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GEOMETRICAL PROPERTIES OF THE FUNDAMENTAL INTERACTIONS.

by

Louis MICHEL and Luigi A. RADICATI*

Institut des Hautes Etudes Scientifiques
91 - BURES-SUR-YVETTE - France

Lecture presented by the first author

on September 19, 1969

at the "CONVEGNO MENDELEVIANO" in Roma

* Permanent address : Scuola Normale Superiore PISA Italy.

Introduction.

One can read in the scientific books written fifty years ago, that our whole universe is made up of three constituents : protons, electrons, and photons. This elegant view was destroyed in 1930 by one experiment performed in this town by Rasetti [1] : the measurement of the spin and statistics of the N^{14} nucleus.

The following year four more particles were predicted : the neutrino by Pauli [2], the antiproton and the antielectron by Dirac [3] and the neutron which was expected by several groups. The last two were discovered after one year [4],[5], the first two, twenty five years later [6],[7]. The rate of discovery of particles has risen so sharply in the sixties that an exact count is at present uncertain. If you add to the photon, 8 leptons, 48 mesons, 111 baryons, and the corresponding antibaryons you reach a provisional total of 279. However, some physicists have not given up the hope of simplicity, and expect to build our universe with three quarks and their antiquarks.

To others, the simplicity appears today in the small number of interactions among this crowd of different particles. As you all know there are in nature only four or five different interactions :

- gravitational, electromagnetic, nuclear or strong, weak and perhaps CP violating.

Gravitational effects are so negligible at the miscroscopic scale that we shall not discuss them here. It is not yet clear whether the CP-violating interaction, discovered only five years ago and observed only in K-meson decay is a new interaction -which has then to be super weak- or a peculiar manifestation of the known ones.

The strong, the electromagnetic and the weak interactions differ by their intensity, their range and their properties with respect to the transformations of the internal symmetry group. In this lecture we will however focus our attention not on these differences but on the striking similarity in the way the three interactions break the underlying symmetry (e.g. SU(3) x SU(3)). We will show that the directions defined in the internal symmetry space by both the electromagnetic and the lepton fields have simple and unique geometrical properties. The situation is more complicated for the strong interaction breaking. There exists several interesting approximations for its symmetry property : some of them have the same type of characteristic geometrical behaviour as the electromagnetic and the weak interactions.

Strong Interaction Symmetry.

Strong interaction alone does not allow us to distinguish between baryons or mesons with different electric charge which belong to the same isospin multiplet. This reflects the property of the strong interaction to be invariant under a group G larger than the Poincaré group $\mathcal{P}$. The group G is the direct product $G = \mathcal{P} \times U(2)$. The U(2) symmetry is broken by the electromagnetic and weak interactions.

In the recent years, larger and larger broken symmetry groups have been proposed for the physics of hadrons. They form the lattice of fig. 1.

SYMMETRY GROUPS and their Dimensions.

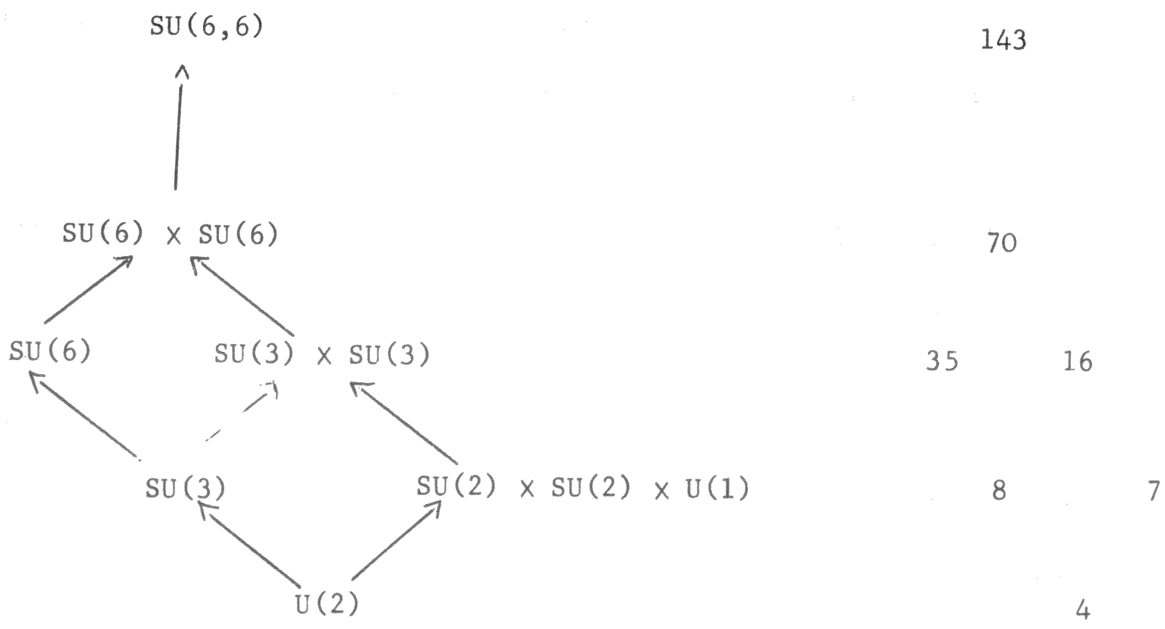

Fig. 1. Lattice of internal symmetry groups used in hadronic physics ; $\longrightarrow$ means injection as subgroups.

The larger is the internal symmetry group, the coarser is the approximation in which this symmetry is exact.

Here we will consider the symmetry corresponding to SU(3) x SU(3) and its subgroups. SU(3) x SU(3) is the smallest group which couples in some way Poincaré invariance and internal symmetry. In the limit where the symmetry is exact, the full symmetry group is indeed the semi direct product (denoted by □)

$$\mathcal{P}_\circ \,\square\, [(SU(3) \times SU(3))_\square \, Z_2] = (\mathcal{P}_\circ \times SU(3) \times SU(3))_\square \, Z_2$$

where $\mathcal{P}_\circ$ is the connected Poincaré group and Z_2 is the two element generated by the space inversion P which acts on $\mathcal{P}_\circ$ by an outer automorphism and permutes the two factors of SU(3) x SU(3) .

The Breaking of U(2) symmetry by the electromagnetic and weak interactions.

We can understand this breaking by analogy with the Zeeman effect. When an atom (which is a physical system with a rotational symmetry) is placed in a magnetic field $\vec{B}$, one must add to the Hamiltonian the interaction term

$$H = \int \vec{B}.\vec{\mu}(x) \, d^3x \qquad (1)$$

where $\vec{\mu}(x)$ is the vector operator representing the magnetic moment density. Of course $\vec{B}.\vec{\mu}$ is invariant under rotations, but the rotational symmetry is broken because the magnetic field marks a preferred direction.

Similarly, we can say that the electromagnetic interaction marks a preferred direction q in the vector space of the Lie algebra of $\mathcal{U}(2)$. The $\mathcal{U}(2)$ Lie algebra is a direct sum $U(1) \oplus SU(2)$: the $U(1)$ corresponds to the hypercharge y , the $SU(2)$ is spanned by three orthonormal isospin vectors t_i $(i = 1, 2, 3)$ which are chosen by a traditional convention. The direction q singled out by the electromagnetic interaction is

$$q = t_3 - \frac{1}{2} y \qquad (2).$$

This is the Gell-Mann-Nishijima relation. [8]

It would make no sense to say that we can recognize a direction if we were not able to recognize all the others and therefore orient ourselves in the $\mathcal{U}(2)$ space. Two other directions t_1 and t_2 can be recognized with the help of the weak hadronic currents for β-decay. The strong mass splitting and the hypercharge conservation by strong and electromagnetic interactions allow us to recognize the direction y .

The analogy with the interaction in a constant external magnetic field goes even deeper. Indeed according to the assumption of Feynmann and Gell-Mann [9] the vector part of the weak hadronic currents $v^{\pm}_{\mu}(x)$ and the electromagnetic current $j_{\mu}(x)$ are three components of the same, $U(2)$ vector operator $\vec{v}(x)$. In analogy to (1) , the electromagnetic and weak interaction terms of the Hamiltonian can be written :

$$H_{em} = e \int \vec{A}^{\mu}(x) . \vec{v}_{\mu}(x) \, d^3 x \tag{3}$$

$$H_w = \frac{G}{\sqrt{2}} \int \sum_{\varepsilon=\pm 1} \vec{L}^{\mu}_{(\varepsilon)}(x) . \vec{v}^{(\varepsilon)}_{\mu}(x) \quad d^3x \tag{3'}$$

where $L^{\mu}_{(\pm)}(x)$ are the charged $\pm$ leptonic currents which single out the directions $t_1 \pm i\, t_2$ of the complexified vector space of $\mathcal{U}(2)$ in the same way as the electromagnetic field $A^{\mu}(x)$ distinguishes the direction q .

Thus even though we can not change the directions of the external fields (as we could do for the magnetic field of our example) we can nevertheless orient ourselves completely in the $\mathcal{U}(2)$ space since the vectors y , q_1 , t_1 and t_2 span the whole space of the $\mathcal{U}(2)$ Lie algebra.

Currents as tensor-operators.

Let us recall the definition of tensor-operators, these fundamental tools for the expression of group invariance in quantum mechanics. Given a Lie group G represented by unitary operators $\in \mathcal{L}(\mathcal{H})$ acting on the Hilbert space $\mathcal{H}$ of states, an $\mathcal{E}$-tensor operator is an intertwinning operator between the vector space $\mathcal{E}$ of a linear representation of G and the space $\mathcal{L}(\mathcal{H})$. If $\mathcal{G}$ is the real vector space of the Lie algebra of G , there is a remarkable tensor operator F whose image is the set of self adjoint "group generators". They form a representation up to a factor i of the Lie algebra

$$\forall a , b \in \mathcal{G} , [F(a),F(b)] = iF(a \wedge b) \qquad (4)$$

where $\wedge$ denotes the Lie algebra law. Let the representation of $\mathcal{G}$ on $\mathcal{E}$ be : $a \rightsquigarrow D(a)$; an $\mathcal{E}$-tensor operator $\mathcal{E} \xrightarrow{T} \mathcal{L}(\mathcal{H})$ can be equivalently defined by the relation

$$\forall a \in \mathcal{G} , \forall \underset{\sim}{r} \in \mathcal{E} , [F(a),T(\underset{\sim}{r})] = i\, T(D(a)\underset{\sim}{r}) \qquad (5) .$$

As we have seen, electromagnetic and hadronic weak vector currents are the images in the directions q and $t_1 \pm i\, t_2$ of the same $\mathcal{G}$-tensor operator where $\mathcal{G}$ is the Lie algebra of $\mathcal{U}(2)$. We will thus write : $v_\mu(x,q)$ and $\frac{1}{\sqrt{2}}\, v_\mu(x,t_1 \pm i\, t_2)$ for the electromagnetic and weak vector currents respectively. After the extension of the strong interaction invariance group[10] from $\mathcal{U}(2)$ to SU(3) Cabibbo's hypothesis[11] extended the variance of these currents. They became octet-operators where the octet-space is the eight dimensional space $\mathcal{E}_8$ of the Lie algebra of SU(3) . With the Cartan Killing form this is an euclidean space and from now on, we shall use only unit length vectors (belonging to the unit sphere $S_7 \subset \mathcal{E}_8$.)

Explicitely,

electromagnetic currents	$= \frac{2}{\sqrt{3}}\, v^\mu(x,q)$	(6)
weak vector currents	$= \frac{1}{\sqrt{2}}\, v^\mu(x,c_1 \pm ic_2)$	(6')
weak axial vector currents	$= \frac{1}{\sqrt{2}}\, a^\mu(x,c_1 \pm ic_2)$	(6") .

Indeed Cabibbo also assumed that the axial vector and weak vector currents are two different octet operators, but in the <u>same</u> directions

c_1 and c_2 . These directions are different from t_1 and t_2 in order to take into account the hypercharge violating weak transitions ; they make an angle θ (=Cabibbo's angle) with the former directions. The total hadronic weak current that we denote by

$$h^{\mu}(x,c_i) = v^{\mu}(x,c_i) - a^{\mu}(x,c_i) \qquad (7)$$

is therefore another octet-operator.

In the next section we will show the geometrical properties of the physical directions appearing in (6) (6') and (6") .

Some geometrical concepts. Their application to the SU(3) octet.

The SU(3) linear action in the octet space $\mathcal{E}_8$ distinguishes some directions.[12] When a group G acts on a set M , the set of all transforms of a given point $p \in M$ is called the orbit $G(p)$ of p and the set of all group elements leaving p fixed is a subgroup $G_p \subset G$ called the little group or the isotropy group of p . Two points of a same orbit have conjugated little groups. We call stratum the set of all points with the same little group up to a conjugation and the orbits of a stratum are said of the same type. To summarize, by the action of G , the set M is partitionned into strata which are partitionned into orbits of the same type. When G is a compact Lie group acting differentiably on the compact manifold M , there is a stratum (called generic) which is open dense. For example in the action of SU(3) on the unit sphere

$S_7 \subset \mathcal{E}_8$, the generic stratum is a one parameter set of 6 dimensional orbits, whose little groups are the Cartan subgroups ($\sim U_1 \times U_1$) of SU(3) . There is one more stratum composed of two four-dimensional orbits. Their elements will be called exceptional vectors ; they have a larger little group : $\mathcal{U}(2)$. The hypercharge and electric charge directions y and q are such exceptional vectors. Their little groups are denoted by $\mathcal{U}_y(2)$ and $\mathcal{U}_q(2)$ and their semi-simple part $SU_y(2), SU_q(2)$ are called the isospin and u-spin groups in the physics litterature. The orbits of exceptional vectors have the following properties :

1° <u>they are critical</u> ; this means : consider a differentiable real function f on M , invariant by G (i.e. constant on the orbits). Its differential df_p at p is an element of the dual of the vector space $T_p(M)$, the plane tangent to M at p . An orbit of G on M is critical if for <u>all</u> G-invariant real differentiable functions f on M , df = 0 on the orbit. In table 1 we list all examples of critical orbits which we discuss here.

2° <u>their vectors are idempotents of the canonical symmetrical algebra</u> ; consider a compact Lie group G and an irreducible linear representation D on $\mathcal{E}$. If in the reduction of the tensor representation $D \otimes D$ on $\mathcal{E} \otimes \mathcal{E}$, the irreducible representation D appears once and once only in the symmetrical part of the tensor product, the corresponding intertwinning operator V (defined up to a multiplicative factor) $\mathcal{E} \otimes \mathcal{E} \xrightarrow{V} \mathcal{E}$ defines a symmetrical algebra

$$V(x \otimes y) = x \vee y = y \vee x \tag{8}$$

which has G as group of automorphisms. Such a canonical symmetrical algebra exists for the adjoint representation of the classical simple Lie groups only for the SU(n) , $n \geq 3$. That of SU(3) has been studied by Gell-Mann who denoted by d_{ijk} its structure constants.

An idempotent vector of this algebra satisfies

$$x \vee x = \lambda x \qquad (9) .$$

This equation is obtained in all SU(3)-invariant bootstrap models that we have seen published.[13]

Properties 1° and 2° are not unrelated. For every real irreducible linear representation of a compact Lie group G , there exists an invariant euclidean scalar product. For a representation of G , on $\mathcal{E}$, which possesses a canonical symmetrical algebra, one can define :

$$\{x,y,z\} = (x \vee y,z) \qquad (10).$$

This is a trilinear invariant, completely symmetrical under the permutation of x , y , z and any differentiable function $f(\mu)$ with $\mu = \{x,x,x\}$ defined on the unit sphere $(x,x)-1 = 0$, is G-invariant. In order to find its extrema, one has to vary the function $f(\{x,x,x\}) - \lambda'(1-(x,x))$ where λ' is a Lagrange multiplier. This yields equation (9) with $\lambda = \frac{2}{3} \lambda'(df/d\mu)^{-1}$.

Since we have been interested only in the directions of the vectors of $\mathcal{E}_8$ we should probably consider, instead of the unit sphere S_7 , the real projective space P_7 which is the set of directions of $\mathcal{E}_8$.

(P_7 is obtained from S_7 by identifying the two points of each diameter). In the action of SU(3) in P_7 there are 3 strata : the generic one (little group $U(1) \times U(1)$); another one consisting of the critical orbit of exceptional vectors (defined up to a sign, little group $\mathcal{U}(2)$) and finally the stratum which contains the orbit of the root-vectors of the SU(3)-Lie algebra (little group $(U_1 \times U_2)_{\square} Z_2$). This orbit is also critical, but equation (9) defined on $\mathcal{E}_8$, does not apply to its vectors. It is remarkable that the physical directions c_1 and c_2 belong to this new critical orbit.

Given a unit root-vector s, then $\sqrt{3}\ s \vee s$ is a unit exceptional vector. The exceptional vector z defined by the weak interaction :

$$z(c_i,c_j) = \sqrt{3}\ c_i \vee c_j \tag{11}$$

with

$$i,j = 1,\ 2,\ 3;\ c_3 = c_1 \wedge c_2 \tag{12}$$

is sometimes called the direction of the weak hypercharge. It is possible that this direction can be observed physically. Indeed the most commonly proposed form of non leptonic weak interaction is

$$H_{N.L.} = \frac{G}{\sqrt{2}} \int h^{\mu}(x, c_1 + i\ c_2)\ h_{\mu}(x, c_1 - i\ c_2)\ d^3x \tag{13}$$

with the drawback that $H_{N.L.}$ is the image of a reducible tensor operator with some component in the "27" irreducible representation of SU(3) . The $\Delta T = \frac{1}{2}$ rule for the weak transitions with $|\Delta y| = 1$ suggests that this 27 component is negligible compared to the octet component. According to the proposal of one of us,[14] the non-leptonic Hamiltonian would instead be

$$H_{N.L.} = \frac{G}{\sqrt{2}} \int (h^{\mu}(x) \vee h_{\mu}(x)) (z) \, d^3x \quad , \tag{14}$$

i.e. it would be the component along the weak hypercharge z of an irreducible octet-tensor operator. It is compatible with the known experimental data.

Let us finally note that y and z define q :

$$q(1 - (y,z)) = \sqrt{3} \quad y \vee z + \frac{1}{2} (y + z) \tag{15}$$

where

$$(y,z) = 1 - \frac{3}{2} \sin \Theta \tag{15'}$$

The SU(3) x SU(3)$_{\square}$ Z_2 symmetry. [15]

The SU(3) x SU(3) symmetry becomes an exact symmetry of the hadronic world when the masses of the octet of pseudo scalar mesons are neglected. Remark that it is not much more drastic to neglect these masses than to neglect their differences as it is already implied by SU(3) . As a matter of fact a much milder approximation than SU(3) is to neglect only the π-meson mass (only 140 MeV which is smaller than mass differences within the octet of pseudo scalar mesons). This corresponds to considering a SU(2) x SU(2) x U(1) subgroup of SU(3) x SU(3) . The SU(3) discussed in the previous section is the diagonal subgroup of SU(3) x SU(3) . The two SU(3) factors are called the chiral $SU^{\pm}(3)$. The vector space $\mathcal{E}_{16}$ of the Lie algebra of SU(3) x SU(3) is a direct sum of two octet spaces

$$\mathcal{E}_{16} = \mathcal{E}_8^{(+)} \oplus \mathcal{E}_8^{(-)} \tag{16}$$

whose vectors we denote by

$$\tilde{a} = a^+ \oplus a^- \qquad (17).$$

On this space there is a SU(3) x SU(3) invariant euclidean scalar product

$$(\tilde{a},\tilde{b}) = \frac{1}{2}(a^+,b^+) + \frac{1}{2}(a^-,b^-) \qquad (18).$$

The Lie algebra is

$$\tilde{a}_\wedge \tilde{b} = (a^+{}_\wedge b^+) \oplus (a^-{}_\wedge b^-) \qquad (19)$$

and the canonical symmetrical algebra is

$$a_\vee b = (a^+{}_\vee b^+) \oplus (a^-{}_\vee b^-) \qquad (20).$$

The electromagnetic current, the vector part and the axial part of the weak hadronic currents are all in the image of the same $\mathcal{E}_{16}$-tensor operator $h(x,\tilde{a})$ for the directions :

$\tilde{q} = q \oplus q$ for the electromagnetic current

$\tilde{c}_1 \pm i\,\tilde{c}_2$, with $\tilde{c}_i = 0 \oplus c_i$ for the weak current

and $\tilde{z} = 0 \oplus z$ is the direction of the weak hypercharge.

The symmetrical algebra has only two different types of idempotents and $\tilde{q}$ and $\tilde{z}$ are two examples of these two types.

The orbit of $\tilde{z}$ on S_{15} is critical for the SU(3) x SU(3) action, whereas the orbit of $\tilde{q}$ is not. It becomes critical when the group is extended to $(SU(3) \times SU(3))_\square Z_2$ .i.e., as we have seen, when one takes into account space reflexions.

When we go to the projective space P_{15} , two new types of critical orbits appear.

- One corresponds to the root vectors of SU(3) x SU(3) and the physical directions $\tilde{c}_1$ and $\tilde{c}_2$ are illustrations of this case ;

- The other type can be represented by the orbit of unit vectors (up to sign) such as

$$s \oplus \pm s$$

where s is a root vector of SU(3). This orbit contains vectors of opposite parity. *

The Breaking of the $(SU(3) \times SU(3))_{\square} Z_2$ by the Strong Interaction.

In the paper in which he introduced this larger symmetry, Gell-Mann [10] proposed that the breaking of SU(3) x SU(3) due to the strong interactions occurs through a tensor operator of the

$$(3,\bar{3}) \oplus (\bar{3},3)$$

representation.

* For instance the image of this orbit by the current, $h_\mu(x,\tilde{a})$ contains both vectors $(\tilde{a} = \pm\, s \oplus s)$ and pseudovectors $(\tilde{a} = \pm\, s \oplus -\, s)$ which are transformed into each other by SU(3) x SU(3).

This 18-dimensioned representation, which is irreducible for $(SU(3) \times SU(3))_{\square} Z_2$, appears naturally in the quark model and SU(3) (the diagonal subgroup of $SU(3) \times SU(3)$) is then the little group of the direction of $\mathcal{E}_{18}$ along which the symmetry is broken in the SU(3)-approximation. In a more recent paper Gell-Mann, Oakes and Remmer [16] have suggested that the direction of the breaking in $\mathcal{E}_{18}$ is in fact closer to the direction invariant under $SU_y^+(2) \times SU_y^-(2) \times U_y^d(1)$ (where d means diagonal). On the space $\mathcal{E}_{18}$ of the $(3,\bar{3}) \oplus (\bar{3},3)$ representation there exists a symmetrical canonical algebra with two kinds of idempotents,

$$\underset{\sim}{x} \top \underset{\sim}{x} = \lambda \underset{\sim}{x} \qquad (21).$$

One kind of unit vector solution $(\lambda = \sqrt{\frac{2}{3}})$ corresponds to two critical orbits (see table I) of $(SU(3) \times SU(3))_{\square} Z_2$ on S_{17} with little group $SU^d(3) \times Z_2$.

The other kind ($\lambda = 0$, nilpotent elements) corresponds to a critical orbit with little group

$$(SU_y^+(2) \times SU_y^-(2) \times U_y(1))_{\square} Z_2.$$

These two solutions of (21) are parity conserving and correspond to the two physically interesting breakings of $SU(3) \times SU(3)$ with approximate SU(3) or $SU(2) \times SU(2) \times U(1)$ invariance for the strong interactions.

Open Problems.

The Cabibbo angle Θ establishes the relative orientation of the frames in the spaces of the two representations : the adjoint representation $(1,8) \oplus (8,1)$ and the $(3,\bar{3}) \oplus (\bar{3},3)$. Probably Θ is a projective invariant built with the vectors which are singled out in these two spaces by the different interactions. We do not expect however that the value of Θ could be obtained from purely geometrical considerations.

The nature of the CP-violating interaction is another open question. It might be a new interaction or, as in many models that have been proposed, it may correspond to a small T-violating term of the weak or electromagnetic interactions. It may even be that CP-violation is, as Good, Michel and de Rafael[17] have proposed, a collective effect of the three interactions (weak, electromagnetic and strong) which disappears when some parts of any one of them is turned off.

In any case one expects that the CP-violating transitions will also single out some direction of a $SU(3) \times SU(3)$ representation space. As we have seen, all the critical orbits of $(SU(3) \times SU(3))_{\square} Z_2$ on P_{15} except for one, correspond to the breakings of $SU(3) \times SU(3)$ by the electromagnetic and the weak interactions. It might be that the CP-violating transitions choose the unused type of critical orbit listed in table 1 .

Conclusion.

The breakings of $(SU(3) \times SU(3))_{\square} Z_2$ induced by the coupling of hadrons to the electromagnetic field and to the lepton currents occur respectively in the directions $\tilde{q}$ and $\tilde{c}_1$, $\tilde{c}_2$ of the space $\mathcal{E}_{16}$ of the adjoint representation $((8.1) \oplus (1.8)$. We have shown that in the action of $(SU(3) \times SU(3))_{\square} Z_2$ on the corresponding projective space P_{15} the orbits to which the above directions belong are critical. In the same space P_{15} there are two other critical orbits. One of them contains the weak hypercharge direction $\tilde{z}$ and one of us has suggested[14] that it could be associated with the non-leptonic weak interactions. We wonder if the preferred direction which appears in the CP-violating interaction might not belong to the remaining critical orbit as suggested by the model of ref. 17.

On the unit sphere $S_{17} \subset \mathcal{E}_{18}$ (the space of the $(3.\bar{3}) \oplus (\bar{3}.3)$ representation) the action of $(SU(3) \times SU(3))_{\square} Z_2$ creates two types of critical orbits* . Their little groups (up to a conjugation) are $SU^d(3)$ and $SU_y^+(2) \times SU_y^-(2) \times U_y^d(1)$. They correspond to the two interesting approximations for the breaking of $SU(3) \times SU(3)$ by the strong interaction. The actual direction of the strong breaking is in between those two and its determination is probably related to that of the Cabibbo angle.

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* One more type of critical orbit appears when one goes to the corresponding projective space P_{17} . It corresponds to $(U_y(2) \times Z_2)_{\square} Z_2$ invariance. More details on this case will be given in a forthcoming paper.

In our opinion the notion of critical orbit is very important for understanding the breaking of symmetries. For hadronic physics this notion appears only when one enlarges the isospin group. Indeed in the three dimensional space all directions are transformed into each other by the action of the rotation group.

As we have seen critical orbits appear naturally in a bootstrap model or in a model blending a variational principle with group invariance. The success or the failure of such models will then be independent of dynamical details but rest only on the physical choice of the invariance group and of the space on which it acts.

In conclusion, we believe that the qualitative mathematical concepts which appear in the study of group action on manifolds have helped us to formulate the empirical laws of the breaking of the internal symmetry in an aesthetical and concise form. This formulation might prove to be well adapted to a future more fundamental explanation.

TABLE 1.

Group	Manifold	Number - Dimension of critical Orbits in the Stratum		Little Group	Example of Vectors belonging to the critical Orbit
a) SU(3)	S_7	2	4	U(2)	$\pm y, \pm q, \pm z$
b) SU(3)	P_7	1 1	4 6	U(2) $C = (U_1 \times U_1)_\square Z_2$	y, q, z c_1 , c_2
c) $(SU(3) \times SU(3))_\square Z_2$	S_{15}	2 2	4 8	$SU(3) \times U(2)$ $(U(2) \times U(2))_\square Z_2$	$\pm \tilde{z} = C \oplus z$ $\pm \tilde{q} = q \oplus q$
d) $(SU(3) \times SU(3))_\square Z_2$	P_{15}	1 1 1 1	4 8 6 12	$SU(3) \times U(2)$ $(U(2) \times U(2))_\square Z_2$ $SU(3) \times C$ $(C \times C)_\square Z_2$	$\tilde{z}$ $\tilde{q}$ $\tilde{c}_i = 0 \oplus c_i$ $\tilde{s} = s \oplus \pm s$
e) $(SU(3) \times SU(3))_\square Z_2$	S_{17}	1 2	9 8	$SU^+_y(2) \times SU^-_y(2) \times U^d_y(1)$ $SU^d(3)$	

This table lists the critical orbits for the group action on real manifolds used in text. The critical orbits are manifolds of smaller dimension. The last column gives examples of points of the critical orbits ; they all correspond to the directions of the interactions with the possible exception of the last one.

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Septembre 1969.