THE GROUP OF AUTOMORPHISMS OF THE POINCARÉ GROUP*

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The group of continuous automorphisms of the Poincaré group is generally known to physicists. The aim of this paper is to show that every automorphism of the Poincaré group is a continuous automorphism. To prove it we use two theorems, one of Zeeman,[1] one of Wigner[2] established without topology.

Notations

We denote by a, b, c ... the elements of $\mathcal{T}$, the translation group. Let G, $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, $g_{\alpha\beta}=0$ where $\alpha \neq \beta$, be the metric tensor of Minkowski space. We denote by $\mathcal{L}''$ the group of homogeneous linear transformation which preserves this metric, i.e.,

$$A \in \mathcal{L}'' \Leftrightarrow A G A^T = G \tag{1}$$

where A^T is the transpose of A.

We can consider $\mathcal{T}$ as a Minkowski space, we denote Aa the transform of a by A, and we denote the Minkowski scalar product by

$$(Aa)^2 = (Aa \cdot Aa) = a \cdot a = a^2 . \tag{2}$$

Lorentz transformations such that $A_0^0 \geqq 1$ form a subgroup of index two $\mathcal{L}'$ of $\mathcal{L}''$. This group $\mathcal{L}'$ is the Lorentz orthochronous group. For the connected Lorentz group $\mathcal{L}$ (subgroup of index two of $\mathcal{L}'$) the A's also satisfy $\det A = 1$.

We define the corresponding Poincaré groups $\mathcal{P}$, $\mathcal{P}'$, $\mathcal{P}''$ as semi-direct product of the corresponding $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$ by $\mathcal{T}$; i.e., we have the multiplication law

$$(a,A)\ (b,B) = (a + Ab,\ AB). \tag{3}$$

We define a dilatation by a number $\lambda > 0$, as

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$$(a, A) \rightarrow (\lambda a, A). \tag{4}$$

The dilatations form a group $\mathcal{D}$ isomorphic to the multiplicative group of positive real numbers. Since every dilatation λ commutes with every Lorentz transformation A, we can consider the direct products

$$\mathcal{G} = \mathcal{P} \times \mathcal{D}, \quad \mathcal{G}' = \mathcal{P}' \times \mathcal{D}, \quad \mathcal{G}'' = \mathcal{P}'' \times \mathcal{D}. \tag{5}$$

<u>THEOREM</u>:

$$\mathrm{Aut}\,\mathcal{P} = \mathrm{Aut}\,\mathcal{P}' = \mathrm{Aut}\,\mathcal{P}'' = \mathcal{G}''$$

(where Aut $\mathcal{P}$ means the group of automorphisms of the "abstract" group $\mathcal{P}$, i.e., neglecting its topology).

<u>PROOF OF THE THEOREM</u>: Recently, E. C. Zeeman[1] proved the remarkable theorm that $\mathcal{G}'$ is the group of <u>all</u> transformations on Minkowski space preserving the causal order relation. He also gives an equivalent formulation:

<u>THEOREM Z</u>: The group g" is the group of <u>all</u> transformations which transform light vectors into light vectors.

We shall need the following lemma:

LEMMA 1: Given a light vector <u>b</u> there is a $A \in \mathcal{L}$ such that $Ab = \alpha b$ with $\alpha > 0$.

Let t be the unit time vector, then $b = \beta(t + n)$ where n is a unit space vector: $t^2 = 1 = -n^2$, $t \cdot n = 0$. The skew symmetric tensor $t \wedge n$ has a matrix $t_\mu n_\nu - n_\mu t_\nu$ with zero trace. Hence the matrix $e^{-\alpha(t \wedge n)}$ has determinant 1. It is a Lorentz transformation which satisfies our conditions:

$$e^{-\alpha(t \wedge n)} \in \mathcal{L}, \quad e^{-\alpha(t \wedge n)} b = \alpha b. \tag{6}$$

The translation group $\mathcal{T}$ is the only proper invariant subgroup of $\mathcal{P}$. An automorphism of a group transforms an invariant proper subgroup into an invariant proper subgroup. Hence $\mathcal{T}$ is a characteristic subgroup of $\mathcal{P}$, i.e., it is transformed into itself by all automorphisms of $\mathcal{P}$.[3] Therefore, in order to define an automorphism Φ of $\mathcal{P}''$ (or $\mathcal{P}$ or $\mathcal{P}'$) we need three mappings:

$$\mathcal{T} \overset{f}{\dashrightarrow} \mathcal{T}, \quad \mathcal{L}'' \overset{g}{\dashrightarrow} \mathcal{T}, \quad \mathcal{L}'' \overset{G}{\dashrightarrow} \mathcal{L}'' \tag{7}$$

which define the image of the elements (a, 1) and (0, A) of $\mathcal{P}''$

$$\Phi(a, 1) = (f(a), 1) \tag{8}$$

$$\Phi(1, A) = (g(A), G(A)). \tag{9}$$

From the unicity of the decomposition of every element of $\mathcal{P}''$:

$$(a,A) = (a,1)(0,A) \tag{10}$$

since Φ is an automorphism, we obtain from (3)

$$\Phi(a,A) = (f(a) + g(A), G(A)). \tag{11}$$

The transformation of the group law (3) by the automorphism Φ yields

$$f(a) + g(A) + G(A)[f(b) + g(B)] = f(a + Ab) + g(AB) \tag{12}$$

$$G(A)\, G(B) = G(AB). \tag{13}$$

Equation (13) shows that G is an automorphism of the Lorentz group $\mathcal{L}''$. Equation (12) is equivalent to the three following equations obtained with the respective choices $A=B=1$, $a=b=0$; $a=0$, $B=1$

$$f(a) + f(b) = f(a + b) \tag{14}$$

$$g(A) + G(A)\, g(B) = g(AB) \tag{15}$$

$$G(A)\, f(b) = f(Ab) \tag{16}$$

Equation (14) shows that f is an automorphism of $\mathcal{T}$.

Let us first study Eq. (16). We choose an arbitrary light vector b and we choose a Lorentz transformation A such that $Ab=2b$. Then (16) reads

$$G(A)\, f(b) = 2f(b); \tag{17}$$

indeed $f(Ab) = f(2b) = f(b + b) = 2f(b)$ since $f \in \mathrm{Aut}\,\mathcal{T}$. So $\ell = f(b)$ is a proper vector of the Lorentz transformation G(A) with proper value 2. It is a light vector $((2\ell)^2 = 4\ell^2 = \ell^2 \quad \ell^2 = 0)$. Hence f transforms any light vector into a light vector and by the Zeeman theorem,

$$f \in \text{"}\mathcal{L}\text{"} \times \mathcal{D} \subset \mathrm{Aut}\,\mathcal{T}. \tag{18}$$

This determines all possible f's.

Equation (16) can also be written as a relation among automorphisms of $\mathcal{T}$: $G(A) \circ f = f \circ A$ or

$$G(A) = f \circ A \circ f^{-1}. \tag{19}$$

This fixes completely G(A). If $f \in \mathcal{D}$, then G is the identity automorphism of $\mathcal{L}''$. If $f \in \mathcal{L}''$ then G(A) is the inner automorphism of $\mathcal{L}''$

induced by f. (Note that Int $\mathcal{L}'' = \mathcal{L}'$ when Int means group of inner automorphisms.)

We are left with the study of Eq. (15). Without using the topology of the Poincaré group Wigner, in his fundamental paper on the Poincaré group[4] has shown (with a slight and nonessential change in its proof) that every mapping $\mathcal{L} \xrightarrow{g} \mathcal{T}$ which satisfies (14) is of the form

$$g(A) = g - G(A)g = (1 - G(A))g \tag{20}$$

where g is a fixed translation.[5] This form corresponds exactly to the inner automorphism of $\mathcal{P}''$ induced by (g,G); indeed

$$(g,G)\,(a,A)\,(g,G)^{-1} = (Ga + (1 - GAG^{-1})g,\ GAG^{-1}) \tag{21}$$

If we consider that $\mathcal{P}$ (resp $\mathcal{P}'$) is characteristic subgroup of $\mathcal{P}'$ and $\mathcal{P}''$ (resp. $\mathcal{P}''$), this ends the proof of our theorem.

The Wigner[2] and Zeeman[1] results and their consequence that every automorphism of the Poincaré group is continuous must be significant for physics. The reader who does not wonder at these results should consider for one moment the huge group of automorphisms of $\mathcal{L}$ or $\mathcal{T}$ both (~ permutation group of a set whose power is the continuum) one obtains when their topology is forgotten![6]

Acknowledgments

I am grateful to Professor Barut and the organizers of the Boulder symposium on Lorentz group for a short visit there. Although this paper is not directly related to the two lectures I gave, I think it fit to publish this theorem on automorphism on Poincaré group in a volume celebrating the twenty-fifth anniversary of Professor Wigner's fundamental paper on this group.

References

1. E. C. Zeeman, J. Math. Phys. 5, 491 (1964).
2. E. P. Wigner, Ann. Math. 40, 149 (1939).
3. The definition of an invariant subgroup $\mathcal{T}$ of a group $\mathcal{P}$ is that $\mathcal{T}$ is transformed into itself by inner automorphisms of $\mathcal{G}$.
4. See Reference 2. Compare his Eqs. (38), (39) with our Eqs. (14) and (19). His proof is on pages 175, 176.
5. Condition (14) is well known in the theory of group extensions (e.g., Baer automorphismen von Enventerungs-gruppen, Act. sci. Ind No 205 Hermann Paris 1935). In cohomology language, (14) $\Rightarrow$ (19) means that every one-cocycle $g \in Z^1(\mathcal{L}, \mathcal{T})$ is a coboundary. So, in fact, Wigner has proven, without topology, that $H^1(\mathcal{L}, \mathcal{T}) = 0$.
6. L. Michel, Nuclear Physics 57, 356 (1964), §7.

NOTE ADDED IN PROOF

The proof given here is incomplete, because Theorem Z' is not correctly quoted. What is needed is an unpublished equivalent of the theorem "The group $\mathcal{G}''$ is the group of all permutations of space-time which preserve the nature of separation, time-like, space-like, light-like, of two points." It is easy to show that $f \in \mathrm{Aut}\,\mathcal{T}$ in equation (16) is of such a nature.

Since this manuscript was written, the author has established a more direct proof of the theorem. It will be published in the lecture notes of the Brandeis Summer School 1965.