ISOSPIN CONSTRAINTS BETWEEN THREE CROSS SECTIONS AND TWO POLARIZATION DENSITY MATRICES

M.G. DONCEL*, L. MICHEL** and P. MINNAERT***
CERN, Geneva, Switzerland

Received 17 August 1972

We give the relation which isospin conservation imposes between the three cross sections and two polarization density matrices, for three reactions related by two isospin channels. It is valid for arbitrary spin and for spin correlations. Its application to the most usual cases of spin 1 and $\frac{3}{2}$ is given.

Consider three reactions of the type

$$1_\alpha + 2_\alpha \to 3_\alpha + 4_\alpha + \ldots \qquad (\alpha = 1, 2, 3) \tag{1}$$

which have their corresponding particles (or resonances) in the same isospin multiplet, and which go through two isospin channels. For given energy momenta of the particles, each reaction is described by a transition matrix T_α from the initial to the final polarization spaces. Then, isospin conservation imposes a linear relation between the three T_α's

$$\sum_{\alpha=1}^{3} \gamma_\alpha T_\alpha = 0, \tag{2}$$

where the real numbers γ_α are simple combinations of Clebsch-Gordan coefficients.

Let us denote by σ_α the differential cross sections and by s_α the weighted cross sections

$$s_\alpha = \gamma_\alpha^2 \sigma_\alpha . \tag{3}$$

It is well known that eq. (2) imposes a "triangular relation" between the square roots of the weighted cross sections, which can be written

$$|s_3 - s_1 - s_2|(2\sqrt{s_1 s_2})^{-1} \leqslant 1. \tag{4}$$

In this letter we give the best relation imposed by isospin conservation on the three weighted cross sections s_1, s_2, s_3 and the two polarization density matrices ρ_1 and ρ_2 of the final state†. This relation is

$$|s_3 - s_1 - s_2|(2\sqrt{s_1 s_2})^{-1} \leqslant \mathrm{tr} \sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} . \tag{5}$$

Note that for positive, trace one matrices ρ_1 and ρ_2, one has

$$\mathrm{tr} \sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} = \mathrm{tr} \sqrt{\sqrt{\rho_2}\rho_1\sqrt{\rho_2}} \leqslant 1. \tag{6}$$

Eq. (5) can be applied in the following physical situations. The initial state is unpolarized or is equally polarized in the three reactions. The measured density matrices ρ_α may be:

i) The joint density matrix of all final particles;
ii) The joint density matrix of some of them;
iii) The density matrix of a single final particle, say 3_α;
iv) The even multipole part of this matrix (i.e., that part which is most usually measured when 3_α is a strongly decaying resonance).

When two cross sections and two polarizations are measured, eq. (5) yields the best bounds on the third cross section. When the three cross sections and one polarization are measured, eq. (5) defines the allowed domain for a second polarization.

First we sketch the proof of eq. (5). Then we give the explicit expression of its right hand side for the usually measured density matrices of spin 1 or $\frac{3}{2}$ resonances‡.

We denote by ρ_0 the polarization density matrix of the initial state. It is convenient to introduce the weighted transition and density matrices

* Departamento de Física Teórica, Universitad Autónoma de Barcelona, Sardanyola, Barcelona, Spain.
** Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette, France.
*** Laboratoire de Physique Théorique, Université de Bordeaux I, 33400 - Talence, France.

† There are no relations, besides eq. (4), between three cross sections and only one polarization density matrix.

‡ We do not consider here the case of spin $\frac{1}{2}$ particles since we have already treated it completely in refs. [1, 2].

$$M_\alpha = \gamma_\alpha T_\alpha \sqrt{\rho_0}, \qquad R_\alpha = s_\alpha \rho_\alpha, \tag{7}$$

so that one simply has

$$R_\alpha = M_\alpha M_\alpha^*, \qquad s_\alpha = \mathrm{tr}\, R_\alpha. \tag{8}$$

Then eq. (2) reads

$$\sum_{\alpha=1}^{3} M_\alpha = 0. \tag{9}$$

This equation can be written $-M_3 = M_1 + M_2$, multiplication on the right by the adjoint equation yields

$$R_3 - R_1 - R_2 = M_1 M_2^* + M_2 M_1^* \equiv 2\,\mathrm{Re}\,(M_2 M_1^*). \tag{10}$$

The trace of this expression and the use of Cauchy-Schwarz inequality yield eq. (4), but eq. (5) cannot be obtained by means of Cauchy-Schwarz inequality †. It is derived from the polar decomposition [e.g., 4] of the transition operator M:

$$M_\alpha = \sqrt{M_\alpha M_\alpha^*} \qquad U_\alpha = \sqrt{R_\alpha U_\alpha}, \tag{11}$$

where R_α is a positive operator acting on the final polarization space and U_α is a partially isometric operator from the initial to the final polarization spaces. Then eq. (10) reads

$$R_3 - R_1 - R_2 = 2\,\mathrm{Re}\,\sqrt{R_2}\, U_{21} \sqrt{R_1}, \tag{12}$$

where the unknown operator $U_{21} = U_2 U_1^*$ can be any operator of norm $\leqslant 1$. Taking the absolute value of the trace of eq. (12) one obtains

$$|s_3 - s_1 - s_2| (2\sqrt{s_1 s_2})^{-1} = |\mathrm{Re}\ \mathrm{tr} \sqrt{\rho_1}\sqrt{\rho_2}\, U_{21}|\,. \tag{13}$$

To compute the maximum of the right hand side for any possible U_{21} we use now the polar decomposition for the operator $\sqrt{\rho_1}\sqrt{\rho_2}$, i.e.,

$$\sqrt{\rho_1}\sqrt{\rho_2} = \sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\, V_{12} \tag{14}$$

and we get

$$|\mathrm{Re}\ \mathrm{tr} \sqrt{\rho_1}\sqrt{\rho_2}\, U_{12}| \leqslant \mathrm{Max}|\ \mathrm{Re}\ \mathrm{tr} \sqrt{\sqrt{\rho_1}\,\rho_2\sqrt{\rho_1}}\, V_{12} U_{21}$$
$$= \mathrm{tr} \sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\,. \tag{15}$$

Indeed the maximum is reached for $U_{21} = V_{12}^*$; therefore eq. (5) gives the best relation.

The proof given above is valid for the case i). To extend it for the cases ii) and iii) one may consider the non observed final particles as unpolarized initial particles. Joseph [5] proved the validity of eq. (5) for the case iv).

For 2 × 2 matrices the right hand side of eq. (5) reduces to

$$\mathrm{tr} \sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} = \sqrt{\mathrm{tr}\ \rho_1\rho_2 + 2\sqrt{\det \rho_1\ \det \rho_2}}. \tag{16}$$

This expression allows one to treat completely the usual cases of spin 1 and $\frac{3}{2}$ particles produced in "B – symmetric reactions", i.e., in parity conserving quasi two body reactions with unpolarized beam and target. Indeed for these cases, the density matrix, in transversity quantization, can be written as a direct sum of matrices of dimension 1 or 2. For simplicity we restrict ourselves to the usual situation where only the even part of the density matrix is measured (case iv).

a) *Spin 1, B-symmetric, even polarization.* We use the three orthonormal polarization parameters X, Y, Z which are related to the density matrix elements in transversity (T) and in helicity (H) quantization by

$$Z = 3\ {}^{\mathrm{T}}\rho_{11} - 1 = \tfrac{1}{2} - \tfrac{3}{2}({}^{\mathrm{H}}\rho_{11} + \mathrm{Re}\ {}^{\mathrm{H}}\rho_{1-1}) \tag{17a}$$

$$X = \sqrt{3}\ \mathrm{Re}\ {}^{\mathrm{T}}\rho_{1-1} = \tfrac{1}{2}\sqrt{3}(-3\ {}^{\mathrm{H}}\rho_{11} + \mathrm{Re}\ {}^{\mathrm{H}}\rho_{1-1} + 1) \tag{17b}$$

$$Y = -\sqrt{3}\ \mathrm{Im}\ {}^{\mathrm{T}}\rho_{1-1} = -\sqrt{6}\ \mathrm{Re}\ {}^{\mathrm{H}}\rho_{10}\,. \tag{17c}$$

The parameters must satisfy the positivity conditions

$$-1 \leqslant Z \leqslant \tfrac{1}{2}\,, \tag{18a}$$

$$C(X, Y, Z) \equiv (Z+1)^2 - 3(X^2+Y^2) \geqslant 0\,, \tag{18b}$$

which are the equations of a truncated cone [6]. In terms of these parameters for the matrices ρ_1 and ρ_2, eq. (5) can be written

† This was felt by Kamei and Sasaki [3] ; they have applied Cauchy-Schwarz inequality to each element of the matrix equation (10). Taking afterwards the trace, they obtained the relation $|s_3 - s_1 - s_2|(2\sqrt{s_1 s_2})^{-1} \leqslant \Sigma_i \sqrt{(\rho_1)_{ii}(\rho_2)_{ii}} \leqslant 1$, and they remarked "there might exist a covariant expression of [this relation] ; however we could not find it".

$$3|s_3-s_1-s_2|(2\sqrt{s_1s_2})^{-1} \leqslant \sqrt{1-2Z_1}\sqrt{1-2Z_2} + $$
$$+\sqrt{2}\{(1+Z_1)(1+Z_2)+3(X_1X_2+Y_1Y_2) + $$
$$+\sqrt{C(X_1,Y_1,Z_1)C(X_2,Y_2,Z_2)}\}^{1/2}.$$

b) *Spin $\frac{3}{2}$, B-symmetric, even polarization.* The orthonormal parameters X, Y, Z that we use now are related to the density matrix elements by

$$Z = \tfrac{4}{3}\sqrt{3}({}^{\mathrm{T}}\rho_{33}-\tfrac{1}{4}) = -\tfrac{2}{3}\sqrt{3}({}^{\mathrm{H}}\rho_{33}-\tfrac{1}{4}) - 2\,\mathrm{Re}\,{}^{\mathrm{H}}\rho_{3-1} \quad (20a)$$

$$X = \tfrac{4}{3}\sqrt{3}\,\mathrm{Re}\,{}^{\mathrm{T}}\rho_{3-1} = -2({}^{\mathrm{H}}\rho_{33}-\tfrac{1}{4}) + \tfrac{2}{3}\sqrt{3}\,\mathrm{Re}\,{}^{\mathrm{H}}\rho_{3-1} \quad (20b)$$

$$Y = -\tfrac{4}{3}\sqrt{3}\,\mathrm{Im}\,{}^{\mathrm{T}}\rho_{3-1} = -\tfrac{4}{3}\sqrt{3}\,\mathrm{Re}\,{}^{\mathrm{H}}\rho_{31}\,. \quad (20c)$$

They satisfy the positivity condition

$$S(X,Y,Z) \equiv 1-3(X^2+Y^2+Z^2) \geqslant 0\,, \quad (21)$$

which is the equation of a sphere [6]. Eq. (5) becomes in this case

$$|s_3-s_1-s_2|(2\sqrt{s_1s_2})^{-1} \leqslant$$
$$\sqrt{\tfrac{1}{2}}\{1+3(X_1X_2+Y_1Y_2+Z_1Z_2) + $$
$$+\sqrt{S(X_1,Y_1,Z_1)S(X_2,Y_2,Z_2)}\}^{1/2}. \quad (22)$$

If we look at eqs. (19) and (22) as constraints on ρ_2 (i.e., on X_2, Y_2, Z_2) when s_1, s_2, s_3 and ρ_1 are known, the allowed domain for ρ_2 is a pearl shaped convex domain which grows around the point ρ_1 when $|s_3-s_1-s_2|(2\sqrt{s_1s_2})^{-1}$ decreases, and is contained in the truncated positivity cone for case a) or in the positivity sphere for case b). A more detailed study of this pearl shaped domain and of the relation between three cross sections and three polarizations will be carried on in a forthcoming publication.

The authors are grateful to the Theory Division and to the Track Chamber Division of CERN for their hospitality. M.G. Doncel thanks the "Grupo Interuniversitario de Física Teórica" and P. Minnaert thanks the "Commission des Grands Accélérateurs" for their financial help.

References

[1] L. Michel, Nuovo Cim. 22 (1961) 203, and 1965–Brandeis Summer Institute in Theoretical physics (Gordon and Breach, New York, 1966) Vol. I, p. 347.
[2] M.G. Doncel, L. Michel and P. Minnaert, Phys. Lett. 38B (1972) 42.
[3] O. Kamei and S. Sasaki, Nuovo Cim. 59 (1969) 535.
[4] J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien (Gauthiers-Villars, Paris, 1957). Specially app. III.
[5] A. Joseph, private communication.
[6] M.G. Doncel, L. Michel and P. Minnaert, Matrices densité de polarisation, Ecole d'été de Gif-sur-Yvette, 1970, ed. R. Salmeron.