CONSTRAINTS ON SPIN ROTATION PARAMETERS DUE TO ISOSPIN CONSERVATION

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We made a complete study of the relations between the three cross sections and the three sets of spin rotation parameters P, A, R for three reactions related by internal symmetry via two channels.

1. The transition matrix T of a reaction involving spin 0 and spin $\frac{1}{2}$ particles:

$$0 + \tfrac{1}{2} \to 0' + \tfrac{1}{2}' , \tag{1}$$

can be written $f + ig\boldsymbol{\sigma}\cdot\boldsymbol{n}$ where f and g are respectively the non-spin-flip and the spin-flip amplitudes. The most usual reactions of this kind (e.g. πN, KN) are going through two channels of isospin‡; hence, for three reactions which differ only by the isospin components of the particle isomultiplets, the transition matrices satisfy a linear relation:

$$\sum_{\alpha=1}^{3} \gamma_\alpha T_\alpha = 0 , \tag{2}$$

where each γ_α is a homogeneous fourth degree polynomial of Clebsch-Gordan coefficients.

In this letter we derive all relations imposed by eq. (2) on the cross-sections and on the spin rotation parameter A, P, R [1-3]. It is convenient to consider $\gamma_\alpha f_\alpha$ and $\gamma_\alpha g_\alpha$ as the components of an element $|\alpha\rangle$ of a two dimensional Hilbert space. Then, denoting by σ_α the cross-section, one has ($\boldsymbol{\sigma}$ is the set of the three Pauli matrices)

$$M_\alpha = |\alpha\rangle\langle\alpha| = \tfrac{1}{2} s_\alpha(1 + \zeta_\alpha\cdot\boldsymbol{\sigma}) , \tag{3}$$

where

$$s_\alpha = \langle\alpha|\alpha\rangle = \gamma_\alpha^2 \sigma_\alpha > 0 , \tag{4}$$

and

$$\zeta_\alpha = \frac{1}{s_\alpha}\langle\alpha|\boldsymbol{\sigma}|\alpha\rangle = (A_\alpha, P_\alpha, R_\alpha) , \tag{5}$$

i.e. the components of ζ are the spin rotation parameters of the reaction α. They satisfy

$$\zeta_\alpha^2 = 1 = A_\alpha^2 + P_\alpha^2 + R_\alpha^2 . \tag{6}$$

The vector ζ will be called the *spin rotation vector*.

2. From now on, the three indices α, β, γ represent any permutation of 1, 2, 3. The linear relation on the vectors $|\alpha\rangle$, corresponding to eq. (2), is

$$|\alpha\rangle + |\beta\rangle + |\gamma\rangle = 0 . \tag{7}$$

Each $|\alpha\rangle$ with spin rotation vector ζ_α has an orthogonal element $|\alpha^\perp\rangle$ with same s_α and spin rotation vector $-\zeta_\alpha$. The scalar product of eq. (7) with $\langle\alpha^\perp|$ gives

$$\langle\alpha^\perp|\beta\rangle = -\langle\alpha^\perp|\gamma\rangle . \tag{8}$$

From

$$|\langle\alpha^\perp|\beta\rangle|^2 = \mathrm{tr}\, M_{\alpha^\perp} M_\beta = \tfrac{1}{2} s_\alpha s_\beta(1 - \zeta_\alpha\cdot\zeta_\beta) = = \mathrm{tr}\, M_\alpha M_{\beta^\perp} = |\langle\alpha|\beta^\perp\rangle|^2 , \tag{9}$$

and from eq. (8) we obtain

$$\boxed{s_\alpha s_\beta(1 - \zeta_\alpha\cdot\zeta_\beta) = \tfrac{1}{2}H \geq 0 ,} \tag{10}$$

where *H is a constant* independent of α, β, γ. Since the ζ have unit length, we can write:

‡ Those reactions commonly go also through two channels of U spin and V spin, and in some cases, such as π^+p^+, through two channels of the full unitary spin. The considerations of this letter can be extended to these invariances and to the cases when the 0-spin particles are replaced by unpolarized particles.

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$$0 \leqslant (\xi_\alpha \times \xi_\beta \cdot \xi_\gamma)^2 \leqslant 1 \; ; \tag{11}$$

with the use of eq. (10), eq. (11) is equivalent to:

$$\boxed{0 \leqslant H \leqslant -\Delta(s_\alpha, s_\beta, s_\gamma) \leqslant 4(s_\alpha s_\beta s_\gamma)^2 H^{-2} + H} \tag{12}$$

where

$$\Delta(s_\alpha, s_\beta, s_\gamma) \equiv s_\alpha^2 + s_\gamma^2 + s_\gamma^2 - 2 s_\alpha s_\beta - 2 s_\beta s_\gamma - 2 s_\gamma s_\alpha \; . \tag{12'}$$

When $H \geqslant 0$ and $s_\alpha > 0$, the last inequality in eq. (12) is always satisfied; the equality holds only when $s_\alpha = s_\beta = s_\gamma$ and $\theta_{\alpha\beta} = \theta_{\beta\gamma} = \theta_{\gamma\alpha} = \frac{1}{2}\pi$, where $\theta_{\alpha\beta}$ is the angle between ξ_α and ξ_β. Note that

$$0 \leqslant \theta_{\alpha\beta} \leqslant \pi \; ; \quad \cos\theta_{\alpha\beta} = \xi_\alpha \cdot \xi_\beta \; . \tag{13}$$

In the following we will say that ξ is described equivalently by a unit vector or a point on the unit sphere.

3. Eqs. (10) and (12) are sufficient for the study of any experimental situation. For instance:

(i) *One knows only* s_α, s_β.

From $0 \leqslant H \leqslant 4 s_\alpha s_\beta$ and from eq. (12) the cross sections $\sigma_\alpha = s_\alpha \gamma \bar{\alpha}^2$ must satisfy:

$$\Delta(s_\alpha, s_\beta, s_\gamma) \leqslant 0 \; . \tag{14}$$

This is the well-known condition that the three $\sqrt{s_\alpha}$ must form a triangle. This condition gives the bounds for s_γ:

$$|s_\gamma - s_\alpha - s_\beta| \leqslant 2\sqrt{s_\alpha s_\beta} \; . \tag{14'}$$

(ii) *One knows* $s_\alpha, s_\beta, \xi_\alpha, \xi_\beta$.

Better bounds on s_γ are given by eq. (12): $-\Delta \geqslant H$; they are

$$|s_\gamma - s_\alpha - s_\beta| \leqslant 2\sqrt{\tfrac{1}{2} s_\alpha s_\beta (1 + \xi_\alpha \cdot \xi_\beta)} = 2\sqrt{s_\alpha s_\beta}\cos\tfrac{1}{2}\theta_{\alpha\beta} \; . \tag{15}$$

This condition (15) is always stricter than condition (14'), except in the case $\xi_\alpha = \xi_\beta$; then $H = 0$ and $\xi_\gamma = \xi_\alpha = \xi_\beta$; this happens when the transition matrix of one of the two isospin channels vanishes.

Eq. (15) can also be written in the two equivalent forms

$$|\cos \omega_{\alpha\beta}| \leqslant \cos\tfrac{1}{2}\theta_{\alpha\beta} \; , \tag{15'}$$

$$0 \leqslant \tfrac{1}{2}\theta_{\alpha\beta} \leqslant \omega_{\alpha\beta} \leqslant \pi - \tfrac{1}{2}\theta_{\alpha\beta} \; , \tag{15''}$$

where $\omega_{\alpha\beta}$ is the angle between the sides $\sqrt{s_\alpha}$, $\sqrt{s_\beta}$ of the triangle defined by eq. (14).

(iii) *One knows* $s_\alpha, s_\beta, s_\gamma$ *satisfying* eq. (14) *and* ξ_α.

The eqs. (15) give the domain of ξ_β; it is, on the unit sphere, a circular portion whose center is ξ_α; its aperture is $\theta_{\alpha\beta}$ such that

$$0 \leqslant \theta_{\alpha\beta} \leqslant \mathrm{Min}(2\omega_{\alpha\beta}, \; 2(\pi - \omega_{\alpha\beta})) \; . \tag{16}$$

Note that there is no restriction on $\theta_{\alpha\beta}$ when $\omega_{\alpha\beta} = \frac{1}{2}\pi$.

(iv) *One knows* $s_\alpha, s_\beta, s_\gamma$ *satisfying* eq. (14) ξ_α, ξ_β *satisfying* eq. (15).

The point on the unit sphere which defines ξ_γ must be, according to eq. (10), at the intersection of the two circles whose centers and apertures are:

$$\xi_\alpha \, , \quad \theta_{\alpha\gamma} = \cos^{-1}(1 - (s_\beta/s_\gamma)(1 - \xi_\alpha \cdot \xi_\beta)) \; ; \tag{17a}$$

$$\xi_\beta \, , \quad \theta_{\beta\gamma} = \cos^{-1}(1 - (s_\alpha/s_\gamma)(1 - \xi_\alpha \cdot \xi_\beta)) \; . \tag{17b}$$

That these two circles intersect is a consequence of eqs. (14) and (15). In general, they have two common points, representing two distinct values of ξ_γ. These two values become equal when the equalities hold in eqs. (15) and (15').

There are two exceptional cases when the two circles coincide; this happens when they have the same axis i.e. $\xi_\alpha = \pm\xi_\beta$. Eq. (10) shows that when $\xi_\alpha = \xi_\beta$ the two circles reduce to one point i.e. $\xi_\alpha = \xi_\beta = \xi_\gamma$. When $\xi_\alpha + \xi_\beta = 0$, eq. (15) reads $s_\gamma = s_\alpha + s_\beta$ which, with eq. (17), yields

$$\cos\theta_{\alpha\beta} = (s_\alpha - s_\beta)/(s_\alpha + s_\beta) = -\cos\theta_{\beta\gamma} \; . \tag{18}$$

This completely defines the common circle.

Experimental situations are more varied than these four typical cases. For example:

(v) *One knows:* $s_\alpha > s_\beta$ and ξ_α:

The triangle relation requires $0 \leqslant \omega_{\alpha\gamma} \leqslant \sin^{-1}\sqrt{s_\beta/s_\alpha}$, and from eq. (15') this yields for ξ_γ:

$$(s_\alpha - 2s_\beta)/s_\alpha \leqslant \xi_\gamma \xi_\alpha \; . \tag{19}$$

Eqs. (10) and (12) are also sufficient to deal with experimental data with partial information on some of the spin-rotation vectors (i.e. not all their components are known); see ref. [3].

4. For each one of the three reactions, the measurement of the cross section and of the spin rotation parameters determine the scattering amplitudes f_α and g_α, up to an unobservable common phase factor $\exp(i\varphi_\alpha)$:

$$\begin{aligned} \gamma_\alpha f_\alpha &= \exp(i\varphi_\alpha)\,[\tfrac{1}{2} s_\alpha(1 + R_\alpha)]^{1/2} \; , \\ \gamma_\alpha g_\alpha &= \exp(i\varphi_\alpha)\,[\tfrac{1}{2} s_\alpha(1 - R_\alpha)]^{1/2} \exp(i\chi_\alpha) \; , \end{aligned} \tag{20}$$

with

$$\chi_\alpha = \tan^{-1}(-P_\alpha/A_\alpha) \; . \tag{20'}$$

We consider the angles $\varphi_{\alpha\beta}$ defined by

$$\exp(i\varphi_{\alpha\beta}) = \frac{\langle\alpha|\beta\rangle}{|\langle\alpha|\beta\rangle|} . \tag{21}$$

They satisfy the relations

$$\varphi_{\alpha\beta} + \varphi_{\beta\alpha} = 0 , \tag{22a}$$

$$\exp\{i(\varphi_{\alpha\beta} + \varphi_{\beta\alpha} + \varphi_{\gamma\alpha})\} = \frac{1 + \xi_\alpha\cdot\xi_\beta + \xi_\beta\cdot\xi_\gamma + \xi_\gamma\cdot\xi_\alpha + i(\xi_\alpha\times\xi_\beta\cdot\xi_\gamma)}{[2(1+\xi_\alpha\cdot\xi_\beta)(1+\xi_\beta\cdot\xi_\gamma)(1+\xi_\gamma\cdot\xi_\alpha)]^{1/2}} \tag{22b}$$

and they are related to the relative phases $\varphi_\beta - \varphi_\alpha$ of the amplitudes by

$$\varphi_{\alpha\beta} = \varphi_\beta - \varphi_\alpha + \mathrm{Arg}\left[1 + \sqrt{\frac{(1-R_\alpha)(1-R_\beta)}{(1+R_\alpha)(1+R_\beta)}}\exp\{i(\chi_\beta - \chi_\alpha)\}\right] . \tag{23}$$

Let us show that by using the isospin conservation condition (3) one can determine the angled $\varphi_{\alpha\beta}$ and hence that *the phases between the amplitudes of different reactions are observable*. Eq. (7) can be written

$$-|\gamma\rangle = |\alpha\rangle + |\beta\rangle .$$

Multiplying this equation by its Hermitian conjugate we obtain, when* $\mathrm{tr}\, M_\alpha M_\beta \neq 0$:

$$M_\gamma = M_\alpha + M_\beta + (X_{\alpha\beta})^{-1/2} \times (M_\alpha M_\beta \exp(-i\varphi_{\alpha\beta}) + M_\beta M_\alpha \exp(-i\varphi_{\beta\alpha})) , \tag{24}$$

with

$$X_{\alpha\beta} = \mathrm{tr}(M_\alpha M_\beta) = \tfrac{1}{2} s_\alpha s_\beta (1 + \xi_\alpha\cdot\xi_\beta) , \tag{25}$$

Using the explicit form (3) of M_α, the trace of eq. (23) yields

$$-\cos\omega_{\alpha\beta} = \cos\tfrac{1}{2}\theta_{\alpha\beta}\cos\varphi_{\alpha\beta} \tag{26}$$

and the trace with $\boldsymbol{\sigma}$ gives the vector relation

$$s_\gamma\xi_\gamma = s_\alpha\xi_\alpha + s_\beta\xi_\beta + 2(s_\alpha s_\beta)^{1/2}\cos\varphi_{\alpha\beta}\,\hat{\boldsymbol{l}}_{\alpha\beta} + (H)^{1/2}\sin\varphi_{\alpha\beta}\,\hat{\boldsymbol{k}}_{\alpha\beta} \tag{27}$$

where H is defined in eq. (10) and

$$\hat{\boldsymbol{l}}_{\alpha\beta} = \frac{\xi_\alpha + \xi_\beta}{|\xi_\alpha + \xi_\beta|}, \quad \hat{\boldsymbol{k}}_{\alpha\beta} = \frac{\xi_\alpha \times \xi_\beta}{|\xi_\alpha \times \xi_\beta|} . \tag{28}$$

If the cross sections s_α, s_β, s_γ and the spin rotation vectors ξ_α, ξ_β are known, eq. (26) allows to determine $\cos\varphi_{\alpha\beta}$. If furthermore ξ_γ is known, eq. (27) yields the sign of $\sin\varphi_{\alpha\beta}$; indeed the scalar product of eq. (27) with $\hat{\boldsymbol{k}}_{\alpha\beta}$ gives

$$\mathrm{sign}(\sin\varphi_{\alpha\beta}) = \mathrm{sign}(\xi_\alpha \times \xi_\beta\cdot\xi_\gamma) . \tag{29}$$

Note that all solutions to the problems settled in section 3 can be obtained from eqs. (26) and (27). For instance, if one knows s_α, s_β, ξ_α and ξ_β, these equations show that the values of s_γ and ξ_γ depend only on one parameter which is the angle $\varphi_{\alpha\beta}$.

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* When $\mathrm{tr}\, M_\alpha M_\beta = |\langle\alpha|\beta\rangle|^2 = 0$ then, from eq. (7), $\langle\alpha|\gamma\rangle = -\langle\alpha|\alpha\rangle \neq 0$ and $\langle\beta|\gamma\rangle = -\langle\beta|\beta\rangle \neq 0$, so $\varphi_{\alpha\gamma} = \varphi_{\gamma\alpha} = \pi = \varphi_{\beta\gamma} = \varphi_{\gamma\beta}$. This exceptional case was already met in section 3 (iv). For the pairs β,γ or γ,α of indices, eq. (26) gives $s_\gamma = s_\alpha + s_\beta$ and eq. (27) gives the circle of solutions for ξ_γ. The angle $\varphi_{\alpha\beta}$ parametrizes this circle.

References

[1] L. Michel solved this problem for P alone: N. Cim. 22 (1961) 203 and "1965-Brandeis Summer Institute in Theoretical physics" (Gordon and Breach, New York, 1966) Vol. 1, p. 347;
R. J. N. Phillips, N. Cim. 26 (1962) 103, remarked that this work can be extended to A or R separately;
M. Korkea-Aho and N. Törnqvist, preprint Helsinki 1971, remarked that it is also valid for any linear combination $\alpha P + \beta A + \gamma R$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$.

[2] Some comparisons of results of ref. [1] with data have been made by O. Kamei and S. Sasaki, N. Cim. 59 (1969) 535; and by
G. V. Dass, J. Froyland, F. Halzen, A. Martin, C. Michael and S. M. Roy, Phys. Letters 36B (1971) 339.

[3] F. Halzen and C. Michael, Phys. Letters 36B (1971) 367, study a case of partial information on P and R only and make a comparison with data. Although we do not treat this case explicitly, it is implicitly contained in this letter. A complete study for arbitrary spin of the relations between internal symmetry and polarization will appear as a forthcoming issue of our work "Polarization density matrix".

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