ON SYMMETRY BREAKING†

Louis Michel
Institut des Hautes Etudes Scientifiques
91 Bures-Sur-Yvette
France

Introduction

I wish to present here several ideas, some mathematical tools and a new mathematical theorem related to the breaking of physical symmetries. I will first give examples, then the general theorem, and its application to the breaking of the SU(3) symmetry, to which an increasing number of papers have recently been devoted.

I want to stress that all the original content of this lecture is the result of a collaboration with Luigi Radicati. Part of this common work has been published in the Proceedings of the Fifth Coral Gables Conference (January 1968).[1] More will be published later.

Broken Symmetry

Since every direction of the three dimensional space can be transformed into any other by the group SO(3), of rotations around the origin, a function $f(\vec{r})$ invariant under rotation depends only on $\vec{r}^2$ and is constant for instance on the unit sphere $\vec{r}^2 = 1$.

Although the theory of magnetism is invariant by rotation, it might happen that the lowest energy state of a homogeneous magnetic material in the vacuum (not in the earth magnetic field!) has a magnetization pointing in a well defined direction so the SO(3) invariance is reduced to SO(2) for this system.

Similarly, the interactions between ions are invariant under the Euclidean group E(3), but the symmetry group of a crystal is only a subgroup of E(3).

In both cases the SO(3) and E(3) symmetry is broken: the equilibrium state corresponds to a solution of lesser symmetry than that of the equations. This breaking is called spontaneous. (The particular state is generally induced by inhomogeneities which may be suppressed afterward.) By applying the invariance group (respectively SO(3)

---

†Presented at the THEORETICAL PHYSICS INSTITUTE, University of Colorado, Summer 1968.

or E(3)) to the particular solution one obtains a full set of similar solutions in both cases considered. May I remind you that might not always be the case. Consider for instance a lithium atom in the fundamental state s^2p. Let us neglect the spin of the p electron to simplify the argument. The atom can be in a P_1^1 or P_0^1 or P_{-1}^1 state, which are invariant by SO(2) only. Note that rotations of SO(3) can transform the P_1^1 state into the P_{-1}^1 but not into the P_0^1.

The symmetry breaking I want to consider is of a different nature and I would like to call it "dynamical breaking" although it is often also called spontaneous. A dynamical breaking is that obtained for instance from a bootstrap formalism (see for instance Cutkosky and Tarjanne[2]) or from a variational principle (for example Domokos and Suranyi;[3] de Mottoni and Fabri.[4] But the type of solution found by these different physical approaches is in fact imposed only by the geometrical aspect of the problem and is independent of the detailed dynamics. This was first suggested, I believe, by R. Brout,[5] but it seems to me that he did not find the true geometrical nature of the problem.

Some Mathematical Facts

Consider a group G acting on a space (= set of points) M. For each point a $\in$ M one can define G_a, the little group of a (also called the isotropy group of a) which is the set of elements g $\in$ G which leave a fixed: g[a] = a. For each a $\in$ M we can also define the orbit of a: it is the set of all points g[a] transformed of a by all elements of the group. Two different orbits cannot have a common point so the action of G on M partitions M into orbits.

If b belongs to the orbit of a, there is g $\in$ G such that b = g[a]. It is easy to verify: $G_b = g\,G_a\,g^{-1}$. Hence the little groups of two points of the same orbits are conjugated. Conversely two points m, n $\in$ M whose little groups G_m, G_n are conjugated ($\exists g$, $G_m = g\,G_n g^{-1}$) are not necessarily on the same orbit. We shall call stratum* any subset of M made of all points which have the same little group up to a conjugation. Hence M is partitioned in strata and each stratum is partitioned into orbits of the same nature. With the partial ordering defined by inclusion, the subgroups of a group form a lattice. This is still true if we consider subgroups modulo a conjugation. This partial ordering induces a partial ordering on the strata of M; we call minimal strata those which correspond to maximal little groups.

Finally if G is a Lie group acting differentiably on a manifold M and if G_a is closed in G, the corresponding orbit is a submanifold of M. Its dimension is: (dim G - dim G_a).

*The concept of strata on M used here is a very simple example of the notion of stratified manifold defined by R. Thom (Ensignement mathematique 1962).

Examples

1. SO(3) acting on the 3-dimensional real vector space $\mathcal{E}(3)$. There are two strata: the origin 0, which is minimal stratum and $\mathcal{E}(3)-0$, decomposed into sphere S_2 as orbits (the corresponding little group is SO(2)).
2. Lorentz group $\mathcal{L}$ acting on the 4 dimensional real vector space $\mathcal{E}(4)$. There are 4 strata: the origin 0, minimal stratum, little group $\mathcal{L}$ and three maximal strata,
   (i) the light cone-0, (one orbit only and little group E(2))
   (ii) the inside of light cone (orbits = 2 sheeted hyperboloids and little group O(3))
   (iii) the outside of the light cone (orbits = 1 sheet-hyperboloids, little group O(2,1))
3. SO(2) acting on the unit sphere $S_2 \subset \mathcal{E}(3)$. There are two strata: minimal stratum being made up of two fixed points (the two poles) and little group SO(2); the other stratum being the union of orbits $= S_1$ (= circles) and little group 1, unit of SO(2).

This third example is an illustration of the theorem of D. Montgomery and C. T. Yang. If a compact Lie group G acts differentiably on a compact manifold M, there is one stratum (the "generic" stratum) which is open dense in M. Radicati and I conjecture the following theorem: Consider a real valued differentiable function f defined on the compact manifold M and invariant by the differentiable action of the compact Lie group G on M. Then f has at least one extremum (grad f = 0) on each connected piece of each minimal stratum.

To our knowledge this theorem was not in the mathematical literature. A sketch of the proof was given to us by A. Borel. I shall not give it here. However, I shall prove this theorem for the example we shall now study: G = SU(3), M is S_7 the unit sphere of the octet space $\mathcal{E}(8)$(= vector space of the Lie algebra of SU(3) with the Killing-Cartan bilinear form as Euclidean scalar product).

Note that a function on M, invariant under G is constant on every orbit of G, so it has also orbits of extrema. We will first prove that SU(3) acting on S_7 divides it into the generic strata (open dense in S_7) and a special stratum made of two orbits of dimension four. The directions in $\mathcal{E}(8)$ corresponding to the points of this minimal stratum have remarkable properties and our theorem predicts that they will very likely be preferred solutions in any SU(3) invariant theories on the unit vectors of the octet space using a variation principle. This geometrical result is independent of the detailed nature of the expression (for example, Lagrangian) to be minimized.

Application to SU(3)

We want to study the action of SU(3) on $\mathcal{E}(8)$, the octet space of of the adjoint representation.

$\mathcal{E}(8)$ can be realized as the real vector space of the 3 x 3 hermitian $(x = x^*)$ and traceless $(\mathrm{tr}\, x = 0)$ matrices. The action of $u \in SU(3)$ on x is given by

$$x \to u\, x\, u^* = u\, x\, u^{-1}.$$

This action leaves the Euclidean scalar product

$$(x, y) = \tfrac{1}{2}\, \mathrm{tr}\, xy$$

invariant (~ Cartan-Killing form of the Lie algebra).

The Lie algebra of SU(3) is realized by the law

$$x \wedge y = \frac{-i}{2}(xy - yx) = -\tfrac{1}{2}[x, y]$$

There is also a symmetric algebra which has SU(3) as automorphism group: (I = unit matrix)

$$x \vee y = \tfrac{1}{2}(xy + yx) - \tfrac{1}{3} I\, \mathrm{tr}\, xy = \tfrac{1}{2}\{x, y\} - \tfrac{2}{3}(x, y) I$$

To relate this to a notation familiar to physicsts, consider the trilinear invariant forms

$$[x, y, z] = (x \wedge y, z) = (x, y \wedge z) = [y, z, x] = -[z, y, z]$$

and

$$\{x, y, z\} = (x \vee y, z) = (x, y \vee z) = \{y, z, x\} = \{z, y, x\}$$

If we choose an orthonormal base of matrices:

$$(\lambda_a, \lambda_b) = \delta_{ab} \qquad (a, b = 1, \ldots, 8)$$

then $[\lambda_a, \lambda_b, \lambda_c] = f_{abc}$, $\{\lambda_a, \lambda_b, \lambda_c\} = d_{abc}$.

What Are the Orbits of SU(3) on $\mathcal{E}(8)$?

The transformation $x \to u\, x\, u^{-1}$ does not change the eigenvalues, or equivalently the characteristic polynome of the matrix

$$x^3 - x\gamma(x) - \mu(x) = 0 \qquad (1)$$

where $\gamma(x) = (x,x)$ $\mu(x) = \frac{2}{3}(x \quad x_1 x) = \frac{2}{3}\{x,x,x\}$ (2)

Since x is hermitian, it has 3 real eigenvalues, so

$$4\,\gamma(x)^3 \geq 27\,\mu(x)^2 \tag{3}$$

Since any hermitian matrix can be diagonalized by a unitary (unimodular) conjugation $x \to u\,x\,u^{-1}$, the orbits are exactly labelled by the two invariants $\gamma(x)$, $\mu(x)$ satisfying (3).

Generic Stratum

If $4\gamma(x)^3 > 27\mu(x)^2$, x has three distinct eigenvalues; the little group in SU(3) of such a diagonal matrix is $u(1) \times u(1)$. It has dimension 2, so the orbits are of dimension 8 - 2 = 6. The stratum is then a 2 parameter-family of 6 dimensional orbits, the parameters being $\gamma(x)$ and $\mu(x)$.

Special Stratum

One param ter family of orbits with $4\gamma(x)^3 = 27\mu(x)^2$: It contains x with a d ıble eigenvalue. So they satisfy a second degree equation whicl s easily found to be

$$x \vee x = \eta(x)\, x \tag{4}$$

with $\eta(x) = (\mu(x)/2)^{\frac{1}{3}}$ (4')

The most general unitary unimodular matrix u commuting with:

$$x = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{pmatrix} \qquad \text{is } u = \left(\begin{array}{c|c} U & 0 \\ \hline 0 & (\det U)^{-1} \end{array}\right)$$

where U is a 2 by 2 unitary matrix. So the little group for this special stratum is U(2). We will call the elements of this special stratum charges. It can be easily verified indeed that they are also characterized by the following properties: their eigenvalues λ in the octet (i. e. for the eigen 3×3 matrix a given by $x \vee a = \lambda a$), are proportional to 0, 1, -1. This is the case of Y, Q and C the hypercharge, electric charge and Cabibbo charge operators respectively. This last charge is defined as follows: let $C_\pm$ [$= (\lambda_1 \pm i\lambda_2)\cos\theta + (\lambda_4 \pm i\lambda_5)\sin\theta$ in the customary basis] be the directions of the weak hadronic currents coupled to leptons in the octet; then $\sqrt{3}C = C_+ \vee C_+ = C_- \vee C_-$. The little group of Y is generated by isospin and hypercharge gauge transformations; the little group of Q is generated by u-spin and electric gauge transformations. The little group U(2) of Y, Q and $\check{C}$ are respectively the invariance group of hadronic semi-strong, electric and maybe weak interactions. Of course the origin $x = 0$ is a minimal stratum with SU(3) itself as little group.

SU(3) Invariant Function on the Octet

Let f be an infinite differentiable function on $\rho(8)$ invariant under SU(3), i. e., :$\forall u \in SU(3)$, $f(x) = f(u\, x\, u^{-1})$. It can be shown that f is a function of x only through the invariants $\gamma(x)$ and $\mu(x)$, i. e., $f(x) \equiv f(\gamma(x), \mu(x))$. Let us compute grad f:

$$\text{grad } f(x) = \frac{\partial f}{\partial \gamma} x + \frac{\partial f}{\partial \mu} x \vee x \tag{5}$$

So grad f = 0, if x = 0. There are also two other cases where grad f = 0:

1) x is a charge; it satisfies (4), then

$$\text{grad } f(x) = 0 \Leftrightarrow \frac{\partial f}{\partial \gamma} \eta(x) + \frac{\partial f}{\partial \mu} = 0.$$

2) x is such that $\frac{\partial f}{\partial \gamma}(x) = 0 = \frac{\partial f}{\partial \mu}(x)$.

To satisfy these last two equations is "more difficult" than to satisfy the one of case 1.

All physics papers I read, which study the breaking of SU(3) from the octet find solution 1). It is really independent of the model.

Let us assume now that f depends only on the unit vectors ($\gamma(x) = 1$). They form the sphere S_7. The function f depends only on $\mu(x)$. The generic stratum is made of one parameter family ($-2^{\frac{2}{3}}/3 < \mu < 2^{\frac{2}{3}}/3$) of 6 dimensional orbits. The minimal stratum is made of the two four-dimensional orbits $\mu(x) = \pm 2^{\frac{2}{3}}/3$ of normalized charges. grad f is tangent to S_7: $x \cdot \text{grad } f(x) = 0$. So

$$\text{grad } f = (x \vee x - \mu(x)\, x) \frac{\partial f}{\partial \mu}.$$

Then grad f does vanish on both orbits of the minimal stratum of S_7. This proves our theorem in this particular case.

Nature of the Equilibrium

From this purely geometrical study, can we say something about the nature of the equilibrium? The answer is yes. We can always choose in a neighborhood of x on the manifold M (here S_7) a local coordinate system u^ℓ ($\ell = 1, \dots, \dim M$) with origin in x. Then if grad $f(x) = 0$

$$f(x + u^\ell) = f(x) + \frac{\partial^2 f}{\partial u^\ell \partial u^r} u^\ell u^r + \text{terms of the third degree in the u's}$$

If $\frac{\partial^2 f}{\partial u^\ell \partial u^r} \neq 0$ for some ℓ, r, the extremum will be called quadratic; it will be called regular if $\det \frac{\partial^2 f}{\partial u^\ell \partial u^r} \neq 0$ i.e., the quadratic form $\frac{\partial^2 f}{\partial u^\ell \partial u^r} u^\ell u^r$ is nondegenerate. As is well known, it is always possible to choose the coordinate system such that the quadratic form is diagonal. Then the number of positive elements, p, and the number of negative elements, q, are independent of the choice of coordinates; q = n will correspond to an absolute maximum and p = n to an absolute minimum. Let r = n - p - y. In the general case we will say that an extremum is a p times stable, a q times unstable and a r times neutral equilibrium.

For a SU(3) invariant function on S_7, an extremum on the special stratum has r = 4, on the generic stratum r = 6. (More generally, r = dimension of the orbit.) So, for the generic stratum an extremum is of the type

$$p \text{ or } q = 1, \; r = 6$$

To study the nature of an extremum at x on the special stratum, one can study the orbits of the little group $U(2)_x$ in the neighborhood of x. One finds sphere S_3. This shows that the extremum is of the type:

$$p \text{ or } q = 3, \; r = 4$$

Finally, in the case we are studying we can even give some relations on the number and nature of orbits of extrema. Indeed the invariant function for S_7 is a function of the parameter μ only, defined in the interval

$$\frac{-2^{\frac{2}{3}}}{3} \leq \mu \leq \frac{2^{\frac{2}{3}}}{3}$$

whose extremities $\mu = \pm 2^{\frac{2}{3}}/3$ correspond to the singular stratum. So if f has only quadratic extrema with M and m, M′ and m′ as the number of maxima and minima orbits, on the special generic stratum is

$$M + m = 2, \; M - m + 2(M' - m') = 0$$

This is an immediate result of well known properties of one variable continuous and differentiable functions defined on a compact interval.**

Induced Dynamical Breaking

We have well understood that in a SU(3) invariant theory all octet directions (= points of S_7) do not play the same role and one might expect the directions belonging to the special strata (charges) to give a special, more fundamental family of solutions. The hypercharge Y is one of those directions forming the special strata. How can the direction Y be defined? Is the situation similar to that of magnetism when an arbitrary small magnetic field can orient the magnetization vector? Cabibbo[6)] has asked the following question: can the electromagnetic interaction in the direction Q and the weak interactions in the direction C_+, C_- destroy the 4-fold neutral equilibrium of the charge orbit and pick up the Y direction? Indeed a neutral equilibrium can be destroyed by the smallest perturbation and one could hope that the 4-dimensional orbit of extrema is resolved by the perturbation into a finite number of isolated extrema.

Cabibbo and also Radicati and I have found by a variational principle that Y must be a solution of the equation m $\mathcal{E}(8)$:

$$\beta_1 y + \beta_2 y \vee y + a \vee y + b = 0$$

where β_1, β_2 can be a function of $\mu(y)$ and a, b are vectors of $U(2)_c$, the $U(2)_c$ Lie algebra being generated by C_+, C_-, Q. Furthermore, Radicati and I have proven that this equation (and even a more general one with a supplementary term $a' \wedge y$, with $a' \in U(2)_c$) cannot have a charge as solution. So the electromagnetic and weak interactions are such that they displace all the extrema from the U-dimensional charge orbit and none are left!

**There is a full-fledged mathematical theory, that of M. Morse, which gives the existing relations between the number and the types of extrema of a real valued differentiable function on M (with only isolated quadratic regular extrema) and the Betti numbers of the compact manifold M. There exist also powerful theorems of the converse type. For instance: if a differentiable real valued function on the compact manifold M has only two extrema, M is homeomorphic to a sphere (proven by Reeb in 1952 if the extrema are quadratic regular and by Milner and also Rosen in 1960 for the general case). The extension of Morse theory to invariant functions by a group G acting on M has been made by A. Wasserman, "Morse theory for G manifolds," Bull. Am. Math. Soc. 71, 384(1964).

We could consider the neighborhood of one charge orbit as a 7-dimensional gutter, with 3-dimensional walls and a 4-dimensional flat bottom. The electromagnetic and weak interaction deform and tilt this gutter such that the lowest points (few isolated ones) are now on what was the wall (and not far from what was the bottom). This has been emphasized by Pais.[7] The distance from y to a minimum can be of the order of the weak coupling constant.

Conclusions

These views on the SU(3)-breaking are probably oversimplified and too schematic. I feel however that they cannot be ignored by the physicists interested in this problem. They also cannot ignore the mathematical tools presented here or those existing in the mathematical literature (see for instance footnote **) for solving similar problems.

Acknowledgments

I am grateful to the Boulder Summer Institute for Theoretical Physics and to its director, Dr. K. T. Mahanthappa, for an opportunity to expose the ideas contained in these notes which are the result of a collaboration with L. Radicati.

References

1. L. Michel and L. Radicati, Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energy, ed. B. Kursunoghi et al., W.H. Benjamin, Inc., New York, (1965).
2. R. E. Cutkosky and P. Tarjanne, Phys. Rev. 132, 1354 (1963).
3. G. Domokos and P. Suranyi, Sov. J. Nucl. Phys. 2, 361 (1966).
4. P. de Mottoni and E. Fabri, Nuovo Cimento 54A, 42 (1968).
5. R. Brout, Nuovo Cimento 47A, 932 (1967).
6. N. Cabibbo, Internal Report #141, Istituto di Fisica G. Marconi, Universita di Roma (1967).
7. A. Pais, Phys. Rev. 173, 1587 (1968).