Nuclear Physics B53 (1973) 303–312. North-Holland Publishing Company

SOME REMARKS ON POLARIZATION MEASUREMENT AND POLARIZATION DOMAIN

M. DAUMENS, G. MASSAS and P. MINNAERT
Laboratoire de Physique Théorique, Université de Bordeaux I †,
33170 Gradignan, France

and

L. MICHEL
Institut des Hautes Etudes Scientifiques,
91440 Bures Sur Yvette, France

Received 4 December 1972

Abstract: We emphasize that any analysis of polarization measurement must be done in terms of the polarization domain which is much smaller than the domain of physical bounds. We propose a possible quantitative procedure to estimate the precision of a measurement and its compatibility with the positivity condition. We illustrate our discussion with the case of spin-$\frac{3}{2}$ particles and as an application we study some experimental results on $Y^*(1385)$.

1. Introduction

In this paper we make some comments on the measurement of particle polarization in high-energy physics. In collaboration with M.G. Doncel, two of us (L.M. and P.M.) have lectured and written detailed notes on this subject [1, 2]. However, we feel necessary to refer again to this problem and emphasize some important points.

To illustrate our discussion we consider the non-trivial case of a spin-$\frac{3}{2}$ resonance, produced in a B-symmetric reaction [1, 2] (i.e. a parity-conserving reaction with unpolarized beam and target which is either a quasi-two-body reaction or a reaction analysed inclusively). As an application we study the results of an interesting experimental paper [3] which appeared recently. This paper, which will be referred to as AW, deals with the polarization of a $Y^*(1385)$ produced in the reactions:

$$\pi^+ p \to K^+ Y^{*+}, \qquad K^- p \to \pi^- Y^{*+}, \tag{1}$$

$$\pi^+ p \to X^+ Y^{*+}, \qquad K^- p \to X^- Y^{*+}. \tag{2}$$

† Equipe de recherche associée au CNRS.

This $Y^{*+}(1385)$ is specially interesting, because from an observation of the cascade decays $Y^{*+} \to \Lambda^0\pi^+$ (strong), $\Lambda^0 \to p\pi^-$ (weak) one can measure completely the density matrix of the $Y^*(1385)$ (refs. [1, 4, 5]).

2. Polarization parameters and parity conservation

The choice of the frame (helicity or transversity) and the choice of the polarization parameters (density matrix elements or multipole parameters) are a matter of taste and have no fundamental consequence. However, like the authors of AW, we find it convenient to use the transversity multipole parameters. Indeed we have defined [1, 2] a set of real parameters $r_M^{(L)}$ which are proportional to the real or imaginary parts of the T_M^L's of AW. For a spin-$\frac{3}{2}$ resonance the polarization is described by 15 parameters $r_M^{(L)}$, $L = 1, 2, 3; M = -L, \ldots, +L$. If the resonance is produced in a B-symmetric reaction, in transversity quantization the 8 parameters $r_M^{(L)}$ with M odd vanish [1, 2], and we are left with the 7 parameters: $r_0^{(L)}(L = 1, 2, 3)$, and $r_{\pm 2}^{(L)}(L = 2, 3)$. They are related to the parameters T_M^L of AW by:

$$r_0^{(L)} = \frac{2}{\sqrt{3}} T_0^L, \tag{3a}$$

$$r_2^{(L)} = \sqrt{\tfrac{8}{3}} \operatorname{Re} T_2^L, \qquad r_{-2}^{(L)} = \sqrt{\tfrac{8}{3}} \operatorname{Im} T_2^L. \tag{3b}$$

Of course, if the 15 polarization parameters of the spin-$\frac{3}{2}$ particle are measurable, as is the case for the $Y^*(1385)$, *one must verify that the 8 parameters $r_M^{(L)}$ with M odd are zero* †. This check has been done in AW, and the result is that these $r_M^{(L)}$'s are compatible with zero to within 2 standard deviations.

3. The polarization domain

For spin-j particles the density matrix ρ is represented by a point in an N-dimensional Euclidean space, $N = (2j + 1)^2 - 1$. The coordinates of ρ must satisfy constraints due to the positivity of the density matrix. The polarization domain $\mathcal{D}$ is the domain of definition of the polarization parameters. It has a well defined shape (independently of the choice of coordinates). The main properties of $\mathcal{D}$ are: (i) $\mathcal{D}$ is convex; (ii) its interior represents the density matrices of maximal rank $2j + 1$ (among them is the unpolarized density matrix $\rho_0 = 1/(2j + 1)$); (iii) its boundary $\partial\mathcal{D}$ represents the density matrices of rank $< 2j + 1$.

The polarization degree d_ρ of $\rho \in \mathcal{D}$ is given by the distance between the representative points ρ and ρ_0. Its range of values is from 0 (for the unpolarized state)

† If this is not satisfied, most likely the presence of non-expected $Y_M^{(L)}$ in the decay angular distribution reveals the existence of interference between the resonance channel and the background

to 1 (for the pure states). Hence $\mathcal{D}$ is inscribed inside the N-dimensional unit sphere S_N, centered at ρ_0. The expression of d_ρ in terms of multipole parameters $r_M^{(L)}$ is:

$$d_\rho = \Big(\sum_{L,M} (r_M^{(L)})^2 \Big)^{\frac{1}{2}} . \tag{4}$$

As we recalled previously the density matrix of B-symmetric spin-$\frac{3}{2}$ particles is described by 7 non-vanishing parameters $r_M^{(L)}$. The seven-dimensional domain $\mathcal{D}$ has been described in ref. [1]. It is the intersection of two quadrics C_ϵ, and it is defined by the relations ($\epsilon = \pm 1$)

$$\big([r_2^{(2)} + \epsilon r_2^{(3)}]^2 + [r_{-2}^{(2)} + \epsilon r_{-2}^{(3)}]^2 + [r_0^{(2)} + \epsilon \frac{1}{\sqrt{5}}(2r_0^{(1)} - r_0^{(3)})]^2\big)^{\frac{1}{2}}$$

$$\leqslant \frac{1}{\sqrt{3}} + \epsilon \frac{1}{\sqrt{5}}(r_0^{(1)} + 2r_0^{(3)}). \tag{5}$$

It is interesting to compare the volume of $\mathcal{D}$ to that of the unit sphere S_7. For this it is convenient to make an orthogonal transformation in the 7-dimensional polarization space.

One defines

$$\xi_\epsilon^{(1)} = \frac{1}{\sqrt{2}} [r_0^{(2)} + \epsilon \frac{1}{\sqrt{5}}(2r_0^{(1)} - r_0^{(3)})],$$

$$\xi_\epsilon^{(2)} = \frac{1}{\sqrt{2}} [r_2^{(2)} + \epsilon r_2^{(3)}],$$

$$\xi_\epsilon^{(3)} = \frac{1}{\sqrt{2}} [r_{-2}^{(2)} + \epsilon r_{-2}^{(3)}],$$

$$t = \frac{1}{\sqrt{5}} [r_0^{(1)} + 2r_0^{(3)}]. \tag{6}$$

In these variables the polarization degree reads

$$d_\rho^2 = t^2 + \xi_+^2 + \xi_-^2, \tag{7}$$

and the equations of the sphere S_7 and of the domain $\mathcal{D}$ are

$$S_7 : t^2 + \xi_+^2 + \xi_-^2 = 1, \tag{8}$$

$$\mathcal{D} : -\frac{1}{\sqrt{3}} \leqslant t \leqslant \frac{1}{\sqrt{3}}, \qquad \xi_\epsilon^2 \leqslant \tfrac{1}{6}(1 + \epsilon\sqrt{3}t)^2, \tag{9}$$

where ξ_ϵ^2 is a short notation for $\Sigma_{i=1}^3 (\xi_\epsilon^{(i)})^2$.

We emphasize that $\mathcal{D}$ is much smaller than S_7. Indeed the volume of $\mathcal{D}$ is

$$V(\mathcal{D}) = \int_{-1/\sqrt{3}}^{+1/\sqrt{3}} \left(\frac{4\pi}{3}\right)^2 \left(\frac{1+t\sqrt{3}}{\sqrt{6}}\right)^3 \left(\frac{1-t\sqrt{3}}{\sqrt{6}}\right)^3 \mathrm{d}t = \frac{64\,\pi^2}{8505\sqrt{3}} = 0.04288, \quad (10)$$

while the volume of the interior of S_7 is

$$V(S_7) = \frac{\pi^{\frac{7}{2}}}{\Gamma(\frac{7}{2}+1)} = 4.7248. \quad (11)$$

Hence, the ratio of the volumes is

$$V(S_7)/V(\mathcal{D}) = \tfrac{1}{4} \times 81\sqrt{3}\pi = 110.188. \quad (12)$$

The authors of AW do not verify that their three measured points belong to $\mathcal{D}$. They only verify that each parameter $r_M^{(L)}$ is inside the physical bounds which are [6]

$$|r_M^{(L)}| \leqslant \begin{cases} \sqrt{\frac{3}{5}} \text{ for } r_0^{(1)} \text{ and } r_0^{(3)}, \\ \frac{1}{\sqrt{3}} \text{ for the 5 other parameters.} \end{cases} \quad (13)$$

These conditions yield a domain P whose volume is

$$V(P) = \left(\frac{2}{\sqrt{3}}\right)^5 (2\sqrt{\tfrac{3}{5}})^2 = 4.9267. \quad (14)$$

Note that P is bigger than S_7, and is more than one hundred times bigger than $\mathcal{D}$ (!);

$$V(P)/V(\mathcal{D}) = 114.898. \quad (15)$$

Consequently, if it is advisable to check that $\rho \in P$ and $\rho \in S_7$, *it is absolutely necessary to verify that the representative point of the measured polarization belongs to the polarization domain* $\mathcal{D}$.

4. Tests of the precision and the positivity of a polarization measurement

To estimate the precision of an experimental result $\rho_{\text{exp}} = \{r_M^{(L)}\}$, it is interesting to compare the relative sizes of the statistical† errors $\Delta r_M^{(L)}$ and of the polarization domain $\mathcal{D}$. We propose here a quantitative procedure. For given $r_M^{(L)}$ and $\Delta r_M^{(L)}$, the equation

$$\sum_{L,M} \left(\frac{x_M^{(L)} - r_M^{(L)}}{\Delta r_M^{(L)}}\right)^2 = \chi^2 \quad (16)$$

† We assume that the systematic errors are negligible. This assumption seems frequently made in experimental papers.

defines an ellipsoid E_{χ^2} in the N variables $x_M^{(L)}$. For N degrees of freedom, to each value of χ^2 corresponds a confidence level for the points on E_{χ^2} to be compatible with the experimental point $\rho_{\exp}$. The volume of E_{χ^2} is

$$V(E_{\chi^2}) = V(S_N) \left(\prod_{L,M} \Delta r_M^{(L)} \right) (\chi^2)^{\frac{1}{2}N}, \tag{17}$$

where S_N is the N-dimensional unit sphere.

To have a feeling for the precision of a measure, we choose χ^2 such that the confidence level is $\frac{1}{2}$(e.g. for $N = 7$, $\chi^2 = 6.346$), we denote by $E(\frac{1}{2})$ the corresponding ellipsoid, and we compute the ratio

$$\lambda = V(E(\tfrac{1}{2}))/V(\mathcal{D}); \tag{18}$$

the smaller this ratio, the better the precision.

Table 1 gives the value of λ (computed from eqs. (12), (17) and (18)) for the three points measured by AW (their tables 2, 3 and 4). Of course, when a measurement is made from a small number of events, it is not astonishing that $E(\frac{1}{2})$ could be bigger than $\mathcal{D}$ (e.g. for point 1, $\lambda = 7.41$).

Table 1
Values of the parameters λ and μ for the experimental results of AW, ref. [3]

Point	1	2	3
Table of AW	2	3	4
Number of events	37	320	282
λ	7.41	0.10	0.02
$\mu \leqslant$	0.003	0.05	0.15

Furthermore we would like to take into account the statistical errors to estimate the positivity of a polarization measurement. For this we consider the intersection $\mathcal{D} \cap E(\frac{1}{2})$ of the polarization domain $\mathcal{D}$ with the ellipsoid $E(\frac{1}{2})$. The ratio μ of the volume of $\mathcal{D} \cap E(\frac{1}{2})$ to that of $E(\frac{1}{2})$,

$$\mu = V(\mathcal{D} \cap E(\tfrac{1}{2}))/V(E(\tfrac{1}{2})) \leqslant 1, \tag{19}$$

gives some quantitative information on the compatibility of the measure with the positivity condition. The bigger the ratio, the better the compatibility. Table 1 gives an upper limit to the ratio μ (computed by a Monte-Carlo method) for the three points measured by AW.

5. Comparison with theory – Two-dimensional plots

The comparison of an experimental result $\rho_{exp} = \{r_M^{(L)}\}$ (with statistical errors $\Delta r_M^{(L)}$) with a theoretical prediction $\rho_{th} = \{x_M^{(L)}\}$, is generally made by computing the χ^2 value (from eq. (16)) and the corresponding confidence level for N degrees of freedom.

We think that this is completely insufficient for polarization measurements. Indeed the χ^2 test studies *only* the compatibility between ρ_{exp} and ρ_{th}. In the usual procedure (eq. (16)), the confidence level improves when the errors increase. In the limit a measurement with infinite errors is perfectly compatible with any theoretical measurement. It is therefore necessary to know first the accuracy of the experiment one *cannot forget* that the $r_M^{(L)}$'s are not free parameters, they are constrained to represent a point ρ_{exp} of the polarization domain $\mathcal{D}$. Hence if the size of the errors and the size of $\mathcal{D}$ are of the same order of magnitude (e.g. $\lambda > 0.2$), the comparison of ρ_{exp} with ρ_{th} is not very significant.

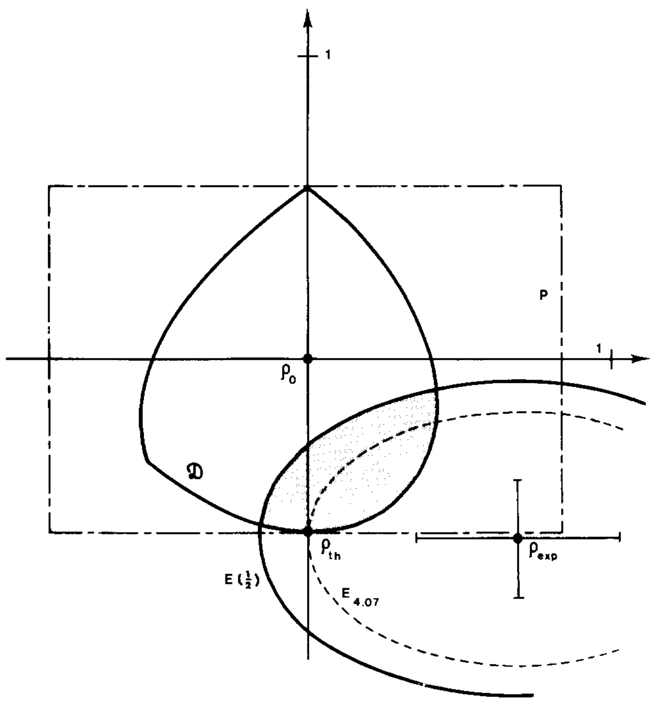

Fig. 1. Two-dimensional plot for the point 1 (table 2 of ref. [3]). The two-plane of the plot is defined by the 3 points ρ_0, ρ_{exp} and ρ_{th}. This figure shows the section by this plane of the physical bounds domain P, the polarization domain $\mathcal{D}$ and the two ellipsoids E($\frac{1}{2}$) (whose points have a level of confidence $\geqslant \frac{1}{2}$) and $E_{4.07}$ (which passes through ρ_{th}, eq. (20)).

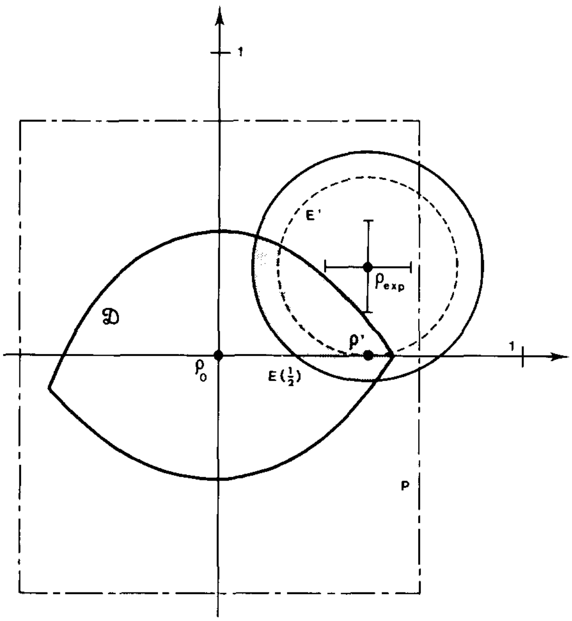

Fig. 2. Two-dimensional plot for the point 2 (table 3 of ref. [3]). The two-plane of the plot is defined by three points ρ_0, ρ_{exp} and ρ' which is the closest to ρ_{exp} of the set of points predicted by the quark model eq. (21). This figure shows the section by this plane of the physical bounds domain P, the polarization domain $\mathcal{D}$ and the two ellipsoids E($\frac{1}{2}$) (whose points have a level of confidence $\geqslant \frac{1}{2}$) and E' (which passes through ρ').

The authors of ref. [1] have emphasized the necessity to know the polarization domain $\mathcal{D}$ for each experiment and they have proposed a procedure for plotting on it the experimental points with their errors. To visualize easily the position of ρ_{exp} and ρ_{th} with respect to $\mathcal{D}$ when the dimension N of the polarization space is large, we propose here a procedure using a two-dimensional plot. We draw the intersections of $\mathcal{D}$ and E($\frac{1}{2}$) with the two-dimensional plane defined by the three points ρ_0, ρ_{exp} and ρ_{th} †. For spin $\frac{3}{2}$, the intersections of this two-plane with the quadrics C_+ and C_- (eq. (5)) which bound $\mathcal{D}$ are conics, and the intersection with E($\frac{1}{2}$) is an ellipse.

For reactions (1), because of angular momentum conservation one has rank $\rho = 2$, at any s and t (ref. [1]). Then the representative point is on the boundary $\partial\mathcal{D}$ of the polarization domain, at the intersection of the quadrics C_+ and C_-, eq. (5). More precisely, for these reactions, several simple models (e.g. 1^- meson exchange with magnetic coupling [7], SU $(6)_W$ invariance [8], quark model [9]) predict, for any s and t

$$\rho_{th} : x_0^{(2)} = -\frac{1}{\sqrt{3}}, \qquad \text{all other } x_M^{(L)} = 0. \tag{20}$$

† Of course the two-dimensional plot, which is a section in the N-dimensional polarization space, does not contain a complete information on the N measured parameters.

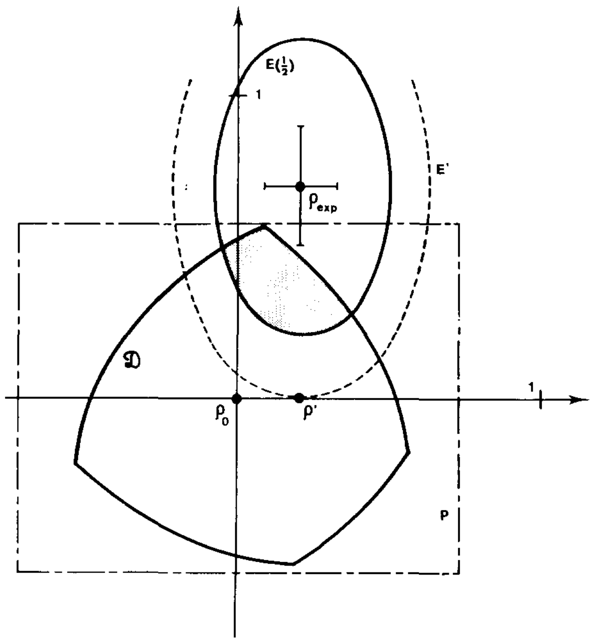

Fig. 3. Two-dimensional plot for the point 3 (table 4 of ref. [3]). See caption of fig. 2.

In AW there is one experimental point (point 1) for the two quasi-two-body reactions (1). They find this point compatible with ρ_{th} of eq. (17); the χ^2 is 4.07 and the corresponding confidence level, for 7 degrees of freedom, is 77%. Fig. 1 shows the two-dimensional plot for this point. The experimental point is outside the polarization domain $\mathcal{D}$ but the statistical errors are so large † (the corresponding $\lambda = 7.41$) that the point ρ_{exp} is compatible with most points of $\mathcal{D}$. This situation shows, as we noticed, that a good χ^2 value is not a sufficient criterium for testing a theory.

For reactions (2), the prediction of the quark model [10] is

$$\rho_{\text{th}} : r_M^{(3)} = 0, \qquad \text{no conditions for } r_M^{(L)}, \qquad L = 1, 2. \tag{21}$$

Conditions (21) do not define a single theoretical point, they define a four-plane in the seven-dimensional polarization space. We denote by ρ' the orthogonal projection of ρ_{exp} on this four-plane, and the two-dimensional plot for this case is the section of $\mathcal{D}$ and $\text{E}(\frac{1}{2})$ by the two-plane defined by $\rho_0, \rho_{\text{exp}}, \rho'$. In AW there is one experimental point (points 2 and 3) for each inclusive reaction (2). They give the value of χ^2 for point 2, the corresponding confidence level (3 degrees of freedom) is 28%. For point 3

† Whatever the statistical errors we notice that the median value $r_0^{(2)} = -(2/\sqrt{3})\,0.52$ corresponds to an angular distribution of the decay $\text{Y}^* \to \Lambda\pi$ which is not positive definite. See ref. [1b], subsect. 3.2.3 and fig. 6.

the level of confidence would be less than 0.5%. Figs. 2 and 3 show the two-dimensional plots for points 2 and 3. These experimental results are based on a number of events 10 times bigger than that of point 1 and the size of errors is definitely smaller than the size of $\mathcal{D}$.

Although the experimental points are both outside the polarization domain, they are very close to it. This suggests that the true polarization is close to the boundary of $\mathcal{D}$ and not very sensitive to $p_{\parallel}$ and $p_{\perp}$ and to the variables which are summed over in the inclusive reaction. Indeed in a convex domain if the barycenter of a set of points is close to a curved boundary, most of the points of the set should be near the barycenter.

Furthermore, we remark that the experimental points, specially point 3, are in disagreement with the quark-model prediction.

6. Conclusion

The representative point of a positive density matrix belongs to a domain $\mathcal{D}$ in the N-dimensional polarization space. In this paper, we have emphasized that *any analysis of polarization measurement must be done in terms of the polarization domain* which is smaller than the domain of the physical bounds. This can be done graphically as proposed in ref. [1]. Here for experimental results given with statistical errors we have defined two parameters λ and μ which estimate the precision of the measure and its compatibility with the positivity condition. Moreover we have shown that the χ^2 test alone is not meaningful for the comparison of an experimental result with a theoretical prediction when the errors are too big. To visualize directly this comparison we have proposed a two-dimensional plot which shows the position of the theoretical and experimental points with respect to $\mathcal{D}$.

To illustrate our discussion we have studied the interesting experimental results of AW (ref. [3]), on the complete density matrix of the $Y^*(1385)$ produced in quasi-two-body and in inclusive reactions. For the case of quasi-two-body reactions we hope to have soon results with better statistics, because of course, it is hopeless to measure precisely 7 parameters from 37 events.

The authors are grateful to Professor J. Donohue for reading the manuscript.
They also thank Hewlett-Packard France for their help with the desk calculator and plotter used for the figures.

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