Topological Classification of symmetry defects in ordered media

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Science, and in particular the spectacular progress of physics have been possible because there is symmetry and order in nature. All of us here, interested by group theoretical methods, we have a keen interest on symmetries. Some of us are fascinated by their spontaneous breaking. And true lovers of symmetries have studied their defects.

I have been asked to report here on a new field of application of group theory to physics : the classification of topologically stable defects in ordered media. It is still possible to give a fairly complete list of references (see Table 1).

Of course physicists classifying crystal dislocations by the Burger's circuit were using homotopy just as Mr Jourdain was speaking prose [7]. As far as I know D. Rogula was the first to have recognized it explicitly [1]. Toulouse and Kléman [2] had the merit to introduce a general scheme for all ordered media and they made from it a specific prediction : that the vertex lines in superfluid A phase of He^3 will annihilate by pair. Later, but independently, Volovik and Mineev made a similar work.

Three slightly different schemes are introduced by the authors of table 1; to present them to you I label them provisorily a, b, c in chronological order ; their correspondance in the reference table is as follows :

$$a \text{ in } [1] \quad , \quad b \text{ in } [2, 4, 6, 9] \quad , \quad c \text{ in } [3, 5] \qquad (1)$$

([7] and [8] are in fact independent from the chosen scheme).

I will present first scheme c that I believe is the most general. To the two dozens of you who attended my last lecture of the Montreal summer school associated with the previous colloquium, I beg to excuse my repetition.

TABLE 1.

REFERENCES On Homotopic classification of symmetry defects in ordered media. (Dates given are those of the preprint when it exists).

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[3]	Juin 1976	Louis MICHEL, Twelfth lecture of the course "Les brisures spontanées de symétrie" Centre de Mathématiques de Montréal ; to be published by "Presses de l'Université de Montréal".
[4]	Juin 1976	Gérard TOULOUSE " Pourquoi et comment classer les défauts dans les milieux ordonnés" Conférence publiée dans Bulletin de la Société Française de Physique, 19, (1976).
[5]	Oct. 1976	Maurice KLEMAN, Louis MICHEL, Gérard TOULOUSE "Classification of Topologically Stable defects in ordered media". J. Phys. Lettres 38 L 195 (1977).
[6]	Nov. 1976	Gérard TOULOUSE "Pour les nématiques biaxes" J. Phys. Lettres 38 L 67 (1977).
[7]	Déc. 1976	Maurice KLEMAN "Relationship between Burgers'circuit, Volterra process and homotopy groups" J. Phys. Lettres, 38L 199 (1977).
[8]	Déc. 1976	Valentin POENARU, Gérard TOULOUSE "The crossing of defects in ordered media and the topology of 3-manifolds" J. Phys. 38 887 ((1977).
[9]	Déc. 1976	G.E. VOLOVIK, V.P. MINEEV "Study of singularities in ordered systems by homotopic topology methods" Preprint Landau Institute Moscow.

Consider an ordered media and forget for a while about its boundary i.e. assume it extends indefinitly ; it can be a magnetic material, a crystal, a liquid crystal, a superfluid liquid, etc... ; every one of you can think of its pet example. Assume moreover that it is in a perfect state, which is a broken symmetry state.

In the example of isotropic magnetic material, there is a spontaneous magnetisation $\vec{B}$, constant everywhere in space, which breaks the rotational (= SO(3)) invariance G . More generally, in all examples, the symmetry is broken from G (the symmetry group of physical laws) into H , subgroup of G , the invariance group of the state (SO(2) for the magnetisation $\vec{B}$ so $G/H = S_2$). For instance G is $E(3) = O(3) \square T$, ($\square$ = semi-direct product) the 3-dimensional Euclidean group and for crystals, H belongs to one of the 230 crystallographic classes, while for nematics H contains all translations T and is the semi-direct product $D_{\infty h} \square T$. For superfluid He^4 , G is U(1) the group of phases and H = 1 .

To summarize, the perfect state of our pet medium is represented by a point of the orbit G/H . By the action of G (e.g. by rotating or translating the crystal) you produce any other "similar" state (e.g. it is the same crystal, but in a different position) representable by a point of the orbit G/H . When the medium is not in a perfect state, you may still recognize a local state (representable by a point of G/H) and this local state varies from place to place. This dependence can be described by a function Φ , defined in our 3-dimensional space and valued in the orbit G/H . This function is not defined outside the volume V occupied by your pet medium ; there might be also points inside the volume where Φ is not defined : they corresponds to defects ; those might be isolated points or they may form lines, surfaces (we will also say "walls") or any subdomain of V. Look at Fig. 1 and 2 which idealizes two 2-dimensional ordered media with defects.

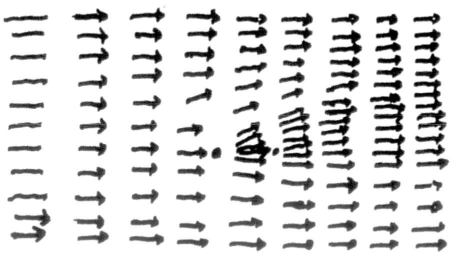

Fig.1

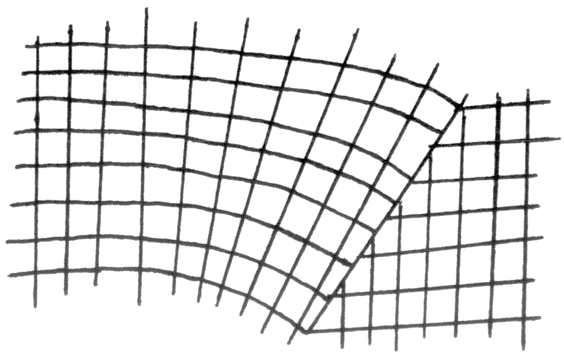

Fig.2

The very existence of defects requires energy. But we will not discuss here their stability from possible energy barriers. It is our aim to study topological stability. By definition, if it is possible to extend continuously the function Φ over a defect, it is said topologically unstable. There is a mathematical theory which study the obtructions to such a continuous extension of function : this is homotopy theory. For instance a constant function can always be continuously extended (with the same value) every where. If necessary the reader is refer to the appendix for the definition of homotopic functions. If on a 2-dimensional sphere S_2 , the function Φ is not homotopic to a constant, it is not possible to extend it inside the sphere and there must exist at least a point defect. Similarly if Φ is not homotopic to a constant on a circle (= one dimensional sphere S_1), this circle must surround a line defect. More generally, for a d dimensional ordered media (in [2] Toulouse and Kléman did consider models of arbitrary dimension in statistical mechanics) the knowledge of Φ on a sphere S_n of dimension n yield information on the d' dimensional defects with

$$d' = d-n-1 \qquad . \qquad (2)$$

Homotopy is doing more for us: for each $n \geq 0$, the homotopy classes of functions from S_n to G/H form a group usually denoted $\pi_n(G/H)$. This allows us not only to classify defects, but also, as we shall see, to study how they coalesce and if eventually they can cross.

I wish to make several remarks on the observation of the function Φ . As is clear for a (axial-) vector density $\vec{B}$ not only its direction, but also its magnitude can vary between some limits $c_0 \leq |\vec{B}| \leq c_1$. Then the domain of values of Φ is really the topological product $S_2' = S_2 \times [c_0, c_1]$ but, as is recalled in the appendix, this has same homotopy as the sphere S_2 . More generally we can always add an "intensive" values which cannot vanish, to each point of G/H . This transforms the orbit into a homotopically equivalent topological space. One can also say that the domain of value of Φ is the orbit G'/H when $G' = G \times R_+$ where R_+ , the multiplicative group of the reals is a scaling group. For crystal we simply recall that the crystallographic class of H is defined up to a conjugation into the general (connected) linear group : this does include dilations. Since a crystal and more generally matter are essentially discrete (built from atoms) it is obvious that continuity of functions has to be defined correctly; every physicist knows intuitively how to do it so I will not be more formal here (e.g. looking at a head, the hair define a vector field on the skull if you consider the right scale length, not if you look at it with a miscroscope!).

Let us first give the classification of defects for crystals [5]. Let E_0 be the connected component of the Euclidean group E(3) and $\bar{E}_0$ its universal (double-) covering ; let us denote by $\bar{E}_0 \xrightarrow{\theta} E_0$ the corresponding group homomorphism (Ker $\theta = Z_2$, generated by the "rotation of 2π ") and $H_0 = H \cap E_0$. Then, as one can deduce from the appendix or read in [5] :

$$\text{for crystals} \quad \Pi_0 = \begin{cases} Z_2 & \text{if } H = H_0 \\ 0 & \text{otherwise} \end{cases}, \quad \Pi_1 = \theta^{-1}(H_0) = \bar{H}_0 , \quad \Pi_2 = 0 \qquad (3)$$

This means that there are no point defects ($\pi_2 = 0$) ; the elements of π_1 which correspond to translations class the dislocations while those corresponding to rotations are called disclinations. Of course there are many more corresponding to the product of rotation and translations, although they are more difficult to observe since they need higher energy excitation. Finally when $H = H_0$, (i.e. the point group of H contains no reflections) π_0 is not trivial and is isomorphic to Z_2 ; the non trivial element corresponds to wall defects : they are the twin boundaries (twin by reticular merihedry in the terminology of G. Friedel) : indeed, according to the abelian group law of Z_2 , they annihilate by pair.

When π_1 is not abelian, the line defects are classified only by its conjugation classes (see appendix). The product of their conjugation classes contains several conjugation classes, but the knowledge of the function Φ allows to find which one is obtained by the coalescence of two line defects.

It is time that I explain the difference between this approach and that of scheme a) and b) of (1). D. Rogula [1] considered only crystals ; for him $G = Aff(3)$, the affine group in three dimensions, acting transitivelly on the set of all lattices. The little group of such a lattice is $H = Z_2 \times [SL(3,Z)\square Z^3] = Z_2 \times H_0$ (consider a basis of three vectors generating this lattice). The author does not compute the homotopy of G_0/H_0 . We can do it from the appendix : if $\bar{G}_0$ is the universal (double -) covering of G_0 and $\theta : \bar{G}_0 \xrightarrow{\theta} G_0 \longrightarrow 1$, then $\pi_1(G_0/H_0) = \theta^{-1}(H_0) = \bar{H}_0$: it is huge and independent of the symmetry class of crystal. Of course $\pi_2 = 0$ and $\pi_0 = 0$ (he misses the twin boundaries). As you see I disagree with this author, but you have to form your own opinion. Scheme b, that of Toulouse and Kléman is general and can be applied to many media : the space of internal states which play the role of the orbit G/H in their scheme is the set of values of the "order parameter" ; it is the same parameter, also called Landau parameter, that one considers in phase transition from the disordered (isotropic phase) to the ordered phase of the medium we consider. (The order parameter considered in Professor Birman's lecture [10], is used for second order transition from a less ordered, but non isotropic phase ; it should not be used here). In many cases this scheme b is a particular case of scheme c ; this is obvious for nematics, (2,5,9) : indeed

$$G/H = E(3)/(D_{\infty h}\square T) = O(3)/D_{\infty h} = P(2,R) \qquad , \qquad (4)$$

the real projective plane, while the order parameter is a line element (i.e. the axis of the aligned molecules). The homotopy is :

$$\pi_0 = \{0\} \quad \pi_1 = Z_2 \quad , \quad \pi_2 = Z \qquad . \qquad (5)$$

The line singularities annihilate by pair. There are an infinite number of classes of point defects. However, as Volovik and Mineev emphasized in [9], since $\pi_1 = Z_2$ acts on $\pi_2 = Z$ by the non trivial automorphism of Z , isolated point defects are classified only by non negative integers ; but when they coalesce, one can observe their relative sign.

The equivalence of scheme band c is also obvious for the still to be observe biaxial nematics : the order parameter space is that of unequal axis quadrupole and $G/H = O(3)/D_{2h}$. It is also true for smectics [26]. It is not clear for cholesterics. Table 2 summarizes the results. The first seven examples correspond to the breaking of Euclidean invariance, the last four correspond to the gauge breaking in superfludity. You are welcome to work out more examples.

TABLE 2. Classification of defects by homotopy groups for several 3 dimensional ordered media.

Medium	Ref	G/H or parameter space	wall defects π_0	line defects π_1	point defects π_2
isotropic ferromagnetism	2,9	$SO(3)/SO(2)=S_2$	{0}	{1}	Z
crystal	5	$E(3)/H$	$H/H_0=\{0\}$ or Z_2	$\bar{H}_0$	{0}
nematics	2,5,9	$E(3)/(D_{\infty h} \square T) = O(3)/D_{\infty h} = P(R,2)$	{0}	Z_2	Z #
smectics A	26	$E(3)/(D_{\infty h} \square (Z \times R^2))$	{0}	$Z^2 \square Z$	{Z}
smectics C	26	$E(3)/(C_{2h} \square (Z \times R^2))$	{0}	$Z^4 \square Z$	{0}
cholesterics	9	$O(3)/D_{2h}$	{0}	Q	{0}
biaxial nematics	5,6	$O(3)/D_{2h}$	{0}	Q	{0}
He^4	2,9	$U_1/\{1\}=P(R,1)=S_1$	{0}	Z	{0}
He^3A	2,9	$SO(3)/\{1\}=P(R,3)$	{0}	Z_2	{0}
(+)	9	$(SO(3) \times SO(3))/(SO(2) \times Z_2)$	{0}	Z_4	Z
He^3B	9	$U_1 \times SO(3)/\{1\} = S_1 \times P(R,3)$	{0}	$Z_2 \times Z$	{0}

Notations and notes : $E_0(3)$ = connected component of 3 dimensional Euclidean group $E(3)$, its universal covering is $\bar{E}_0 \xrightarrow{\theta} E_0 \longrightarrow$, H = symmetry group of crystal $\bar{H}_0 = \theta^{-1}(H_0)$ with $H_0 = H/E_0$. $Q = \theta^{-1}(D_{2h})$ is the quatermonic group (generated by $\pm i\tau_k$ where τ_k are the Pauli matrices). # $\pi_1 = Z_2$ acts non trivially on Z , ref [T9] ; + when the volume is small enough new types of defects appear! P(R,n) is the real projective space in n dimensions.

Of course topological stability is only the first logical step for the study of symmetry defects ; we will soon see how fruitful it is. When is topological stability no longer valid? We already noted that the orbit G/H or the space of internal states has to be thicken by a scalar (intensive) parameter, the Landau parameter $\eta > 0$. This parameter is a function of T which vanishes at the critical temperature T_c of phase transition to the isotropic phase. As pointed out in [5]. when the temperature T is increased enough, although still below T_c , the local fluctuations of η to the value $\eta = 0$ can no longer be neglected and the space of states $\bigcup_{0 \leq \eta \leq \eta_1}(G/H)(n)$ becomes contractible so its homotopy becomes trivial, and the topological stability of defects dissapears. This is the annealing process for crystals.

I already pointed out how homotopy classes of defects combine when the defects coalesce. In [9] Poenaru and Toulouse ask the question : can two defects cross! Their answer for line defects is yes when, and only when, they correspond to commuting elements of π_1 (this is always the case when π_1 is abelian). For higher dimensions they wrote a following paper [11] in which they use the Whitehead product on the homotopy groups (it defines a super algebra)for answering this question.

As you may know, homotopy theory has already been used in physics for the classification of t'Hooft-Polyakov magnetic monopoles (π_2 of the orbit of the Higgs field [12,13]) and the classification of instantons by π_3 of the gauge group [14]. It is most appropriate to mention here the pioneer work of Finkelstein on kinks ten years ago [15]. In a preprint [16] , this author, with Weil, study the relevance of kinks of the magnetic field for non resistive plasmas and to the problems of galaxy formation : the dynamical equations require that the magnetic field vanishes nowhere in space so, as in the first line of Table 2, $\pi_2=Z$; this classifies the kinks. π_2 and π_3 are also useful to classify solitons.

It seems to me that topological classification of defects (and of solitons) is a first step in a right direction and may become soon a classical chapter of physics. But do not forget that it is somewhat a crude analysis of the physical phenomena and it cannot replace the beautiful work done and to be done on the physics of defects. However the topological classification has some predictive power and it gives a new insight into the subject. Let us hope that this approach will be fruitful. In any case it seems to me an easy guess to predict that applications of homotopy to physics will be a lively subject in the near future and be one of the topics of the VIIth Colloquium two years from now.

APPENDIX. - Short tourist guide on homotopy.

Physicists are not yet expected to be fluent in homotopy.So I wrote this appendix to complete my lecture and make it intelligible. This cannot replace the serious study of a good mathematical book ; this is rather easy to the motivated physicist (I hope that this lecture has strenghthened yourmotivation!). The 24 pages of § 15, 16, 17 of Steenrod's book [17] are the best adapted to our need. Books [18], [19], [20], [21] could be useful ([19] is condensed : few pages to read for the proofs of this appendix, but the special case of Lie group is not studied, [21] is a teaching book in the modern style).

We consider topological spaces and continuous functions between them (children in 9^{th} grade at Berkeley will tell you that we use a category). Two functions f, g from T to X are said homotopic if one can pass continuously from one to the other. Technically, $\exists$ (= there exists) a continuous X-valued function $F(t,\lambda)$ defined on $T \times I$ with I the closed segment $[0,1]$ of the real line, such that $F(t,0)=f(t)$, $F(t,1) = g(t)$. To be homotopic is an equivalence relation between functions : we denote it by $f \sim g$. We choose for T the n dimensional cube : $\Gamma_n = \{t_i , 0 \leq t_i \leq 1 , i \leq 1 \leq n\}$; and we denote by $\partial\Gamma_n$ its boundary (one $t_i = 0$ or 1). We consider the continuous functions from $\Gamma_n \xrightarrow{f} X$ such that $f(\partial\Gamma_n) = x_0$ a fixed point of X . A cube Γ_n with all points of its boundary identified is homeomorphic to a sphere S_n (this must be obvious to you for $n = 1$; every Japonese woman knows it for $n = 2$: instead of a handbag they use a square scarf with a special knot taking the whole brim of the scarf), so we really consider continuous maps $S_n \longrightarrow X$. Mathematicians define the following composition law between two functions :

$$f * g\,(t_1,t_2,\ldots,t_n) = \begin{cases} f(2t_1,t_2,\ldots,t_n) & , \quad 0 \leq t_1 \leq 1/2 \\ g(2t_1-1,t_2,\ldots,t_n) & , \quad 1/2 \leq t_1 \leq 1 \end{cases}$$

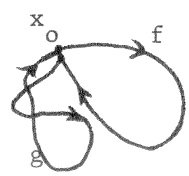

Fig. 3

(For $n = 1$ this means that you run successively in X through the two closed paths,first $f(t)$, then $g(t)$ with a double speed, so this is again a closed path travelled in the unit interval time).

As an exercise prove that $*$ is a group law : the inverse of f is the same function but with the other orientation of S_n , and the neutral element e is the constant function. Since $f \sim f'$, $g \sim g' \Longrightarrow f*g \sim f'*g'$ the $*$ defines a group on the homotopy classes of functions $S_n \xrightarrow{f} X$; they are traditionnaly denoted by

$\pi_n(x_0,X)$. If $n \geq 2$, make a rotation of π in the 2-plane t_1, t_2 ; it is homotopic to the identity $\Gamma_n \xrightarrow{I} \Gamma_n$ and it changes $f * g$ into $g * f$ so for $n \geq 2$ the homotopy groups are abelian. What happens if we change the base point from x_0 to x_1 in X connected? Using a closed path x_0 to x_1 and to x_0 again we see that $\pi_n(x_0,X)$ and $\pi_n(x_1,X)$ are isomorphic but this isomorphism depends on the chosen closed path. Technically : $\pi_1(X)$ acts on the $\pi_n(X)$; in particular it acts on itself by inner automorphisms.

Examples of homotopy groups :

X is contractible e.g. R^n , so every function is $\sim$ to a constant, hence $\pi_n(x,X) = \{e\}$ i.e. trivial homotopy.

$X = S_1 = R^2 - \{0\}$, $\pi_1 = Z$ (and we will see later $\pi_n = \{e\}$ for $n > 1$) A(1)

$X = R^2 - \{0\} - \{0'\}$ π_1 = free group of 2 generators (infinite, non abelian)

$X = S_n$, $k < n$ $\pi_k = \{e\}$, $\pi_n = Z$. A(2)

One could have defined $\pi_0(X)$: it is not a group but a set = set of connected components of X . If $\pi_1(x_0,X) = \{e\}$, the connected component of X containing x_0 is simply connected. We are especially interested here when X is a topological group G ; its connected component G_0 containing $\{e\}$ is an invariant subgroup and $\pi_0(e,G)$ is now a group isomorphic to G/G_0 . We denote $\pi_n(e,G)$ simply by $\pi_n(G)$ ($= \pi_n(G_0)$ for $n > 0$) . We give here the proof that $\pi_1(G)$ is abelian : convention $f(t) = e = g(t)$ when $t \notin]0,1[$; consider $H(t,\lambda) = f((1+\lambda)t).g((1+\lambda)t-\lambda)$ where . is the group law in G . Then $H(t,1) = (f * g)(t)$, $H(t,0) = f(t).g(t)$; then consider $K(t,\lambda) = g(\lambda t).f(t).g(t).g(\lambda t)^{-1}$ and note that $K(t,0) = f(t).g(t)$ and $K(t,1) = g(t).f(t)$. So $\pi_1(G)$ does not act on itself. One also proves that $\pi_1(G)$ acts trivially on $\pi_n(G)$ when G is a Lie group. However $\pi_0(G)$ acts on $\pi_n(G)$ and this action might be non trivial. We now consider two topological spaces X, Y and a fixed continuous map $X \xrightarrow{\theta} Y$; to each $f : S_n \xrightarrow{f} X$ corresponds $S_n \xrightarrow{\theta \circ f} Y$ and this correspondance induces a correspondance $\pi_n(x_0,X) \xrightarrow{\theta *} \pi_n(y_0,Y)$ where $y_0 = \theta(x_0)$. One proves that θ_* is a group homomorphism. Obviously if $Y = X$ and θ is the identity map, then θ_* is the identity isomorphism ; it is also obvious for some 9^{th} grader that each π_n is a functor so to every commutative diagram of continuous maps between topological spaces corresponds the same diagram of homomorphisms between the corresponding π_n . For

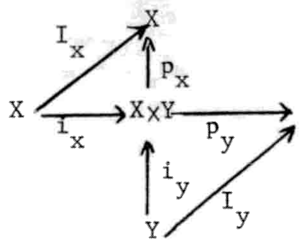

Fig.4

instance to the diagrammatic definition of the topological product (Fig. 4) corresponds the same diagram of homomorphisms between the π_n with two identity maps, i.e.

$$\pi_n((x_0,y_0),X \times Y) = \pi_n(x_0,X) \times \pi_n(y_0,Y) \tag{A1}$$

: the homotopy groups of a topological product are direct products of the homotopy groups of the factors. Let us apply it to a semi-simple real Lie group G . Using the Iwasawa decomposition G_0 = K.A.N. where K is maximal compact, while A (abelian and N (with a nilpotent Lie algebra), have a topology R^m (so trivial homotopy) and where the dots mean both group law and topological product; we obtain $k > 0$, $\pi_k(G) = \pi_k(K)$ e.g. (a semi-direct product of groups is a topological product)

$$\pi_k(O(n)) = \pi_k(GL(n,R)) = \pi_k(Aff(n,R)) \tag{A2}$$

where Aff is the affine group (inhomogeneous G.L.). Stewart [22] generalized this decomposition to the group of diffeomorphisms proving that

$$\pi_k(\text{Diff } n) = \pi_k(O(n)) \tag{A3}$$

E. Cartan proved [23] that for any compact Lie group K , $\pi_2(K) = 0$, so for any

$$\text{real Lie group } G \ , \ \pi_2(G) = \{0\} \tag{A4}$$

I let you translate for all Lie groups the result of Bott [24]

$$K \text{ compact simple Lie group} \qquad \pi_3(K) = Z \quad . \tag{A5}$$

How does the results (A1) for topological product i.e. for trivial bundle $X \times Y$ over base Y (and fiber X) extends to non trivial bundles? Instead of splitting short exact sequences for each n , one obtains a long exact sequence of homomorphisms (exact means: the Image of a homomorphism is the Kernel of the next one) that we write explicitly for the particular case of a real Lie group G considered as bundle over the homogeneous space (= orbit) G/H where H , the fiber, is a closed subgroup of G : (see e.g. 17)

$$\to\pi_3(H)\to\pi_3(G)\to\pi_3(G/H)\to\pi_2(H)\to\pi_2(G)\to\pi_2(G/H)\to\pi_1(H)\to\pi_1(G)\to\pi_1(G/H)\to\pi_0(H) \tag{A6}$$

It will be useful to consider a vocering group $\bar{G}$ of $G : \bar{G} \xrightarrow{p} G \longrightarrow 1$, so that $\bar{G}/\bar{H} \sim G/H$ ($\sim$ = homeomorphic) where $\bar{H} = p^{-1}(H)$. Then we have two equivalent long homotopy sequence.

$$\begin{array}{cccccccccccc}
\to\pi_3(\bar{H})\to & \pi_3(\bar{G})\to & \pi_3(\bar{G}/\bar{H})\to & \pi_2(\bar{H})\to & \pi_2(\bar{G})\to & \pi_2(\bar{G}/\bar{H})\to & \pi_1(\bar{H})\to & \pi_1(\bar{G})\to & \pi_1(\bar{G}/\bar{H})\to & \pi_0(\bar{H}) \\
\downarrow & \downarrow & \downarrow I & \downarrow & \downarrow & \downarrow I & \downarrow & \downarrow & \downarrow I & \downarrow \\
\to\pi_3(H)\to & \pi_3(G)\to & \pi_3(G/H)\to & \pi_2(H)\to & \pi_2(G)\to & \pi_2(G/H)\to & \pi_1(H)\to & \pi_1(G)\to & \pi_1(G/H)\to & \pi_0(H)
\end{array} \qquad (A7)$$

which break with $\{1\}$'s : $\pi_2(G) = \pi_2(\bar{G}) = \pi_2(H) = \pi_2(\bar{H}) = \{0\}$; if furthermore $\bar{G}$ is simply connected (one says that $\bar{G}$ is the universal covering of the connected G) $\pi_1(\bar{G}) = 0$. Moreover when H is discrete, $\bar{H}$ is discrete and

$$k > 0 \ , \ \pi_k(\bar{H}) = \pi_k(H) = 0 \quad \text{so} \quad n > 1 \quad \pi_n(G/H) = \pi_n(G) = \pi_n(\bar{G}) \qquad (A8)$$

for $n = 1$, we have in this case(see e.g. Wolf [25] section 1.8)

$$\pi_1(G/H) = \pi_1(\bar{G}/\bar{H}) = \pi_0(\bar{H}) = \bar{H} \qquad (A9)$$

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