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Covariant Description of Polarization.

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1. – Description of one particle states.

We want to describe a particle of mass m, spin j. For this we use the irreducible representation $[m, j]$ described by WIGHTMAN. The wave function $\chi(p)$ has $2j+1$ components. The $Q(p, \Lambda)$ are $2j+1$ by $2j+1$ unitary matrices, such that

$$(U(a, \Lambda)\chi)(p) = \exp[ipa]\, Q(p, \Lambda)\chi(\Lambda^{-1}p)\,,$$

$\chi(p)$ restricted to $\mathbf{p}^2 = m^2$ span the representation space of $[m, j]$ which is called the Hilbert space of the particle states. For a fixed $\mathbf{p}$, $\chi(p)$ span the Hilbert space of the polarization states for the particle of energy momentum $\mathbf{p}$.

2. – Mixtures of states.

If φ is any vector $\in \mathcal{H}$ one can construct the projection operator on the normalized state $|\varphi\rangle$ $(\langle\varphi|\varphi\rangle = 1)$

$$P_\varphi = |\varphi\rangle\langle\varphi|: \qquad P_\varphi|\psi\rangle = |\varphi\rangle\langle\varphi|\psi\rangle\,.$$

Characteristic properties are:

$$P_\varphi^2 = P_\varphi\,, \qquad P_\varphi^* = P_\varphi\,, \qquad \operatorname{Tr} P_\varphi = \langle\varphi|\varphi\rangle = 1\,.$$

Expectation value of observable $A = \langle\varphi|A|\varphi\rangle = \operatorname{Tr} A P_\varphi = \operatorname{Tr} P_\varphi A$; note that $\varphi \to \exp[i\alpha]\varphi$: $P_\varphi \to P_\varphi$.

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Now consider an incoherent mixture of orthogonal states φ_n each with probability c_n $(0 \leqslant c_n \leqslant 1;\ \sum c_n = 1)$.

The average value of an observable A over this mixture is:

$$\langle A \rangle = \sum c_n \langle \varphi_n | A | \varphi_n \rangle = \sum c_n \operatorname{Tr} A P_{\varphi_n} = \operatorname{Tr} A \sum c_n P_{\varphi_n} = \operatorname{Tr} A \varrho = \operatorname{Tr} \varrho A \,,$$

where we define

$$\varrho = \sum c_n P_{\varphi_n}; \qquad \operatorname{Tr} \varrho = \sum c_n = 1 \qquad\qquad \varrho^* = \varrho;$$

ϱ is called the density matrix of the mixture.

Note that for a true mixture $\operatorname{Tr} \varrho^n < 1$ for each integer $n > 0$; $\varrho^n = \varrho$ for a pure state.

3. – Pure states and mixtures of polarization states.

We consider particles of non-vanishing mass. In the rest system: $p = (m, 0, 0, 0)$ there are $2j+1$ independent states. Therefore, under a rotation

$$\varrho \to \varrho' = D^{(j)} \varrho D^{(j)*} \,,$$

where $D^{(j)}$ is the proper rotation matrix:

$$\varrho'_{\alpha\beta} = D^{(j)}_{\alpha\sigma} \overline{D}^{(j)}_{\beta\tau} \varrho_{\sigma\tau} \,,$$

which means that the elements of ϱ transform like the components of tensors under $D^{(j)} \otimes \overline{D}^{(j)}$. Since $\overline{D}^{(j)} \sim D^{(j)}$; $D^{(j)} \otimes \overline{D}^{(j)} \sim D^{(2j)} + D^{(2j-1)} \ldots + D^{(0)}$, which means that ϱ can be written as a sum of irreducible tensors of ranks 0, ... up to $2j$.

Since space inversion is known to commute with all rotations, by Schur's lemma, it is represented by a multiple of the unit matrix ($\exp[i\alpha]$); thus $D\overline{D}$ is invariant under space reflection, therefore the irreducible tensors are even under space reflection (*e.g.* one has a scalar $\operatorname{Tr} \varrho$, a pseudovector...).

In order to describe a beam of particles one can normalize ϱ to $\operatorname{Tr} \varrho =$ the intensity of the beam instead of unity: the beam is composed of particles which have all been prepared in the same way, so that one may consider that all particles in the beam are described by the same density matrix, the c_n's being now the probabilities for finding particles in a given pure state. The use of a density matrix allows for the description of a system, the knowledge of which is incomplete.

Spin $\frac{1}{2}$:

$$D^{(\frac{1}{2})} \otimes D^{(\frac{1}{2})} = D^{(0)} + D^{(1)} \,.$$

The polarization state is described in terms of a scalar and a pseudo-vector $\boldsymbol{S}$:

$$\varrho = \frac{1}{2}(1 + \boldsymbol{S}\cdot\boldsymbol{\tau}) , \qquad \operatorname{Tr} \varrho^2 = \frac{1 + \boldsymbol{S}^2}{2} \leqslant 1 : \qquad \boldsymbol{S}^2 \leqslant 1 .$$

4. – Relativistic description of polarization.

As we know how to transform χ by any Lorentz transformation (see WIGHTMAN's lecture). (When Λ is in the little group Λ_p isomorphic to the rotation group, for mass $\neq$ zero, one has exactly the ordinary published accounts of « non relativistic theory ».) The transformation law for the density matrix is:

$$\varrho'(p) = Q(p, \Lambda)\varrho(\Lambda^{-1}p)Q^*(p, \Lambda) .$$

Example. Scattering of two particles: let $\varrho_1^{(i)}(p_1) \otimes \varrho_2^{(i)}(p_2)$ be the density matrix for uncorrelated initial particles: after the interaction the density matrix cannot in general be written as a tensor product of two density matrices, that is to say that, although the outgoing particles may not be individually polarized, there may be some correlation between their polarizations: *i.e.* may be $\text{Prob}\begin{pmatrix}\uparrow\\1\end{pmatrix} = \text{Prob}\begin{pmatrix}\downarrow\\1\end{pmatrix}$, $\text{Prob}\begin{pmatrix}\uparrow\\2\end{pmatrix} = \text{Prob}\begin{pmatrix}\downarrow\\2\end{pmatrix}$ but $\text{Prob}\begin{pmatrix}\uparrow\ \uparrow\\1,2\end{pmatrix} \neq$ $\neq \text{Prob}\begin{pmatrix}\uparrow\ \downarrow\\1,2\end{pmatrix}$.

5. – Particles of vanishing mass and finite spin.

The little group is isomorphic to the 2-dimensional Euclidean group. First of all one shall not consider the translations of this group since we want a finite spin; the remaining of the little group is then isomorphic with the group of rotations about $\boldsymbol{p}$ and reflections through planes containing $\boldsymbol{p}$, that is to say the space group of diatomic molecules:

$$\mathcal{D}_j \otimes \mathcal{D}_j \sim \mathcal{D}_{2j} + \mathcal{D}_0^+ + \mathcal{D}_0^-$$

which correspond, to the well known classification $\Sigma^\pm$, π, Δ, Φ, ...

$\mathcal{D}_0^+$ (scalar) corresponds to $\operatorname{Tr} \varrho$ and describes the degree of polarization,

$\mathcal{D}_0^-$ (pseudoscalar) describes the circular polarization (helicity),

$\mathcal{D}_{2j}$ (two dimensional vector *i.e.* an azimuth angle) describes the direction of the transverse polarization $\boldsymbol{n}$ ($\boldsymbol{n}\cdot\boldsymbol{p} = 0$, $\boldsymbol{n}^2 = 1$). It can be written covariantly $\mathbf{p}^2 = 0 = \mathbf{p}\cdot\mathbf{n}$, $\mathbf{n}^2 = -1$; this $\mathbf{n}$ is defined modulo a component along $\mathbf{p}$, *i.e.* $\mathbf{n} + \alpha\mathbf{p} \sim \mathbf{n}$.

6. – Representation by means of the Poincaré sphere.

Poles $P_\pm$ represent pure circular polarization: $\xi = \pm 1$.

Equator E = transverse polarization. Any point of the sphere is of elliptical polarization $-1 \leqslant OG = \xi \leqslant 1$, $\boldsymbol{OM}^2 = -\mathbf{n}^2$. Pure state *i.e.* totally polarized $\xi^2 - \mathbf{n}^2 = 1$. Partially polarized state with degree of polarization η: $0 \leqslant \eta = \sqrt{\xi^2 - \mathbf{n}^2} \leqslant 1$. Center of the sphere = unpolarized light.

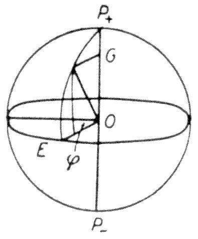

Fig. 1.

Exercise. By $\Lambda \in \Lambda_p$ (little group of p) the Poincaré sphere rotates around the axis of poles.

This is quite different of the case $m \neq 0$ spin $\frac{1}{2}$ where there is also only 2 linearly independent polarization states, but where by $\Lambda \in \Lambda_p$, $\boldsymbol{S}$ can be transformed in *any* $\boldsymbol{S}'$ of same length.

7. – Infinitesimal approach.

We shall now revert to the study of infinitesimal generators of the Lie algebra:

$$\begin{aligned}
[P_\lambda, P_\mu] &= 0 \,, \\
[M_{\mu\nu}, M_{\varrho\sigma}] &= i[g_{\mu\varrho} M_{\nu\sigma} - g_{\nu\varrho} M_{\mu\sigma} - g_{\mu\varrho} M_{\varrho\nu} + g_{\nu\sigma} M_{\mu\varrho}] \,, \\
[P_\lambda, M_{\mu\nu}] &= i[g_{\lambda\mu} P_\nu - g_{\lambda\nu} P_\mu] \,.
\end{aligned}$$

The P's and M's are hermitian operators: as we saw they generate the envelopping associative algebra. The mathematician looks for a maximal abelian subalgebra.

The physicist has the same reflexes, but will call it a complete set of commuting observables.

We know already that $\mathbf{P}^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$ belong to this set $(W_\mu = \frac{1}{2}\varepsilon^{\mu\nu\varrho\sigma} P_\nu M_{\varrho\sigma})$ since they belong to the center. One has

$$[P_\nu, W_\mu] = 0 \,, \qquad \text{but} \qquad [W^\mu, W^\nu] = -i\varepsilon^{\mu\nu\varrho\sigma} P_\varrho W_\sigma \,.$$

Hence one can choose the four P_λ and W^2 and one W_μ. How to choose the last one « covariantly »? We shall now study a basis adapted to particle states of given energy momentum: *i.e.* $(P^\mu = p^\mu \in R^4$, $\mathbf{p}^2 = m^2)$. To every such point p^μ of the spectrum of the P^μ's there corresponds a Hilbert space of polarization states. Consider now the restriction of the preceeding operators to this Hilbert space:

$$\begin{aligned}
P^\mu &\to p^\mu \,, \\
W^\mu &\to w^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\varrho\sigma} P_\nu M_{\varrho\sigma} \,.
\end{aligned}$$

Consider an ortoghonal set of four-vectors: $\mathbf{u}^{(\alpha)}\cdot\mathbf{u}^{(\beta)} = g^{\alpha\beta}$:
(If $m \neq 0$ choose $\mathbf{p}/m$, $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$; then $\mathbf{n}_i$ are space like $\mathbf{p}$).
Then let $S^{(\alpha)} = W^\lambda n_\lambda^{(\alpha)}$; then $W_\lambda = g_{\alpha\beta} S^{(\alpha)} n_\lambda^{(\beta)} = S^{(\alpha)} n_{(\alpha)\lambda}$.
When $m \neq 0$, $W^\lambda = \sum_{i=1}^{3} S^{(i)} n_{(i)}^\lambda$ because $W^\lambda p_\lambda = 0$

$$W^2 = W^\lambda W_\lambda = -\sum_{i=1}^{3} S^{(i)2} ,$$

write now $\{S^i\} = \mathbf{S}$. The commutator of W_λ's yields

$$\left[\frac{S^i}{m}, \frac{S^j}{m}\right] = i\varepsilon^{ijk}\frac{S^k}{m} ,$$

which is to say that the little group is isomorphic to the three dimensional rotation group: hence

$$W^2 = -S^2 = -j(j+1)m^2 .$$

Take furthermore any $n \in \mathbf{n}\cdot\mathbf{p} = 0$, then $\mathbf{W}\cdot\mathbf{n}$ has eigenvalues $-j < m < +j$
Thus we can choose for the center:

$$P_\lambda(P_\lambda P^\lambda = m^2) , \qquad W^2 , \qquad \mathbf{W}\cdot\mathbf{n} .$$

8. – Mass 0 case; $p^2 = 0$.

The only difference from the previous case is that one cannot choose an orthogonal basis which includes $\mathbf{p}$. We choose a time-like vector $\mathbf{t}$ and complete it with $\mathbf{n}_1$, $\mathbf{n}_2$, $\alpha\mathbf{p} - \mathbf{t}$ to form an orthogonal basis $\mathbf{U}^{(\alpha)}$. Writing $S^{(\alpha)} = W^\lambda U_\lambda^{(\alpha)}$, the condition $P_\lambda W^\lambda = 0$ yields

$$S^{(0)} + S^{(3)} = 0 \qquad (\text{write } W_\lambda = S^{(\alpha)} U_\lambda^{(\alpha)}) .$$

Furthermore

$$W^2 = S^{(0)2} - \sum_i (S^i)^2 = -S^{(1)2} - S^{(2)2} .$$

The commutation rules for the S's are found to be

$$[S^{(3)}, S^{(1)}] = \frac{i}{\alpha} S^{(2)} ,$$

$$[S^{(2)}, S^{(3)}] = \frac{i}{\alpha} S^{(1)} ,$$

$$[S^{(2)}, S^{(1)}] = 0 ,$$

which characterizes the Lie algebra of L_p. This Lie algebra is isomorphic to that of the Euclidean two dimensional group: $S^{(1)}$, $S^{(2)}$ stand for the translation generators whereas $S^{(3)}$ stands for the rotation generator (rotations around $\mathbf{p}$).

Two cases must be distinguished:

$W^2 \neq 0$ which corresponds to the infinite spin case.

$W^2 = 0$: $(S^{(1)})^2 = (S^{(2)})^2 = 0$, *i.e.* since $(S^{(1)})^* = S^{(1)}$; $S^{(2)} = S^{(1)} = 0$.

The only generator left is $S^{(3)} = -S^{(0)}$, it defines an abelian group.

9. – Conclusion.

We have characterized the state of a particle of momentum $\mathbf{p}$, mass m ($p^2 = m^2$), spin j $(W^\mu W_\mu = -m^2 j(j+1))$ by means of a polarization operator $W \cdot n$, where n is a unit vector orthogonal to p. The eigenvalues of $\mathbf{W} \cdot \mathbf{n}$ are « magnetic » quantum numbers: $-j \leqslant m \leqslant +j$.

In the zero mass case, only two polarization states are available: $m = \pm j$ these states have opposite helicities.

Examples. Spin $\frac{1}{2}$.

Consider $\mathbf{n}^2 = -1$, $\mathbf{n} \cdot \mathbf{p} = 0$.

The projection operators on states for $\boldsymbol{W} \cdot \boldsymbol{n}/m = \pm \frac{1}{2}$ are $1 \pm ((2\mathbf{W} \cdot \mathbf{n}/m)/2)$. This is the density matrix of the pure state. More generally, we have seen for the density matrix

$$\frac{1 + \sum_i \varphi^i \tau^i}{2} = \frac{1 + 2 \sum_i \varphi^i \sigma^i}{2} \qquad \text{where} \qquad [\sigma^i, \sigma^j] = i\varepsilon_{ijk}\sigma^k .$$

Here

$$\sigma^i = \frac{W^\lambda n^{(i)}_\lambda}{m} = \frac{\boldsymbol{W} \cdot \boldsymbol{n}^{(i)}}{m}$$

as we have seen.

Hence the most general density matrix, if we call $\sum \varphi^{(i)} \mathbf{n}^{(i)} \mathbf{s}$, is $(1 + (2\mathbf{W} \cdot \mathbf{s}/m))/2$ where the four-pseudovector $\mathbf{s}$ is such that $\mathbf{s} \cdot \mathbf{p} = 0$, $0 \leqslant -\mathbf{s}^2 = \sum_i (\varphi^i)^2 =$ (degree of polarization)$^2 \leqslant 1$.

10. – Spin j.

The projection on the state polarized along $\mathbf{n}$ with polarization $(-j \leqslant m \leqslant j)$ is

$$C_{(m)} \prod_{\lambda \neq m}^{j}{}_{\lambda = -j} (\mathbf{W} \cdot \mathbf{n} - \lambda) = P_{2j}(\mathbf{W} \cdot \mathbf{n}) ,$$

where $C^{-1}_{(m)} = (-1)^{j-m}(j-m)!\,(j+m)!$

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It is a polynomial of degree $2j$ in $\mathbf{W}\cdot\mathbf{n}$.

More generally we have seen that the elements of the density *matrix* can be linearly combined into,

$$\frac{1}{2j+1}+\sum_i \varphi^i J^i+\sum_{ij}\varphi^{ij}J^iJ^j+\sum_{ijk}\varphi^{ijk}J^iJ^jJ^k+\dots,$$

with $[J^i, J^j]=i\varepsilon^{ijk}J^k$ and where the tensors φ^i, φ^{ij}, φ^{ijk} transform under the little group (isomorphic to 3-dimensional rotation group) as D_1, D_2, ..., D_{2j}. Hence the most general state of polarization will be covariantly described by the density matrix,

$$\frac{1}{2j+1}+s_\lambda W^\lambda+s_{\lambda\mu}W^\lambda W^\mu+s_{\lambda\mu\nu}W^\lambda W^\mu W^\nu+\dots+s_{\lambda\dots\sigma}W^\lambda\dots W^\sigma\,,\quad 2j \text{ indices}\,.$$

where $\sigma_{\lambda\dots\sigma}$ is a totally symmetric tensor such that $p^\lambda s_{\lambda\dots\sigma}=0$, and $s_{\lambda\dots}{}^{\mu}{}_{\mu\dots\sigma}=0$ and other conditions on $s_{\lambda\dots\sigma}s^{\lambda\dots\sigma}$ fixing the polarization degree.

11. – Application to Dirac theory, mass $\neq 0$.

The Dirac amplitude $u(p)$ satisfying the Dirac equation $(\not p-m)u(p)=0$ is transformed by inhomogeneous Lorentz transformations $(a, \Lambda\in L^\uparrow)$

$$(U(a,\Lambda)(u(p))=\exp[ipa]S(\Lambda)U(\Lambda^{-1}p)\,,$$

where S is a four dimensional representation of the homogeneous Lorentz group $L^\uparrow$. The $U(a,\Lambda)$ is a unitary operator in the Hilbert space of the $u(p)$ with the metric,

$$\int_{\mathscr{H}_m}|u^+(p)u(p)|\,\mathrm{d}\Omega_m=\int|u^+(p)u(p)|\,\frac{\mathrm{d}^3\boldsymbol{p}}{|p^0|}=\int|\bar u(p)\cdot A_i\gamma\cdot u(p)|\,\frac{\mathrm{d}^3\boldsymbol{p}}{|p^0|^2}\,.$$

(The integration is over the hyperboloid $\mathbf{p}^2=m^2>0$. The adjoint spinor $u^+=\bar u A$ where the matrix A is defined by $A^*=A$, det $A=1$. $A\gamma^\mu A^{-1}=-\gamma^{\mu*}$; then $A^i\gamma^0$ is definite positive and is generally taken to be one.)

The infinitesimal operator $iM_{\mu\nu}$ is obtained by derivation of $(U(0,\Lambda)u)(p)$ at the identity, as shown in Section **4**. We obtain

$$M_{\mu\nu}=\frac{1}{2}\sigma_{\mu\nu}+\frac{1}{i}\left(p_\mu\frac{\partial}{\partial p^\nu}-p_\nu\frac{\partial}{\partial p^\mu}\right),$$

where $\sigma^{\mu\nu} = (1/2i)[\gamma^\mu, \gamma^\nu]$. The two terms correspond to the spin part and orbital momentum part, the last part does not give contribution to W^λ.

We shall compute the restriction of W^λ on the (two dimensional) Hilbert space of polarization for energy momentum **n**

$$W^\lambda = \tfrac{1}{2}\varepsilon^{\lambda\mu\nu\varrho} p_\mu M_{\nu\varrho} = \tfrac{1}{4}\varepsilon^{\lambda\mu\nu\varrho}\sigma_{\nu\varrho} p_\mu = \tfrac{1}{2}\gamma^5\sigma^{\lambda\mu} p_\mu$$

(computation of $W^\lambda W_\lambda$ yields $\mathbf{w}^2 = -\frac{3}{4}m^2 = -j(j+1)m^2$ as expected). Now $W^\lambda n_\lambda = \mathbf{W}\cdot\mathbf{n} = \frac{1}{2}\gamma^5\sigma^{\lambda\mu} n_\lambda p_\mu = \frac{1}{2} i\gamma^5 \not{n}\not{p}$.

We check that $(2\mathbf{W}\cdot\mathbf{n})/m = i\gamma^5\not{n}(\not{p}/m)$ has square 1 and $\frac{1}{2}(1 + i\gamma_5\not{s}(\not{p}/m))$ project the solution $u(p)$ of $(\not{p} - m)u(p) = 0$ onto the $u(p, s)$ Dirac amplitude for the state of energy momentum **p** and total polarization along **s**. Taking $Ai\gamma^0$, the projector

$$P(\boldsymbol{p}, \boldsymbol{s}) = \frac{1}{4}\left(1 + i\gamma^5 \not{s}\,\frac{\not{p}}{m}\right)(\not{p} + m)\,\frac{i\gamma^0}{|p^0|} = \frac{1}{4}\,(1 + i\gamma^5\not{s})(\not{p} + m)\,\frac{i\gamma^0}{|p^0|}$$

is hermitian, and projects any Dirac spinor onto the spinor $u(p, s)$. Since it is a rank-one projector $(\mathrm{Tr}\, P = 1)$, one has therefore

$$P_{\alpha\beta}(p, s) = u_\alpha(p, s)\,\bar{u}_\beta(p, s) = u(p, s)\,\bar{u}(p, s) = P(p, s)\,.$$

We have therefore explicitly constructed the density matrix of the pure polarization state p, s. For the general mixture $0 \leqslant -\mathbf{s}^2 \leqslant 1$ is to be added to $\mathbf{s}\cdot\mathbf{p} = 0$.

We can choose for basis of states of polarization $u(p, \varepsilon s)$ with $\varepsilon = \pm 1$. Then any state vector is of the form $\sum_\varepsilon \chi_\varepsilon(p) u(p, \varepsilon s)$. (For a given p; more generally, there can be an integration over p.)

The $\chi_\varepsilon(p)$ are those already studied at the beginning of V. We can therefore describe covariantly to our convenience the polarization states of spin $\frac{1}{2}$ either by a two by two density matrix

$$\varrho_{\varepsilon\eta}(p) = \chi_\varepsilon(p)\,\bar{\chi}_\eta(p)$$

or by the four by four density matrix $P_{\alpha\beta}(p, s)$.

The formula containing all the correspondence is (BOUCHIAT and MICHEL: *Nuclear Physics* (Feb. 1958)), using the usual Pauli matrix representation with τ_3 diagonal, $((\tau_i)_{\varepsilon\eta}$ = elements of Pauli matrices)

$$u_\alpha(\boldsymbol{p}, \eta\boldsymbol{n}^{(3)})\,\bar{\boldsymbol{u}}_\beta(\boldsymbol{p}, \varepsilon\boldsymbol{n}^{(3)}) = \frac{1}{4}\left(1 + i\gamma_5\Big(\sum_i \not{n}^{(i)}(\tau_i)_{\varepsilon\eta}\Big)\right)(\not{p} + m)\,\frac{i\gamma^0}{|p^0|}\,.$$

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Hence, any field theoretical computation of polarization effects can be made covariantly and reduces to trace computation, although it is however possible to describe the computation in terms of the 2 by 2, $\varrho_{\varepsilon\eta}$ density matrices.

12. – Case of $m = 0$.

We do not explain here the limiting process given by MICHEL and WIGHTMAN, but indicate only the result:

$$P(p, s, \xi) = \frac{1}{4}\,(1 + i\gamma_5(\not{s} + \not{\xi}))\not{p}\,\frac{i\gamma^0}{|p^0|}\,,$$

where ξ, **s** have already been defined. We recall that

$$0 \leqslant \xi^2 - \mathbf{S}^2 \leqslant 1\,, \qquad \mathbf{P}^2 = \mathbf{p}\cdot\mathbf{s} = 0\,.$$

13. – Application to the Bargmann-Wigner theory of particles of arbitrary spin.

(BARGMANN-WIGNER: *Proc. Nat. Ac. of Sci.*, **34**, 211 (1948)).

We obtain, according to BARGMANN and WIGNER, the theory of particles of spin $j = n/2$ from the Dirac theory of spin $\frac{1}{2}$ by the following « transport of structure ».

Consider instead of the four dimensional Dirac spinor space $\mathcal{E}_4$ the n-th tensorial symmetric product $\overset{n}{\vee}\mathcal{E}_4$, that is the space of functions $u_{\alpha_1\dots\alpha_n}(p)$, completely symmetrical in the α_i (*i.e.* invariant by any permutation of the indices $\alpha_1 \dots \alpha_n$), and satisfying the equations:

$$\sum_{\alpha_1\dots\alpha_n} (\not{p}^{(k)} - m)_{\alpha'_1\dots\alpha'_n,\alpha_1\dots\alpha_n} u_{\alpha_1\dots\alpha_n}(p) = (\not{p}^{(k)} - m)u(p) = 0\,,$$

where $\not{p}^{(k)} = i\gamma_\mu^{(k)} p^\mu$ and

$$(\gamma_\mu^{(k)})_{\alpha'_1\dots\alpha'_n,\alpha_1\dots\alpha_n} = \delta_{\alpha'_1\alpha_1} \dots \delta_{\alpha'_{k-1}\alpha_{k-1}} (\gamma_\mu)_{\alpha'_k\alpha_k} \dots \delta_{\alpha'_n\alpha_n}$$

that is, in tensorial notation

$$\gamma_\mu^{(k)} = 1 \otimes 1 \otimes \dots \otimes \underset{k\text{-th place}}{\gamma_\mu} \otimes \dots \otimes 1 \quad (n \text{ terms in the product}),$$

The $u_{\alpha_1\dots\alpha_n}(p)$ are transformed by the inhomogeneous Lorentz group according to

$$(U(\mathbf{a}, \Lambda)u)_{\alpha_1\dots\alpha_n}(p) = \sum_{\beta_i} \exp[\overset{n}{\otimes} i\mathbf{p}\cdot\mathbf{a}][S(n)]_{\alpha_1\dots\alpha_n,\beta_1\dots\beta_n} u_{\beta_1\dots\beta_n}(\Lambda^{-1}\mathbf{p})\,,$$

where $\overset{n}{\otimes} S(\Lambda) = S(\Lambda) \otimes \dots n$ times $\dots \otimes S(\Lambda)$.

(We leave to the reader the case of a, $\Lambda \in \mathcal{L}^{\downarrow}$ as we did for the Dirac case.)

This is a unitary representation of $\mathcal{L}^{\uparrow}$ in the Hilbert space of scalar product (here $A = i\gamma^0$, and $Ai\gamma^0 = 1$)

$$(u'(p), u(p)) = \int_{\mathcal{H}_m} |\bar{u}(p)(\overset{n}{\otimes} i\gamma^0)\, u(p)| \frac{\mathrm{d}^3\boldsymbol{p}}{|p^0|} = \int_{\mathcal{H}_m} \bar{u}(p)u(p)\frac{\mathrm{d}^3\boldsymbol{p}}{|p^0|^{n+1}} =$$

$$= \sum_{\alpha_i} \int \bar{u}_{\alpha_1\dots\alpha_n}(p)\, u_{\alpha_1\dots\alpha_n}(p) \frac{\mathrm{d}^3\boldsymbol{p}}{|p^0|^{n+1}}\,.$$

In a fashion analogous to the Dirac case, we find for the infinitesimal operator

$$M_{\mu\nu} = \sum_k \frac{1}{2}\sigma^{(k)}_{\mu\nu} + \frac{1}{i}\left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu}\right),$$

where

$$\sigma^{(k)}_{\mu\nu} = \frac{1}{2i}[\gamma^{(k)}{}_\mu, \gamma^{(k)}{}_\nu]\,.$$

Indeed the derivative of $\overset{n}{\otimes} S$ at the origin is

$$(S'(1)\otimes 1 \dots \otimes 1) + \big(1\otimes (S'(1)) \otimes \dots \otimes 1\big) + \dots + (1 \otimes \dots \otimes S'(1)) = \sum_k S'_k\,.$$

Hence

$$W^\lambda = \tfrac{1}{4}\,\varepsilon^{\lambda\mu\nu\varrho} p_\mu \sum_k \sigma^{(k)}{}_{\nu\varrho} = \sum_k \tfrac{1}{2}\gamma^{(k)5}\sigma^{(k)\lambda\mu}p_\mu\,.$$

We leave to the reader to form the projector $P_{2j}(W\cdot n)$ already defined. Let us just conclude that the polarization states are covariantly described by the set of $2j$ tensors

$$s_\lambda,\ s_{\lambda\mu},\ \dots,\ s_{\lambda\dots\sigma},\ \dots$$

which are the expectation values of the products of 1 to $2j$, W_λ just computed. For pure states described by the amplitude $u(p)$

$$s_{\lambda\dots\sigma} = \bar{u}(p)W_\lambda \dots W_\sigma u(p) = \mathrm{Tr}\, u(p)\otimes \bar{u}(p) W_\lambda \dots W_\sigma\,.$$

For the misture described by the density matrix $P_{\alpha_1\dots\alpha_n,\alpha_1\dots\beta_n}(p)$

$$s_{\lambda\dots\sigma} = \mathrm{Tr}\, P(p) W_\lambda \dots W_\sigma\,.$$