_2$ and $\mathcal{S}L_3$, their equivalence classes in $\mathcal{D}\mathrm{iff}_2$ are separated by the isomorphism classes of their centralizers and normalizers.

All one dimensional sub algebras of $\mathcal{D}\mathrm{iff}_2$ are equivalent; this class is denoted by $[\mathcal{A}_1]$. There are two isomorphic classes of two dimensional algebras, one Abelian: $\mathcal{A}_2$ and one non Abelian: $\mathcal{S}_{1,1}$ ($\mathcal{S}$ is for solvable), with commutation relation $[a,b]=a$. Each of these two isomorphic classes has two equivalent classes in $\mathcal{D}\mathrm{iff}_2$ depending on whether the functional dimension of the subalgebra is 1 (upper index $^+$) or 2 (upper index $^-$). We give here an example of a subalgebra for each of these four classes: $\mathcal{A}_2(\partial_x, y\,\partial_x) \in [\mathcal{A}_2]^+$, $\mathcal{A}_2(\partial_x, \partial_y) \in [\mathcal{A}_2]^-$, $\mathcal{S}_{1,1}(\partial_x, x\,\partial_x) \in [\mathcal{S}_{1,1}]^+$, $\mathcal{S}_{1,1}(\partial_x, x\,\partial_x + y\,\partial_y) \in [\mathcal{S}_{1,1}]^-$.

Table 1 gives the isomorphy classes of non simple Lie algebras of dimensions 3. The nilpotent algebra $\mathcal{N}_{2,1}$ is also called the Heisenberg algebra; its commutation relations can also be written: $[a,c]=0=[b,c], [a,b]=c$. There is one simple Lie algebra of dimension 3: $\mathcal{S}L_2$; we can write its commutation relations: $[h,e_\pm]=\pm 2e_\pm$, $[e_+,e_-]=h$.

Table 2 gives the equivalence classes of the 3 dimensional Lie subalgebras of $\mathcal{D}\mathrm{iff}_2$, their centralisers and normalizers. It also gives the partial order relations between 2 and 3 dimensional Lie subalgebra equivalent classes.

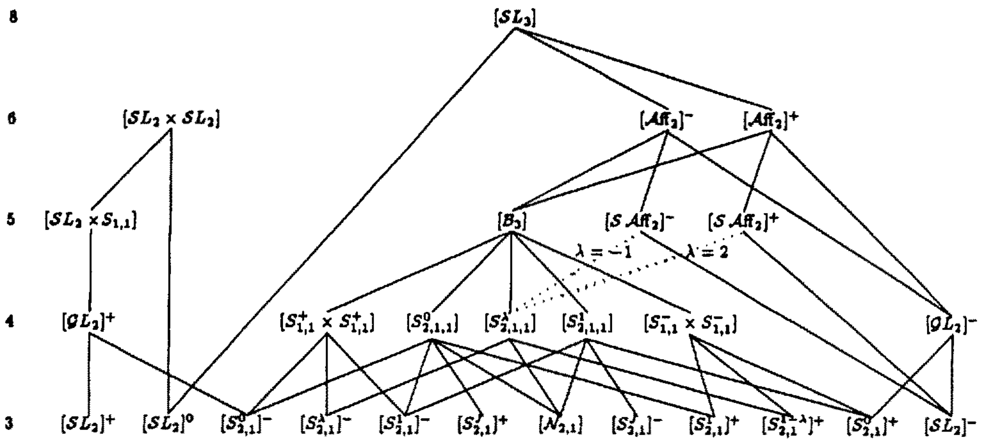

Diagram 3. Partial ordering of the equivalence classes of $\mathcal{D}\mathrm{iff}_2$ subalgebras (of dimension ≥ 3) which are smaller than the two maximal classes: $[\mathcal{S}L_2 \times \mathcal{S}L_2]$ and $[\mathcal{S}L_3]$.

The four direct products in diagram 3 as well as $[\mathcal{G}L_2]^+ = [\mathcal{A}_1 \times \mathcal{S}L_2{}^+]$, have one factor in $\mathcal{D}\mathrm{iff}_1(x)$ and the other in $\mathcal{D}\mathrm{iff}_2(y)$. We introduce a new family of 4-dimensional algebras:

$$\mathcal{S}^\lambda_{2,1,1} \sim \mathcal{N}_{2,1}(a,b,c) \rtimes \mathcal{A}_1(d) : \begin{cases} [a,b]=c, \ [a,c]=[b,c]=0, \\ [d,c]=c, \ [d,a]=\lambda a, \ [d,b]=(1-\lambda)b. \end{cases} \qquad 5(1)$$

One verifies that $S^{\lambda}_{2,1,1}$ and $S^{1-\lambda}_{2,1,1}$ are isomorphic but inequivalent; however, for each value of λ (in the complex plane) there exists a unique equivalence class that we denote by $[S^{\lambda}_{2,1,1}]$. A representative of this class is:

$$\langle a = \partial_y,\ b = y\,\partial_x,\ c = \partial_x,\ d = -(x\,\partial_x + \lambda y\,\partial_y)\rangle \in [S^{\lambda}_{2,1,1}] \qquad 5(2)$$

As we saw in section 3 (before 3(12)) a natural representative of the class $[SL_3]$ is:

$$SL_3 = \langle \partial_x, \partial_y, x\,\partial_x, y\,\partial_y, x\,\partial_y, y\,\partial_x, xD, yD\rangle \text{ with } D = x\,\partial_x + y\,\partial_y. \qquad 5(3)$$

with the natural $\mathcal{G}L_2^{\,-}$ subalgebra:

$$\mathcal{G}L_2^{\,-} = \mathcal{A}_1(D) \times SL_2^{\,-}(y\,\partial_x, x\,\partial_y, x\,\partial_x - y\,\partial_y). \qquad 5(4)$$

The Borel subalgebra (=maximal solvable subalgebra) of SL_3 is denoted by $\mathcal{B}_3$; it is generated by the first five terms of 5(3). The affine algebra $\mathcal{A}\mathrm{ff}_2$ is a maximal subalgebra which belong to two classes:$[\mathcal{A}\mathrm{ff}_2]^{\mp} = [\mathcal{A}_2^{\mp} \rtimes \mathcal{G}L_2^{\,-}]$ corresponding respectively to the classes of stabilizers of points and of lines in the projective plane (indeed $SL_3 \sim SPL_2$, the special projective linear algebra in dimension 2). The two classes: $[\mathcal{A}\mathrm{ff}_2]^{\mp}$ are exchanged by the outer automorphisms of SL_3; their representatives in 5(3) are generated by the first (respectively last) six generators of this equation.

6 The strata of order 2 ODE.

The classification of the symmetries of order 2 ODE has been given by Tresse [10] in 1896 from the study of their differential invariants. This even classifies the orbits, and Lie in [2]-III had already given a characterization of all order 2 ODE on the orbit of the linear ones (for a more precise formulation see e.g. [11]). To conclude this lecture we apply the general sections 2,3,5 to order 2 ODE. This yields very fast the complete list of their strata.

From 3(15) we know that the symmetry algebras of order 2 ODE do not contain an $\mathcal{A}_3$, so they belong to equivalent classes of dim 0,1,2 and those of diagram 3. Consider an arbitrary second order differential equation:

$$\mathcal{E} \equiv y_2 - \omega(x, y, y_1) = 0 \qquad 6(1)$$

and let $\mathcal{G}_{\mathcal{E}}$ be its symmetry algebra. With the prolongation of vector fields, equation 2(5) yields:

$$\partial_y \in \mathcal{G}_{\mathcal{E}} \Leftrightarrow \omega_y = 0;\ x\,\partial_y \in \mathcal{G}_{\mathcal{E}} \Leftrightarrow x\omega_y + \omega_{y_1} = 0;\ y\,\partial_y \in \mathcal{G}_{\mathcal{E}} \Leftrightarrow \omega = y\omega_y + y_1\omega_{y_1}. \qquad 6(2)$$

With these relations we obtain:

$$\mathcal{A}_2(\partial_y, x\,\partial_y)^+ \in \mathcal{G}_{\mathcal{E}} \Rightarrow y_2 = \omega(x), \qquad S_2(\partial_y, y\,\partial_y)^+ \in \mathcal{G}_{\mathcal{E}} \Rightarrow y_2 = \alpha(x)y_1 + \beta(x); \qquad 6(2)$$

i.e. the equation is linear and, as we have seen, its sy mmetry algebra is SL_3. All those results can be summariezd in the

Lemma *Outside SL_3, the symmetry algebra of an ODE of order 2 cannot belong to an equivalence class $\geq [\mathcal{A}_2]^+$ or $\geq [S_{1,1}]^+$ in $\mathcal{L}$.*

From table 2, we see that the only possible $\mathcal{G}_{\mathcal{E}}$ of dimension 3 are

$$[S^{\lambda}_{2,1}]^-, \lambda \neq 0,\ [S^{J}_{2,1}]^-,\ [SL_2]^-,\ [SL_2]^0. \qquad 6(3)$$

Diagram 3 shows that there are no larger symmetry algebra classes outside $[SL_3]$.

Of course "nearly all" second order ODE have no symmetry, but in practical problems we meet mainly equations with symmetry, this property helping to solve them. The equations of the Painlevé Gambier transcendentals are examples of order 2 ODE without symmetry; another family of examples is:

$$\mathcal{G}_{\mathcal{E}} = 0: \ m \neq -3, \quad \omega = Ay^m + f(x), \ f \neq C, \neq C(x+K)^{2m/(1-m)}. \qquad 6(4)$$

For order 2 ODE, even a one dimensional symmetry algebra allows the integration of the equation; indeed this algebra can be transformed into $\mathcal{A}_1(\partial_x)$. By a Riccati transformation $X = y, Y = y_1$ (which is not in $\mathcal{D}\mathrm{iff}_2(x,y)$) we can decrease by one unit the order of the equation (it becomes $YY' = \omega(X,Y)$).

For each strata with non trivial symmetry we give examples of equations:

$$\mathcal{G}_{\mathcal{E}} \in [\mathcal{A}_1], y_2 = \omega(y, y_1); \ \mathcal{G}_{\mathcal{E}} \in [\mathcal{A}_2]^-, y_2 = \omega(y_1); \ \mathcal{G}_{\mathcal{E}} \in [\mathcal{S}_{1,1}]^-, y_2 = \rho(y_1)y^{-\kappa}, \kappa \neq 0. \quad 6(5,6,7)$$

$$\mathcal{G}_{\mathcal{E}} \in [\mathcal{S}_{2,1}^{\lambda}]^-, \lambda \neq 0, \frac{1}{2}, 1, 2: \quad y_2 = Cy_1^{\frac{1-2\lambda}{1-\lambda}}, \qquad \mathcal{G}_{\mathcal{E}} \in [\mathcal{S}_{2,1}^{J}]^-: \quad y_2 = Ce^{-y_1} \qquad 6(8,9)$$

$$\mathcal{G}_{\mathcal{E}} \in [SL_2]^0, \quad y_2 = \frac{4}{3}y_1^2(3 - y_1 + Cy_1^{-1/2})(x-y)^{-1}, \qquad 6(10)$$

$$\mathcal{G}_{\mathcal{E}} \in [SL_2]^-: \quad y_2 = (y_1^2 + C)(2y)^{-1}, \qquad \mathcal{G}_{\mathcal{E}} \in [SL_3]: \quad \text{linear equations.} \qquad 6(11.12)$$

We leave as an exercise to the reader the listing of the strata (with examples of equations for each stratum) for a given order $n \geq 3$ of ODE!

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