

## On the Renormalization of the Axial Vector Coupling Constant in $\beta$ -Decay.

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**Summary.** — The models of the axial vector current discussed by GELL-MANN and LÉVY are examined further. Generalized Ward identities are derived for the axial vector weak vertex. It is then shown that in the  $\sigma$  model and the non-linear model the renormalization factor  $— G_A/G$  may be expressed as a matrix element in the theory of strong interactions. Thus in the  $\sigma$  model, which is renormalizable,  $— G_A/G$  is finite in every order. Since  $— G_A/G$  exhibits divergences in the non-linear model, that model is not renormalizable in the usual sense.

### 1. — Introduction.

A conserved vector current <sup>(1,2)</sup> has been suggested in order to make the renormalization factor  $G_V/G$  of the weak vector current  $V_x$  in  $\beta$  decay equal to unity. The quantity  $V_x$  is taken to be a component of an isotopic vector  $V_x$ .

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(1) S. S. GERSHTEIN and J. B. ZELDOVICH: *Žurn. Éksp. Teor. Fiz.*, **29**, 698 (1955).

(2) R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958).

that is proportional to the isotopic spin current, so that its divergence vanishes.

In an accompanying article <sup>(3)</sup>, a discussion is given of possible theories in which the analogous axial vector current  $P_\nu$  would have its divergence proportional to the pion field:

$$(1) \quad \partial_\nu P_\nu = ia\pi,$$

where  $a$  turns out to be equal to  $-\mu_0^2/f_0$  in each case. Here, the theoretical renormalization factor  $-G_A/G$  is not unity, and the experimental one is not either, having a value of about 1.25. We should like, of course, to be able to calculate this factor, and it is interesting to see how much we can learn about it by methods analogous to those that give the result  $G_V/G=1$  in the vector case. In particular, we shall be able to prove that in the second and third models considered in *A*, the quantity  $-G_A/G$  is expressible as a matrix element of the pion field in the strong interaction theory. Thus in the second model (the  $\sigma$  model), which is renormalizable, the axial vector renormalization factor is finite in all orders. The third model (the non-linear one) gives a logarithmically divergent contribution in second order to  $-G_A/G$  and therefore the corresponding theory of the pion strong interaction cannot be renormalizable in the usual sense.

In our work we shall make use of a «generalized Ward identity», which in the case of the conserved vector current gives immediately <sup>(4)</sup> the result that  $G_V/G=1$ . The identity may be derived by a gauge transformation <sup>(5)</sup> and we shall use the same method to derive the analogous identity for the axial vector current in the models of article *A*. (Generalization to other theories of the axial vector current is not difficult). In the Appendix, the same «generalized Ward identities» for axial vector currents are derived by another method.

## 2. - The identities.

In the vector case, we have generated the current  $V_\alpha$  by means of the infinitesimal gauge transformations of eq. (A.22), such that:

$$(2) \quad N(x) \rightarrow (1 + i\boldsymbol{\tau} \cdot \mathbf{u}(x)) N(x)$$

<sup>(3)</sup> M. GELL-MANN and M. LÉVY: to be referred to as *A*. We shall employ the notation of that article and we shall quote equations from it as (A.1) (A.2), etc.

<sup>(4)</sup> S. OKUBO: *Nuovo Cimento*, **13**, 292 (1959).

<sup>(5)</sup> R. UTIYAMA, S. SUNAKAWA and T. IMAMURA: *Progr. Theor. Phys.*, **8**, 77 (1952).

and

$$(3) \quad \mathcal{L} \rightarrow \mathcal{L} - i\mathbf{V}_\alpha \cdot \partial_\alpha \mathbf{u}.$$

Now the unrenormalized nucleon propagator  $S'_F(p)$  is proportional to the Fourier transform of:

$$(4) \quad S'_F(x-y) = \langle T(N(x)\bar{N}(y)) \rangle_0,$$

where we have taken the expectation value in the physical vacuum of the  $T$ -product of two Heisenberg operators. We may now alter the nucleon fields as in eq. (2) and we can calculate the corresponding change in the propagator (4) by adding a first order perturbation to the Lagrangian, as given in (3). Thus we have:

$$(5) \quad i\boldsymbol{\tau} \cdot \mathbf{u}(x) S'_F(x-y) - i S'_F(x-y) \boldsymbol{\tau} \cdot \mathbf{u}(y) = \Delta S'_F(x-y),$$

where  $\Delta S'_F(x-y)$  is the change induced by the perturbation in (3). If we now go over to momentum space and if we define  $\mathbf{\Gamma}_\alpha(p', p)$  to be the unrenormalized vertex function corresponding to the current  $\mathbf{V}_\alpha$ , we have:

$$(6) \quad i\boldsymbol{\tau} S'_F(p) - i S'_F(p') \boldsymbol{\tau} = - S'_F(p') k_\alpha \mathbf{\Gamma}_\alpha(p', p) S'_F(p),$$

where  $k = p' - p$ . Dividing on the left by  $S'_F(p')$  and on the right by  $S'_F(p)$ , we obtain:

$$(7) \quad i[S'_F(p')]^{-1} \boldsymbol{\tau} - i\boldsymbol{\tau} [S'_F(p)]^{-1} = -k_\alpha \mathbf{\Gamma}_\alpha(p', p).$$

We may now transform to renormalized quantities, by multiplying both sides by  $-iZ_2$ :

$$(8) \quad S_{Fc}^{-1}(p') \boldsymbol{\tau} - \boldsymbol{\tau} S_{Fc}^{-1}(p) = ik_\alpha \hat{\mathbf{\Gamma}}_\alpha(p', p).$$

Here  $\hat{\mathbf{\Gamma}}_\alpha = Z_2 \mathbf{\Gamma}_\alpha$  is the effective value of the vertex, often written as:

$$(9) \quad \hat{\mathbf{\Gamma}}_\alpha = \frac{Z_2}{Z_1} \mathbf{\Gamma}_{\alpha c},$$

where  $\mathbf{\Gamma}_{\alpha c}(p, p)$  between free spinors of equal momentum acts like  $\boldsymbol{\tau} \gamma_\alpha$  with coefficient unity, and  $Z_2/Z_1$  is the renormalization factor  $G_V/G$ . In other words, we have, for free nucleons:

$$(10) \quad \bar{u}_i(p) \hat{\mathbf{\Gamma}}_\alpha(p, p) u_i(p) = \bar{u}_i \boldsymbol{\tau} \gamma_\alpha u_i \frac{G_V}{G}.$$

From the «generalized Ward identity» (8) it is trivial to see that  $G_{\nu}/G = 1$ . We let  $p' \rightarrow p$  or  $k \rightarrow 0$  on each side and to first order in  $k$  we have, since  $\tau$  commutes with  $S'_F$ , the result of Ward:

$$(11) \quad \tau \frac{\partial}{\partial p_\alpha} S_{FC}^{-1}(p) = i\hat{\Gamma}_\nu(p, p).$$

But near the mass shell  $S_{FC}^{-1}(p) = i\gamma \cdot p + \mathbf{m} + O(i\gamma \cdot p + \mathbf{m})^2$ , so that:

$$(12) \quad \bar{u}_f(p) \hat{\Gamma}_\nu(p, p) u_i(p) = \bar{u}_f \tau \gamma_\alpha u_i,$$

and hence by (10) the renormalization factor is unity. (It should be noticed that if we simply take eq. (8) between free states directly we learn only that  $0 = 0$ ).

Now let us obtain the analogous results for the axial vector current in the second and third models of A. The gauge transformations are those of eq. (A.39) and have the properties:

$$(13) \quad \begin{cases} N \rightarrow (1 + i\tau \cdot \mathbf{v} \gamma_5)N, \\ \mathcal{L} \rightarrow \mathcal{L} - i\mathbf{P}_\alpha \cdot \partial_\alpha \mathbf{v} + a\boldsymbol{\pi} \cdot \mathbf{v}. \end{cases}$$

Instead of (8) we get the generalized Ward identity <sup>(6)</sup>:

$$(14) \quad S_{FC}^{-1}(p')\tau\gamma_5 + \tau\gamma_5 S_{FC}^{-1}(p) = ik_\alpha \hat{\Gamma}_{\alpha 5}(p', p) - a\sqrt{Z_3}g_1 \frac{\hat{\Gamma}_5(p', p)}{k^2 + m_\pi^2} d_\pi(k^2),$$

where  $\hat{\Gamma}_5$  is the effective pion vertex and the other quantities are defined as in (A.10). The plus sign on the left hand side of eq. (14) results from the anticommutation of  $\tau\gamma_5$  with the matrix  $\beta$  in  $\bar{N}$ .

In the gradient coupling model, the gauge transformations are those of (A.30) and give:

$$(15) \quad \begin{cases} N \rightarrow N, \\ \mathcal{L} \rightarrow \mathcal{L} - i\mathbf{P}_\alpha \cdot \partial_\alpha \mathbf{v} + a\boldsymbol{\pi} \cdot \mathbf{v}, \end{cases}$$

so that instead of (14) we have:

$$(16) \quad 0 = ik_\alpha \hat{\Gamma}_{\alpha 5}(p', p) - a\sqrt{Z_3}g_1 \hat{\Gamma}_5(p', p) \frac{d_\pi(k^2)}{k^2 + m_\pi^2}.$$

<sup>(6)</sup> Identities of this type have been studied in the limit of a conserved axial vector current by S. WEINBERG (private communication to J. BERNSTEIN).

In contrast to the vector case, we get a non-trivial result if we just take the matrix element of (14) or (16) between free nucleons, namely:

$$(17) \quad 0 = ik_\alpha \bar{u}_f(p') \hat{\Gamma}_{\alpha 5}(p', p) \bar{u}_i(p) - a\sqrt{Z_3} g_1 \frac{d_\pi(k^2) F_\pi(k^2)}{k^2 + m_\pi^2} \bar{u}_f(p') \boldsymbol{\tau} \gamma_5 u_i(p).$$

But this is an equation that we could have derived directly from (1). In fact, it is obvious from eq. (A.10) and the discussion preceding that equation.

Our weak current being invariant under  $GP = CP \exp[i\pi I_y]$ , we may write (7):

$$(18) \quad \bar{u}_f(p') \hat{\Gamma}_{\alpha 5}(p', p) u_i(p) = -\frac{G_A}{G} \bar{u}_f \boldsymbol{\tau} \gamma_\alpha \gamma_5 u_i \alpha(k^2) + i \bar{u}_f \boldsymbol{\tau} \gamma_5 u_i k_\beta \beta(k^2),$$

where  $\alpha(0) = 1$ . We then have, from (17) and (18), the important formula:

$$(19) \quad 2m \left( -\frac{G_A}{G} \right) \alpha(k^2) + k^2 \beta(k^2) = -a\sqrt{Z_3} g_1 \frac{d_\pi(k^2) F_\pi(k^2)}{k^2 + m_\pi^2},$$

which contains some familiar results. When  $k = 0$  we just get eq. (A.11):

$$(20) \quad -\frac{G_A}{G} = -a\sqrt{Z_3} \frac{f_1}{m_\pi^2} d_\pi(0) F_\pi(0).$$

Near the pole, which can occur only in  $\beta$  and not in  $\alpha$ , since it comes from a virtual pion, we have:

$$(21) \quad \beta(k^2) \approx \frac{a\sqrt{Z_3}}{m_\pi^2} g_1 \frac{1}{k^2 + m_\pi^2},$$

and since  $a\sqrt{Z_3}$  is related to the pion decay amplitude by eq. (A.7), this is just the result of GOLDBERGER and TREIMAN<sup>(8)</sup> on the induced pseudoscalar interaction. They assume that the pole term dominates even at the value  $k^2 = m_\mu^2$  relevant to muon capture; such a result is made plausible by dispersion calculations.

So far we have not really used our generalized Ward identities (14) and (16).

### 3. - Use of the identities and results.

Let us now consider the identity (14) characteristic of the second and third models and extract new information from it. We put  $p^2 = p'^2 = -m^2$ , but we do not take matrix elements between free spinors. The  $GP$  invariance then

(7) S. WEINBERG: *Phys. Rev.*, **112**, 1375 (1958).

(8) M. L. GOLDBERGER and S. B. TREIMAN: *Phys. Rev.*, **111**, 354 (1958).

allows us to write the two vertex functions in the following way:

$$(22) \quad \hat{\Gamma}_{\alpha 5} = \tau \{ \gamma_\alpha \gamma_5 F_1(k^2) + \gamma \cdot k \gamma_5 k_\alpha F_2(k^2) + i \gamma_5 k_\alpha F_3(k^2) + \\ + [\gamma_\alpha, i \gamma \cdot l] \gamma_5 F_4(k^2) + [\gamma \cdot k, i \gamma \cdot l] k_\alpha \gamma_5 F_5(k^2) + \gamma \cdot l \gamma_5 l_\alpha F_6(k^2) \},$$

$$(23) \quad \hat{\Gamma}_5 = \tau \left\{ \gamma_5 i \gamma \cdot k \frac{\xi(k^2)}{2m} + \gamma_5 \eta(k^2) + [i \gamma \cdot l, i \lambda \cdot k] \gamma_5 \frac{\zeta(k^2)}{k^2 + 4m^2} \right\},$$

where  $l = (p + p')/2$ . With  $p^2 = -m^2$ , we may write:

$$(24) \quad S_{F_0}^{-1}(i \gamma \cdot p) = (i \gamma \cdot p + m) C,$$

where

$$(25) \quad C = (2m)^{-1} S_{F_0}^{-1}(i \gamma \cdot p = +m).$$

We may now compute  $i k_\alpha \hat{\Gamma}_{\alpha 5}$  as follows:

$$(26) \quad i k_\alpha \hat{\Gamma}_{\alpha 5} = \tau \{ i \gamma \cdot k \gamma_5 (F_1 + k^2 F_2) + \gamma_5 (-k^2 F_3) + [i \gamma \cdot k, i \gamma \cdot l] \gamma_5 (F_4 + k^2 F_5) \}.$$

The identity (14) then yields three equations:

$$(27a) \quad C = F_1(k^2) + k^2 F_2(k^2) - X(k^2) \xi(k^2),$$

$$(27b) \quad 2mC = -k^2 F_3(k^2) + 2mX(k^2) \eta(k^2),$$

$$(27c) \quad 0 = F_4(k^2) + k^2 F_5(k^2) - \frac{2m}{4m^2 + k^2} X(k^2) \zeta(k^2),$$

where

$$(28) \quad X(k^2) = -\frac{a \sqrt{Z_3} g_1}{2m} \frac{d_\pi(k^2)}{k^2 + m_\pi^2} = \frac{\mu_0^2}{k^2 + m_\pi^2} \frac{f_1}{f_0} \sqrt{Z_3} d_\pi(k^2),$$

we have used the fact that  $a = -\mu_0^2/f_0$ .

In particular, for  $k^2 = 0$ , we have:

$$(29a) \quad C = F_1(0) - X(0) \xi(0),$$

$$(29b) \quad -C = -X(0) \eta(0),$$

$$(29c) \quad 0 = 2m F_4(0) - X(0) \zeta(0).$$

In order to apply these results, we note that by taking matrix elements of (22) and (23) between free spinors we obtain:

$$(30) \quad -\frac{G_A}{G} = F_1(0) + 2mF_4(0),$$

$$(31) \quad F_\pi(k^2) = \xi(k^2) + \eta(k^2) + \zeta(k^2).$$

By considering the vertex  $\hat{\Gamma}_5$  in the general form (23), we have split the pion form factor  $F_\pi$  into three parts; between free spinors only the sum is important, since all three matrices in (23) look like  $\gamma_5$  when taken between free nucleons.

If we sum the three relations (29), we simply get back eq. (20). But if we use them separately, we can evaluate the axial vector renormalization factor in terms of the nucleon propagator and the pion vertex:

$$(32) \quad -\frac{G_A}{G} = C \frac{F_\pi(0)}{\eta(0)} = C \frac{F_\pi(0)}{F_\pi(0) - \xi(0) - \zeta(0)}.$$

Thus if our strong interaction theory is renormalizable, as in the  $\sigma$  model, the quantity  $-G_A/G$  is *finite in every order*. For example, up to second order, we may put:  $C = 1 + C_2$ ,  $F_\pi(0) = 1 + F_{\pi_2}$ ,  $\xi(0) = \xi_2(0)$ ,  $\zeta(0) = \zeta_2$ . We find  $\xi_2 = 0$  and:

$$(33) \quad -\frac{G_A}{G} = 1 + C_2 + \zeta_2 = 1 + \frac{1}{\pi} \frac{g_1^2}{4\pi} + \dots,$$

with  $m_\pi$  and  $m_\sigma$  put equal to zero for simplicity. (As  $m_\pi \rightarrow 0$ , both  $C_2$  and  $\zeta_2$  have infrared divergences, but they cancel in the sum). It is easy to see that the power series (33) is not much use for calculation, even though the coefficients are finite.

It is also true that if  $-G_A/G$  exhibits a divergence in second order, as in the non-linear model, then the corresponding strong interaction theory cannot be renormalizable in the usual sense. In that model, the culprit in second order is  $\xi_2$ , which is logarithmically divergent and comes from corrections to the pion vertex due to the term  $2f^2 m_\sigma \pi^2 \bar{N}N$  in the expansion of the Lagrangian  $\mathcal{L}_3$  of eq. (A.46).

In the gradient coupling model, the Ward identity (16) leads to equations just like (29) with  $C$  replaced by zero, but they tell us nothing new.

APPENDIX

An alternate method of deriving (14) and (16) may be sketched as follows <sup>(6,9)</sup>. If  $J_\alpha(y)$  is any current then we may define the vertex  $\Gamma'_\alpha(u, u', y) = \Gamma'_\alpha(u - y, y - u')$  by the equation:

$$(A.1) \quad \langle T(\bar{N}(x)N(x')J_\alpha(y)) \rangle_0 \equiv - \int d^4u d^4u' S'_F(x-u) \Gamma'_\alpha(u-y, y-u') S'_F(u'-x').$$

The notation is as in (4) of the text.

Now it follows from well-known properties of the  $T$  symbol that:

$$(A.2) \quad \frac{\partial}{\partial y_\alpha} \langle T(N(x) \bar{N}(x') J_\alpha(y)) \rangle_0 = \left\langle T \left( \frac{\partial}{\partial y_\alpha} J_\alpha(y) N(x) \bar{N}(x') \right) \right\rangle - \delta(x_0 - y_0) \langle T(\bar{N}(x') [N(x), J_0(x)]) \rangle_0 - \delta(x'_0 - y_0) \langle T(N(x) [\bar{N}(x'), J_0(y)]) \rangle_0.$$

In the familiar case of electrodynamics, and hence also in the case of the vector  $\beta$ -decay current  $V_\alpha$ , one may write:

$$(A.3) \quad V_\alpha = \bar{N} \tau_\alpha N + G_\alpha,$$

where  $G_\alpha$  is a function of fields other than the nucleon field, which commutes with the nucleon field at equal times. Hence the equal time commutators in (A.2) may be evaluated explicitly.

Using the definition of  $S'_F(x-x')$  and the relation  $\partial_\alpha V_\alpha = 0$  we are led at once to (6) when we transform (A.1) and (A.2) into momentum space.

For the second and third models of  $A$  we may also write:

$$(A.4) \quad P_\alpha = \bar{N} \tau_\alpha \gamma_5 N + G_{\alpha 5},$$

where  $G_{\alpha 5}$  commutes, as above, with  $N$ . In this case the equal time commutations give the plus sign noted in (14) and  $\partial_\alpha P_\alpha = i a \pi$  gives the last term on the right hand side of (14).

In the pseudovector model we have:

$$[G_{05}, N] \neq 0,$$

at equal times. This is a familiar property of the gradient coupling. However, in the model under consideration we have:

$$(A.5) \quad P_\alpha = \bar{N} \tau_\alpha \gamma_5 N - \frac{i}{f_0} \partial_\alpha \pi$$

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(9) Y. TAKAHASHI: *Nuovo Cimento*, **6**, 371 (1957).



and it is easy to see from the pseudovector Lagrangian that the component  $P_4$  is proportional to the field momentum canonical to  $\pi$ . Thus the equal time commutators in (A.2) simply vanish in virtue of the canonical commutation relations and one is led to (16).

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#### RIASSUNTO (\*)

Si esaminano ulteriormente i modelli della corrente vettoriale assiale discussi da FEYNMAN, GELL-MANN e LÉVY. Si derivano identità generalizzate di Ward per il vertice debole del vettore assiale. Si mostra poi che nel modello  $\sigma$  e nel modello non lineare il fattore di rinormalizzazione  $-G_A/G$  può essere espresso come un elemento di matrice nella teoria delle interazioni forti. Così nel modello  $\sigma$ , che è rinormalizzabile,  $-G_A/G$  è finito in ogni ordine. Poichè  $-G_A/G$  presenta divergenze nel modello non lineare, questo modello non è rinormalizzabile nel senso usuale.

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(\*) Traduzione a cura della Redazione.