PHYSICAL REVIEW D VOLUME 49, NUMBER 3 1 FEBRUARY 1994

Quark mass hierarchies from the universal seesaw mechanism

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(Received 12 April 1993)

The paper is an extension of the previous work based on the idea of a universal seesaw mechanism to explain the hierarchies in the fermion mass spectrum. A model is proposed within the framework of left-right symmetry with a minimal Higgs system and an axial U(1) symmetry imposed to distinguish the generations. Previous work was confined, for mathematical simplifications, to the case of nonsingular mass matrices. In the present paper, singular matrices are considered. A systematic perturbative technique is developed to display the mass eigenvalues in terms of the vacuum expectation values of the assumed Higgs multiplets. The model successfully correlates the mass hierarchies among the quarks to the assumed hierarchies in the vacuum expectation values without appealing to a hierarchy in the Yukawa-type fermion–Higgs-boson couplings. By considering a general Higgs potential appropriate to the model, we study its minimization and prove that there exists an open subdomain in the parameter space where the orbit of the lowest minima of the potential corresponds to the kind of hierarchy in the vacuum expectation values needed for the success of the model.

PACS number(s): 12.15.Ff, 12.10.Dm

I. INTRODUCTION

The idea of a universal seesaw mechanism [1] has several attractive features that include (1) the generalization of the mechanism to explain the superlightness of the neutrino masses to all fermions, (2) a generalization of the left-right symmetric framework with a minimal Higgs system, (3) a correlation between the smallness of the charged lepton masses and superlightness of the neutrino masses, and (4) the possibility of incorporating the assumed heavy left-right singlet fermions in the framework of a grand unified theory [2]. Indeed, if nature were kind and simple and had we a single generation of quarks and leptons, the universal seesaw mechanism would have provided probably the best explanation for the observed mass hierarchies within the framework of the left-right symmetric standard model.

As in all models, the existence of more than one generation and the bewildering mass hierarchies within and between generations pose difficult problems, and in spite of wide ranging ideas in recent attempts, no convincing explanation has emerged. It is universally recognized that an appeal to hierarchies in the Yukawa-type fermion–Higgs-boson couplings [3] is not a satisfactory way out. On any type of symmetry principles, one expects them to be of similar orders of magnitude and hence incapable of explaining the five or so orders of magnitude differences in the masses of the fundamental fermions. We would like to attribute this phenomenon to the breaking of left-right symmetry leading to parity violation. The generalization of the standard model then provides a natural hierarchy in the vacuum expectation values of the left and right Higgs doublets, which is necessary to account for the known experimental limits on the associated phenomena. By introducing global axial U(1) symmetry quantum numbers to distinguish the generations, we allow ourselves the possibility of also linking two problems, namely, the problem of the strong *CP* violation and the structure of fermion mass matrices, which play such an important role in determining the Cabibbo-Kobayashi-Maskawa angles and the physics of the new flavors.

The present paper is a continuation of the work in which we treated the case of the three generations assuming a nonsingular structure for the submatrix involving vacuum expectation values of the heavy singlets [4]. This was done so that we could then reduce the 6×6 mass matrices to 3×3, express elements of the mass matrix in terms of the masses themselves, and thereby predict the mixing angles. The results, although satisfactory in some respects, showed that the resulting Yukawa-type couplings had a high degree of hierarchy. Our treatment of two generations [1] showed that the singular case avoids this problem and provides a natural hierarchy only in terms of a hierarchy in the vacuum expectation values. We generalize these considerations to the realistic case of three generations by treating the full 6×6 mass matrices in the case of quarks.

The paper is organized as follows. In the next section, we review, for the sake of convenience, our assumptions and the general structure of the mass matrices in our model. In Sec. III we discuss the perturbative technique we employ to derive the eigenvalues of these matrices and express their dependence on the vacuum expectation values analytically to any desired degree of accuracy. In this way the inherent hierarchies are manifest. Section IV is devoted to a discussion of the numerical results, a summary, and conclusions. In the Appendix we discuss the minimization of the relevant Higgs potential and justify our assumption regarding the hierarchy in the vacuum expectation values of the doublet and singlet Higgs bosons.

II. BRIEF REVIEW OF THE MODEL

Working within the framework of left-right symmetry, we shall consider three generations of conventional fermions $f^i_{L,R}$ accompanied by $SU(2)_L\otimes SU(2)_R$ singlet partners $F^i_{L,R}$, $i=1,2,3$. We assume a minimal Higgs system consisting of a left doublet φ_L, a right doublet φ_R, and a left-right singlet σ. At the classical level, we postulate that the Lagrangian containing the above fermions and the Higgs system is invariant, in addition to the usual gauge transformations, under a global, axial, $U(1)_A$ symmetry. Under $U(1)_A$, let the fermions transform as

$$f^i_{L,R}\to e^{\pm i\theta x_i}f^i_{L,R}\,,\quad F^i_{L,R}\to e^{\pm i\theta y_i}F^i_{L,R}\ , \tag{2.1}$$

and the Higgs scalars as

$$\varphi_L\to e^{\pm i\theta h_L}\varphi_L\,,\quad \varphi_R\to e^{\pm\theta h_R}\varphi_R\ , \tag{2.2}$$

where θ is a continuous parameter and x_i, y_i, h_L, and h_R are as yet unspecified $U(1)_A$ charges. As discussed in the previous section, we would like to identify $U(1)_A$ as the Peccei-Quinn (PQ) symmetry and, at the same time, use the associated quantum numbers to characterize and distinguish the generations. Therefore we impose the constraints on the PQ quantum numbers x_i,y_i:

$$x_i\neq x_j\,,\quad y_i\neq y_j\,,\quad \text{if } i\neq j\ . \tag{2.3}$$

Because of the $U(1)_A$ symmetry, a given scalar field φ and its charge conjugate $\tilde\varphi=i\tau_2\varphi^*$ are distinguished since they have opposite $U(1)_A$ charges. When the symmetry is broken spontaneously due to φ acquiring a nonvanishing vacuum expectation value, φ and $\tilde\varphi$ generate fermion masses in different charge sectors. If φ is chosen to be the conventional doublet

$$\begin{bmatrix}\varphi^\dagger\\ \varphi^0\end{bmatrix}\,,$$

it contributes to the down-charge sector and $\tilde\varphi$ contributes to the up-charge sector. Since φ and $\tilde\varphi$ have distinct PQ quantum numbers, we are led in our model to have a minimum of four Higgs doublets,

$$\varphi_L^{h_1}(1,2,1)_1,\quad \varphi_L^{h_2}(1,2,1)_1\ ,$$
$$\varphi_R^{-h_1}(1,1,2)_1,\quad \varphi_R^{-h_2}(1,1,2)_1\ ,$$

where h_i represent the $U(1)_A$ charges that can be normalized to ± 1,

$$h_1=\pm 1,\quad h_2=\mp 1\ . \tag{2.4}$$

Therefore, under $U(1)_A$, the φ's transform according to

$$\varphi_L^{1,-1}\to e^{\pm i\theta}\varphi_L^{1,-1}\,,\quad \varphi_R^{1,-1}\to e^{\pm i\theta}\varphi_R^{1,-1}\ . \tag{2.5}$$

The charge-conjugated fields $\tilde\varphi$ are defined by

$$\tilde\varphi^{+1}=i\tau_2(\varphi^{-1})^*,\quad \tilde\varphi^{-1}=i\tau_2(\varphi^{+1})^*\ . \tag{2.6}$$

Tables I and II describe, respectively, the particle content in our model and the mass-generating, nonvanishing fermion–Higgs-boson couplings.

As discussed in detail in Ref. [4], the quark mass matrices assume the form

TABLE I. Particle content and the quantum number of the particles.

		$SU(3)_c$	$SU(2)_L$	$SU(2)_R$	$U(1)_{B\text{-}L}$	$U(1)_A$
Conventional quarks	$\begin{bmatrix}u\\ d\end{bmatrix}^i_L$	3	2	1	$\frac{1}{3}$	x_i
	$\begin{bmatrix}\bar u\\ \bar d\end{bmatrix}^i_L$	3*	1	2	$-\frac{1}{3}$	x_i
Heavy	U^i_L	3	1	1	$\frac{4}{3}$	y_i
"singlet"	D^i_L	3	1	1	$-\frac{2}{3}$	y_i
quarks	$\bar U^i_L$	3*	1	1	$-\frac{4}{3}$	y_i
	$\bar D^i_L$	3*	1	1	$\frac{2}{3}$	y_i
Higgs	$\varphi_L^{1,2}$	1	2	1	-1	$+1$
bosons	$\varphi_R^{1,2}$	1	1	2	-1	$+1$
	σ	1	1	1	0	0

TABLE II. Nonvanishing fermion–Higgs-boson couplings. K_{ij} represent the sums of the $U(1)_A$ charges of the fermions.

K_{ij}	Mass-generating quark–Higgs-boson couplings	
$+1$	$\bar{q}^i_L \varphi_L^{+1} D^j_R$	$\bar{D}^j_L \varphi_R^{+1} q^i_R$
	$\bar{q}^i_L \tilde{\varphi}_L^{+1} U^j_R$	$\bar{U}^j_L \tilde{\varphi}_R^{+1} q^i_R$
-1	$\bar{q}^i_L \varphi_L^{-1} D^j_R$	$\bar{D}^j_L \tilde{\varphi}_R^{-1} q^i_R$
	$\bar{q}^i_L \tilde{\varphi}_L^{-1} U^j_R$	$\bar{U}^j_L \tilde{\varphi}_R^{-1} q^i_R$

$$M=\begin{bmatrix} 0 & M_L \\ M_R & M_\chi \end{bmatrix}, \tag{2.7}$$

where M_L, M_R, and M_χ are in general 3×3 complex matrices which will contain the vacuum expectation values (VEV's)

$$\begin{aligned} &\langle \varphi_L^1 \rangle = v_L^1, \quad \langle \varphi_L^2 \rangle \equiv \langle \varphi_L^{-1} \rangle = v_L^2 \ , \\ &\langle \varphi_R^1 \rangle = v_R^1, \quad \langle \varphi_R^2 \rangle \equiv \langle \varphi_R^{-1} \rangle = v_R^2 \ , \end{aligned} \tag{2.8}$$

and M_χ will contain the VEV χ of σ. The $U(1)_A$ quantum numbers of σ will depend on how it is coupled to the φ's and will be discussed shortly. A closer analysis of the nonvanishing couplings that enter the mass matrices subject to the constraints (2.3) shows that the matrices M_L and M_R in both the charge sectors (*down sector* consisting of d,s,b quarks and the *up sector* containing u,c,t quarks) are of the Fritzsch type, namely, in the down sector,

$$\begin{aligned} M_L^{(d)} &= \begin{bmatrix} 0 & \alpha_{12}v_L^1 & 0 \\ \alpha_{12}v_L^1 & 0 & \alpha_{23}v_L^2 \\ 0 & \alpha_{23}v_L^2 & \alpha_{33}v_L^1 \end{bmatrix}, \\ M_R^{(d)} &= \begin{bmatrix} 0 & \alpha_{12}v_R^1 & 0 \\ \alpha_{12}v_R^1 & 0 & \alpha_{23}v_R^2 \\ 0 & \alpha_{23}v_R^2 & \alpha_{33}v_R^1 \end{bmatrix}, \end{aligned} \tag{2.9}$$

and, in the up sector,

$$\begin{aligned} M_L^{(u)} &= \begin{bmatrix} 0 & \beta_{12}v_L^{2^*} & 0 \\ \beta_{12}v_L^{2^*} & 0 & \beta_{23}v_L^{1^*} \\ 0 & \beta_{23}v_L^{1^*} & \beta_{33}v_L^{2^*} \end{bmatrix}, \\ M_R^{(u)} &= \begin{bmatrix} 0 & \beta_{12}v_R^{2^*} & 0 \\ \beta_{12}v_R^{2*} & 0 & \beta_{23}v^{1^*} \\ 0 & \beta_{23}v_R^{1^*} & \beta_{33}v_R^{2^*} \end{bmatrix}. \end{aligned} \tag{2.10}$$

In order to reduce the number of parameters in the model, we have assumed, in addition to left-right symmetry, that the Y-type couplings are symmetric. Since the VEV's are arbitrary at the moment, if we write

$$\alpha_{12}v_L^1 = a, \quad \alpha_{23}v_L^2 = A \ , \tag{2.11}$$

and, likewise,

$$\alpha_{12}v_R^1 = b, \quad \alpha_{23}v_R^2 = B \ , \tag{2.12}$$

the 33 matrix elements in $M_L^{(d)}$ and $M_R^{(d)}$, respectively, are $(\alpha_{33}/\alpha_{12})a$ and $(\alpha_{33}/\alpha_{12})b$. Let

$$\left[\frac{\alpha_{33}}{\alpha_{12}}\right] \begin{bmatrix} a \\ b \\ A \\ B \end{bmatrix} = \begin{bmatrix} a' \\ b' \\ A' \\ B' \end{bmatrix}. \tag{2.13}$$

With this simplified notation, the matrices in (2.9) take the form

$$M_L^{(d)} = \begin{bmatrix} 0 & a & 0 \\ a & 0 & A \\ 0 & A & a' \end{bmatrix}, \quad M_R^{(d)} = \begin{bmatrix} 0 & b & 0 \\ b & 0 & B \\ 0 & B & b' \end{bmatrix}. \tag{2.14}$$

As discussed in Ref. [4], the form of M_χ depends on how we choose to couple σ to the Higgs doublets or on the choice of the $U(1)_A$ charge of σ. With a singular M_χ in mind, we have the following choices.

(a) $U(1)_A$ of σ is zero. There are three possible forms depending on the choice of x, $x=2$, 4, or 5:

(i) $x=2$, $$M_\chi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \chi \\ 0 & \chi & 0 \end{bmatrix},$$

(ii) $x=4$, $$M_\chi = \begin{bmatrix} 0 & 0 & \chi \\ 0 & 0 & 0 \\ \chi & 0 & 0 \end{bmatrix},$$

(iii) $x=5$, $$M_\chi = \begin{bmatrix} \chi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) $U(1)_A$ of σ is ± 2. There are two possible cases corresponding to $x=3$ or 5:

(iv) $x=3$, $$\begin{bmatrix} 0 & 0 & \chi \\ 0 & 0 & \chi \\ \chi & \chi & 0 \end{bmatrix},$$

(v) $x=5$, $$\begin{bmatrix} 0 & 0 & \chi \\ 0 & 0 & 0 \\ \chi & 0 & 0 \end{bmatrix}.$$

Again we have absorbed the Y-type coupling in the definition of the VEV of σ and assumed only one χ representing the heavy, singlet fermion mass scale.

In case (a), where the $U(1)_A$ quantum number of σ is zero, while σ breaks L-R symmetry and brings about parity violation, it does not correspond to the singlet that makes an "invisible" axion [5]. In case (b), σ can be the singlet subject to the constraint on $\langle \sigma \rangle \simeq \chi$ arising from cosmological considerations. For determining the eigenvalues of mass matrices, however, these considerations are not relevant. We have four cases to consider in which M_χ is singular. In each case we calculate $MM^\dagger$ and, using the perturbative technique to be described in the next section, compute analytical expressions for the (mass)^2 eigenvalues to the desired order.

Assuming that $\langle\sigma\rangle \simeq \chi$ has the highest value, we divide each matrix under consideration by χ. The mass matrices $MM^\dagger$ have the block form

$$\begin{bmatrix} M_L M_L^T & M_L M_\chi^T \\ M_\chi M_L^T & M_R M_R^T + M_\chi M_\chi^T \end{bmatrix} , \tag{2.15}$$

where each element is a 3×3 matrix. Denoting

$$\frac{a}{\chi}=l,\quad \frac{A}{\chi}=L,\quad \frac{b}{\chi}=r,\quad \frac{B}{\chi}=R ,$$
$$\frac{a'}{\chi}=l',\quad \frac{A'}{\chi}=L',\quad \frac{b'}{\chi}=r,\quad \frac{B'}{\chi}=R' , \tag{2.16}$$

the 6×6 matrices in the down sector that we want to consider are as follows.

(1) $x=2$, $\mathrm{U}(1)_A$ charge of $\sigma=0$:

$$M_{(23)}=\begin{bmatrix} l^2 & 0 & lL & 0 & 0 & l \\ 0 & l^2+L^2 & Ll' & 0 & L & 0 \\ Ll & l'L & L^2+l'^2 & 0 & l' & L \\ 0 & 0 & 0 & r^2 & 0 & rR \\ 0 & L & l' & 0 & r^2+R^2+1 & Rr' \\ l & 0 & L & Rr & r'R & R^2+r'^2+1 \end{bmatrix} . \tag{2.17}$$

(2) $x=4$, $\mathrm{U}(1)_A$ charge of $\sigma=0$ or ±2:

$$M_{(13)}=\begin{bmatrix} l^2 & 0 & lL & 0 & 0 & 0 \\ 0 & l^2+L^2 & Ll' & L & 0 & l \\ Ll & l'L & L^2+l'^2 & l' & 0 & 0 \\ 0 & L & l' & r^2+1 & 0 & rR \\ 0 & 0 & 0 & 0 & r^2+R^2 & Rr' \\ 0 & l & 0 & Rr & r'R & R^2+r'^2+1 \end{bmatrix} . \tag{2.18}$$

(3) $x=5$, $\mathrm{U}(1)_A$ charge of $\sigma=0$:

$$M_{(11)}=\begin{bmatrix} l^2 & 0 & lL & 0 & 0 & 0 \\ 0 & l^2+L^2 & Ll' & l & 0 & 0 \\ Ll & l'L & L^2+l'^2 & 0 & 0 & 0 \\ 0 & l & 0 & r^2+1 & 0 & rR \\ 0 & 0 & 0 & 0 & r^2+R^2 & Rr' \\ 0 & 0 & 0 & Rr & r'R & R^2+r'^2 \end{bmatrix} . \tag{2.19}$$

(4) $x=3$, $\mathrm{U}(1)_A$ charge of $\sigma=\pm2$:

$$M_{(13,23)}=\begin{bmatrix} l^2 & 0 & lL & 0 & 0 & l \\ 0 & l^2+L^2 & Ll' & L & L & l \\ Ll & l'L & L^2+l'^2 & l' & l' & L \\ 0 & L & l' & r^2+1 & 1 & rR \\ 0 & L & l' & 1 & r^2+R^2+1 & Rr' \\ l & l & L & Rr & r'R & R^2+r'+2 \end{bmatrix} . \tag{2.20}$$

In Eqs. (2.17)–(2.20) the matrix elements are dimensionless quantities. $M_{(ij)}$ denotes the matrix derived from the form of the matrix χ in which $\chi_{ij}\neq0$. The (mass)^2 eigenvalues of $MM^\dagger$ are then given by $\chi^2\times$(eigenvalues of $MM^\dagger$).

III. van VLECK PERTURBATION TECHNIQUE TO DETERMINE THE EIGENVALUES

We adopt the van Vleck procedure [6] for carrying out a degenerate perturbation theory calculation to determine the eigenvalues of the mass matrices (2.17)–(2.20) described in the previous section. We note that each matrix element outside the lower right-hand block is by assumption small compared to unity, and we intend to calculate the eigenvalues to *fourth order* in these values.

Let the generic 6×6 matrix M be decomposed as follows:

$$M=M_0+M_1+M_2 , \tag{3.1}$$

where M_0 is block diagonal containing all the zero eigenvalues in its top block. M_1 will contain all the first-order elements in l, L, l', L', r, R, r', and R' and is block off diagonal. M_2 will contain all the second-order elements and has a mixed form. We shall specify this decomposition in each of the four cases separately, but it turns out that, in all the cases, M_0 and M_1 can be written in the form

$$M_0 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & A_1 \\ A_1^\dagger & 0 \end{bmatrix}, \tag{3.2}$$

where $\mathbf{0}$, $\mathbf{1}$, and A_1 are block matrices of appropriately chosen dimensions.

The van Vleck perturbation procedure consists of bringing the off-diagonal higher-order terms into the lower-order diagonal blocks by successive unitary transformations.

A. First van Vleck transformation

Let U_1 be a unitary transformation given by

$$U_1 = \exp(iS_1) , \tag{3.3}$$

where S_1 is Hermitian. Then the transformed matrix $M^{(1)}$ is given by

$$M^{(1)} = e^{-iS_1} M e^{iS_1} = \sum_{n=0}^{\infty} \frac{i^n}{n!} [M,S_1]_n , \tag{3.4}$$

where

$$[M,S_1]_0 = M, \quad [M,S_1]_n = [[M,S_1]_{n-1},S_1] . \tag{3.5}$$

Now let S_1 be so chosen that

$$M_1 + i[M_0,S_1] = 0 . \tag{3.6}$$

From (3.2), it is easily verified that the desired S_1 is given by

$$S_1 = \frac{i}{\alpha} \begin{bmatrix} 0 & -A_1 \\ A_1^\dagger & 0 \end{bmatrix} , \tag{3.7}$$

which implies

$$[M_0,S_1] = iM_1 . \tag{3.8}$$

Using (3.6) and (3.8), we can write

$$M^{(1)} = M_0 + \left[M_2 + \frac{i}{2}[M_1,S_1] \right] + \left[i[M_2,S_1] - \tfrac{1}{3}[M_1,S_1]_2 \right] + \cdots + \left[\frac{i^n}{n!}[M_2,S_1]_n + \frac{i^{n+1}}{(n+2)!}(n+1)[M_1,S_1]_{n+1} \right] + \cdots . \tag{3.9}$$

From the structure of (3.9), it is clear that the first two terms contain all the second-order corrections to the block diagonal part of M and the transformed matrix $M^{(1)}$ now has only second- and higher-order terms in its off-diagonal blocks.

B. Second van Vleck transformation

In view of the above noted fact, we can rewrite $M^{(1)}$ in the form

$$M^{(1)} = M_0 + M_2^{(1)} + M_3^{(1)} + \cdots , \tag{3.10}$$

where we have used the notation $M_n^{(k)}$ to denote nth-order terms in M found after the kth unitary transformation U_k on M.

We can then find another unitary transformation,

$$U_2 = \exp(iS_2) , \tag{3.11}$$

such that

$$M_{2,\mathrm{OD}}^{(1)} + i[M_0,S_2] = 0 , \tag{3.12}$$

where

$$M_2^{(1)} = M_{2,D}^{(1)} + M_{2,\mathrm{OD}}^{(1)} ,$$

with $M_{2,D}^{(1)}$ the block diagonal and $M_{2,\mathrm{OD}}^{(1)}$ the off-diagonal parts of $M_2^{(1)}$:

$$M_{2,D}^{(1)} = \begin{bmatrix} B_2^{(1)} & 0 \\ 0 & C_2^{(1)} \end{bmatrix},$$

$$M_{2,\mathrm{OD}}^{(1)} = \begin{bmatrix} 0 & A_2^{(1)} \\ A_2^{(1)} & 0 \end{bmatrix} .$$

Then

$$S_2 = \frac{i}{\alpha} \begin{bmatrix} 0 & -A_2^{(1)} \\ A_2^{(1)\dagger} & 0 \end{bmatrix} . \tag{3.13}$$

Using this, we find

$$M^{(2)} = M_0 + M_{2,D}^{(1)} + M_3^{(1)} + \left[M_4^{(1)} + \frac{i}{2}[M_{2,\mathrm{OD}}^{(1)},S_2] + i[M_{2,D}^{(1)},S_2] \right] + \cdots . \tag{3.14}$$

Proceeding in this way, we can consistently bring all the terms up to any desired order in the block diagonal elements. We shall carry out the calculations for our model to fourth order. As a check of the calculations, we compute three invariants: Tr(M), Tr(M^2), and Det(M) of the final matrix and make sure that, to the desired order, they agree with the required quantities from the given matrix. In what follows we shall discuss the details of the calculation in one case, namely, the case in (2.17), and sketch the results in the other three cases.

(1) For $M_{(23)}$ in (2.17), the following decomposition into 4×4 and 2×2 block diagonal forms for M_0 and M_2 and 4×2 and 2×4 nonvanishing block forms for M_1 suggests, naturally,

$$M_0=\begin{bmatrix}0 & \\ & 1\end{bmatrix}, \quad \begin{matrix}0 \text{ is } 4\times4 \text{ null matrix,}\\ 1 \text{ is } 2\times2 \text{ identity matrix ,}\end{matrix} \tag{3.15a}$$

$$M_1=\begin{bmatrix}0 & A_1\\ A_1^\dagger & 0\end{bmatrix}, \tag{3.15b}$$

where

$$A_1=\begin{bmatrix}0 & l\\ L & 0\\ l' & L\\ 0 & rR\end{bmatrix},$$

$$M_2=\begin{bmatrix}B_2 & 0\\ 0 & C_2\end{bmatrix}, \tag{3.15c}$$

where

$$B_2=\begin{bmatrix}l^2 & 0 & lL & 0\\ 0 & l^2+L^2 & Ll' & 0\\ lL & Ll' & L^2+l'^2 & 0\\ 0 & 0 & 0 & r^2\end{bmatrix},$$

and

$$C_2=\begin{bmatrix}r^2+R^2 & r'R\\ r'R & r'^2+R^2\end{bmatrix}.$$

Two van Vleck transformations are carried out on $M_{(23)}$. The first, Eq. (3.9), yields second-, third, and fourth-order terms in $M^{(1)}$. The off-diagonal part of $M_2^{(1)}$,

$$A_2^{(1)}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & rR\end{bmatrix},$$

is then used to generate the second transformation, Eq. (3.14).

The results after this second transformation for the 4×4 block are

$$B_2^{(1)}=\begin{bmatrix}0 & 0 & 0 & 0\\ & l^2 & 0 & 0\\ & & 0 & 0\\ & & & r^2\end{bmatrix},$$

$$B_3^{(1)}=\begin{bmatrix}0 & 0 & 0 & -lrR\\ & 0 & 0 & 0\\ & & 0 & -LrR\\ & & & 0\end{bmatrix},$$

$$B_4^{(2)}=\begin{bmatrix}l^2(R^2+r'^2) & lLRr' & l[L(R^2+r'^2)+l'Rr'] & ll'Rr'\\ & -L^2(l^2-r^2-R^2) & L[l'(-\frac{1}{2}l^2+r^2+R^2)+LRr'] & 0\\ & & L[(R^2+r'^2)+l'^2(r^2+R^2)+2Ll'Rr'] & 0\\ & & & -r^2R^2\end{bmatrix}.$$

The eigenvalues of the B matrix are found using perturbation theory. The smallest two eigenvalues, corresponding to the 0 diagonal entries in $B_2^{(1)}$, must be found by diagonalizing a 2×2 matrix. The larger two can be found directly since they are not degenerate to second order. The results are

$$\lambda_{1,2}=\frac{(Lr+l'R)^2}{2}(1+\alpha+\beta)\left\{1\mp\left[1-\frac{4\alpha\beta}{(1+\alpha+\beta)^2}\right]^{1/2}\right\},$$

$$\lambda_3=l^2+\text{fourth-order terms}, \quad \lambda_4=r^2+\text{fourth-order terms},$$

where

$$\alpha=\frac{l^2r'^2}{(Lr'+l'R)^2}, \quad \beta=\frac{l'^2r^2}{(Lr'+l'R)^2}.$$

The eigenvalues of the C matrix are

$$\lambda_{5,6}=1+L^2+R^2+\frac{l^2+l'^2+r^2+r'^2}{2}\mp\left[\left[\frac{l^2-l'^2-r^2+r'^2}{2}\right]^2+(Ll'+Rr')^2\right]^{1/2}, \tag{3.16}$$

where these results are only reported to second order. The full fourth-order results were used to check that

$$\mathrm{Tr}(M)=\sum_{i=1}^{6}\lambda_i \quad (\text{to fourth order}), \quad \mathrm{Tr}(M^2)=\sum_{i=1}^{6}\lambda_i^2 \quad (\text{to fourth order}) ,$$

and

$$\det M=\prod_{i=1}^{6}\lambda_i \quad (\text{to 12th order}) .$$

(2) For $M_{(13)}$ in (2.18), a similar decomposition into a 4×4 and a 2×2 block applies. The results for the lowest four eigenvalues are

$$\begin{aligned}\lambda_{1,2}&=\frac{(lR+Lr)^2}{2}(1+\alpha'+\beta')\left\{1\mp\left[1-\frac{4\alpha'\beta'}{(1+\alpha'-\beta')^2}\right]^{1/2}\right\} ,\\ \lambda_3&=l^2+L^2,\quad \lambda_4=r^2+R^2 ,\end{aligned} \tag{3.17}$$

where

$$\alpha'=\frac{l^2l'^2r^2}{(l^2+L^2)(lR+Lr)^2},\quad \beta'=\frac{l^2r^2r'^2}{(r^2+R^2)(lR+Lr)^2} ,$$

The largest two eigenvalues, to second order, are

$$\lambda_{5,6}=1+\frac{l^2+L^2+l'^2+r^2+R^2+r'^2}{2}\mp\left[\left[\frac{l^2-L^2-l'^2-r^2+R^2+r'^2}{2}\right]^2+(lL+rR)^2\right]^{1/2} . \tag{3.18}$$

Once again, using the full fourth-order results, the eigenvalues reproduce the values of three invariants.

(3) For $M_{(11)}$ in (2.19), the decomposition yields a 5×5 and a 1×1 block. After the first transformation, the 5×5 block still has one zero eigenvalue. The results to lowest nonvanishing order are

$$\begin{aligned}\lambda_1&=\frac{l^4l'^2r^4r'^2}{(l^2L^2+l^2l'^2+L^4)(r^2R^2+r^2r'^2+R^4)} ,\\ \lambda_{2,3}&=\frac{l^2+2L^2+l'^2}{2}\mp\left[\left[\frac{l^2-l'^2}{2}\right]^2+L^2l'^2\right]^{1/2} ,\\ \lambda_{4,5}&=\frac{r^2+2R^2+r'^2}{2}\mp\left[\left[\frac{r^2-r'^2}{2}\right]^2+R^2r'^2\right]^{1/2} ,\\ \lambda_6&=1 .\end{aligned} \tag{3.19}$$

The eigenvalues through fourth order check with the three invariants.

(4) For $M_{(13,23)}$ in (2.20), we must perform a simple 2×2 rotation before starting the van Vleck procedure:

$$M_0=\begin{bmatrix}0&0&0&0&0&0\\ &0&0&0&0&0\\ &&0&1&1&0\\ &&&&1&0\\ &&&&&2\end{bmatrix}$$

can be transformed into

$$\begin{bmatrix}0&0&0&0&0&0\\ &0&0&0&0&0\\ &&0&0&0&0\\ &&&0&0&0\\ &&&&2&0\\ &&&&&2\end{bmatrix} .$$

Using the basis that diagonlizes M_0 allows the decomposition into a 4×4 and a 2×2 block.

The eigenvalues can be found as before. We shall not write out the values of λ_1 and λ_2 since the expression is very lengthy and the value of λ_2 allows us to rule out this case:

$$\lambda_3=l^2+\frac{L^2}{2},\quad \lambda_4=r^2+\frac{R^2}{2},\quad \lambda_{5,6}=2 . \tag{3.20}$$

Once again, the invariants check with the full fourth-order results.

IV. DISCUSSION OF NUMERICAL RESULTS AND CONCLUSIONS

As stated in the Introduction, we would like to relate the mass hierarchies ranging over three to five orders of magnitudes in the two sectors primarily to the hierarchies in the VEV's of the Higgs multiplets and not to the hierarchies in the Y couplings of Higgs bosons to fermions. The reason is simple. Whereas spontaneous symmetry breaking of a Higgs potential does not forbid wide-ranging VEV's, the desired hierarchies in the couplings are generally not possible in any theoretical schemes known so far. Therefore we would like to seek solutions in which all the Y couplings are approximately of the same order of magnitude.

With this in mind, we recall the definitions of l,L,r,R in the down and up sectors of the quark mass matrices.

Down sector:

$$\begin{aligned} l&=\frac{a}{\chi}=\alpha_{12}\frac{v_L^{(1)}}{\chi}, \quad L=\frac{A}{\chi}=\alpha_{23}\frac{v_L^{(2)}}{\chi} , \\ r&=\frac{b}{\chi}=\alpha_{12}\frac{v_R^{(1)}}{\chi}, \quad R=\frac{B}{\chi}=\alpha_{23}\frac{v_R^{(2)}}{\chi} . \end{aligned} \tag{4.1}$$

Up sector: The expressions for the eigenvalues will still be given by identical formulas, but with the following redefinitions of l, L, r, and R,

$$\begin{aligned} l&=\beta_{12}\frac{v_L^{(2)}}{\chi}, \quad L=\beta_{23}\frac{v_L^{(1)}}{\chi} , \\ r&=\beta_{12}\frac{v_R^{(2)}}{\chi}, \quad R=\beta_{23}\frac{v_R^{(1)}}{\chi} , \end{aligned} \tag{4.2}$$

since $v_L^{(1)}\to v_L^{(2)}$, $v_R^{(1)}\to v_R^{(2)}$ with different Y couplings in general as we go from the down to the up sector. To compare with experiments, we shall take the following values for quark masses [7]:

$$\begin{aligned} &m_u=5.1\pm1.5 \text{ GeV}, \quad m_c=1.35\pm0.05 \text{ GeV}, \\ &m_t=225\pm75 \text{ GeV}, \quad m_d=8.9\pm2.6 \text{ GeV} , \\ &m_s=0.175\pm0.055 \text{ GeV}, \quad m_b=5.6\pm0.4 \text{ GeV} . \end{aligned} \tag{4.3}$$

The above values of quark masses are assumed to be at the scale of 1 GeV. The physical mass of the top quark, $m_t^{\text{phys}}\simeq0.6m_t$ (1 GeV)$=135\pm45$ GeV. The eigenvalues λ_i in the previous section are related to $(m_i)^2$ by the relation

$$(m_i)^2=\chi^2\lambda_i . \tag{4.4}$$

With our chosen criteria, three out of the four cases considered in the previous section can be ruled out. Those are the $M_{(13)}$, $M_{(13,23)}$, and $M_{(11)}$ cases. Since the arguments are similar in the three cases, it is sufficient to consider one case. Consider, for instance, the eigenvalues for m_3^2 in the $M_{(13)}$ case:

$$\chi^2\lambda_3^{(d)}=m_3^{(d)^2}=m_b^2=\alpha_{12}^2 v_L^{(1)^2}+\alpha_{23}^2 v_L^{(2)^2}+\cdots , \tag{4.5}$$

$$\chi^2\lambda_3^{(u)}=m_3^{(u)^2}=m_t^2=\beta_{12}^2 v_L^{(2)^2}+\beta_{23}^2 v_L^{(1)^2}+\cdots . \tag{4.6}$$

It is clear that in the extreme case when all the Y couplings are equal, $m_b=m_t$. If we assume a hierarchy $v_L^{(1)}\ll v_L^{(2)}$ or $v_L^{(2)}\ll v_L^{(1)}$, then $m_b/m_t\simeq\alpha_{23}/\beta_{12}$ or α_{12}/β_{23}, leading to an explanation of the hierarchy in mass ratio in terms of a hierarchy in the ratio of Y couplings.

The remaining $M_{(23)}$ case stands out as the most plausible and attractive solution. We shall consider this case in some detail. From (3.16), expressing the λ's in terms of the (eigenmass)^2 values, we can deduce the relations

$$\left[\frac{m_d m_s}{m_b^2}\right]^{1/2}=\alpha_{33}\frac{v_R^{(1)}}{\chi}, \quad \left[\frac{m_u m_c}{m_t^2}\right]^{1/2}=\beta_{33}\frac{v_R^{(2)}}{\chi} , \tag{4.7}$$

$$(m_s-m_d)=\frac{\alpha_{33}\alpha_{23}}{\beta_{12}}\left[m_b\frac{\beta_{12}}{\alpha_{12}}\frac{v_R^{(2)}}{\chi}+m_t\frac{v_R^{(1)}}{\chi}\right] , \tag{4.8}$$

$$(m_c-m_u)=\frac{\beta_{33}\beta_{23}}{\beta_{12}}\left[m_b\frac{\beta_{12}}{\alpha_{12}}\frac{v_R^{(2)}}{\chi}+m_t\frac{v_R^{(1)}}{\chi}\right] , \tag{4.9}$$

$$m_b=\alpha_{12}v_L^{(1)}, \quad m_t=\beta_{12}v_L^{(2)} . \tag{4.10}$$

It is interesting to observe that the four VEV's associate themselves naturally with the hierarchical mass ratios and masses. This makes it possible to fit the mass parameters keeping the Y couplings essentially to be of the same order of magnitude. Indeed, substituting the mass values from (4.3), we find, from (4.7)–(4.9),

$$\alpha_{33}\frac{v_R^{(1)}}{\chi}=7.1\times10^{-3}, \quad \beta_{33}\frac{v_R^{(2)}}{\chi}=3.7\times10^{-4} , \tag{4.11}$$

$$\frac{\alpha_{23}}{\beta_{12}}=0.11, \quad \left[\frac{\beta_{33}}{\alpha_{33}}\right]\left[\frac{\beta_{23}}{\beta_{12}}\right]=0.85 . \tag{4.12}$$

One of the $v_L^{(i)}$'s should correspond to the VEV of the Higgs doublet in the standard model. If we assume $v_L^{(2)}$ to correspond to that doublet, $v_L^{(2)}\approx246$ GeV and, with the given mass of the top quark, $\beta_{12}\approx1$. Choosing $v_L^{(1)}\sim m_b$, we see that $\alpha_{12}\approx1$. Thus we are able to fit all the quark masses keeping the Y couplings within one order of magnitude. It is also worth noting that since the $U(1)_A$ charge of σ in this case is zero, σ is not the Dine-Fischler-Srednicki [5] singlet-breaking $U(1)_{PQ}$ needed to avoid the strong *CP* problem. In our model, the left-right symmetry and $U(1)_A$ are both broken by the largest of the two $v_R^{(i)}$, namely, $v_R^{(1)}$. If this is limited to the $10^{11}-10^{12}$ GeV range from axionic considerations, χ is pushed to the grand unification scale. Clearly, to make the model more precise, further work on mixing angles, CP violation, and experimental restrictions on left-right symmetric models is necessary.

To summarize, we have found a model based predominantly on the idea of universal seesaw mechanism that can explain the quark mass hierarchies. It has a minimal Higgs system within the framework of left-right symmetry, and an axial U(1) symmetry is imposed to distinguish the generations by assigning them distinct quantum numbers, thereby leading to restrictions on the form of the mass matrices. With a hierarchy

$$|v_L^{(1)}|<|v_L^{(2)}|<|v_R^{(2)}|<|v_R^{(1)}|<\chi , \tag{4.13}$$

we are able to explain the mass hierarchies. We will show in the Appendix that this hierarchy is obtainable by examining the absolute minimum of the relevant Higgs potential.

The model, however, is not totally satisfactory in that a certain hierarchy in the Yukawa coupling constants is necessary to explain the mass hierarchies. In addition, the observed mass hierarchies appear to be just correlated with the hierarchies. The number of assumed hierarchies in the vacuum expectation values almost coincides with the number of mass hierarchies that are accounted for. However, the model has now predictions on the Kobayashi-Maskawa angles and the related physics of new flavors. Further study of these matters will decide the fruitfulness of the model.

Finally, a word about the choice of the U(1) symmetry to be axial. In our original considerations [8], we were motivated by grand unification that at some stage had left-right symmetry [SO(10), for example]. If the U(1) symmetry was a part of such grand unified theory (GUT) symmetry, then it has to be axial. However, if one is simply interested in a minimal model beyond the standard model, a vectorial U(1) symmetry combined with the idea of a universal seesaw mechanism is not only possible, but has some new and interesting features which will be reported elsewhere.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy under Contract No. DE-FG02-85ER40231. K.C.W. wishes to thank the hospitality of the Physics Department of Ben-Gurion University of the Negev where this work was completed. He is also indebted to the Dozor Foundation for their research support.

APPENDIX

In this paper we have implicitly assumed the existence of a Higgs potential $\mathcal{H}$ with a symmetry group G containing the continuous symmetry group $G_0=\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_A$ and the discrete left-right symmetry $L \leftrightarrow R$, such that its lowest minimum orbit yields VEV's of the Higgs fields with the hierarchical scale properties specified in (4.13). We prove the validity of this assumption in this appendix.

(1) We shall consider the most general Higgs potential $\mathcal{H}$ with the full required symmetry in the case which provides a satisfactory mass spectrum discussed in the previous section. The Higgs sector consists of the four doublets $\varphi_L^{(1)}$, $\varphi_L^{(2)}$, $\varphi_R^{(1)}$, and $\varphi_R^{(2)}$ and a singlet σ carrying zero $\mathrm{U}(1)_A$ charge. Such a potential depends on 15 real dimensionless parameters. We denote by $\mathcal{D}$ the allowed domain in the parameter space, which is the set of parameter values such that $\mathcal{H}$ is bounded below and, moreover, has a local maximum at the origin where all the field vacuum expectation values vanish.

(2) We shall show that there is an open subdomain $\mathcal{O}$ of $\mathcal{D}$, where the orbit of the lowest minima of $\mathcal{H}$ corresponds to the kind of vacuum expectation values we wish. To find the lowest minimum of $\mathcal{H}$ on $\mathcal{O}$, we list all the extrema on $\mathcal{O}$ (i.e., the zeros of the gradient $\nabla\mathcal{H}$) and for each of them study, through the Hessian, the nature of each G orbit for the extrema. We will not give here the details of the calculations, but will only present an outline of the procedure.

1. Higgs potential $\mathcal{H}$: Its gradient and its Hessian

The most general Higgs potential to be considered is the sum of the following polynomial terms:

$$\mathcal{H}=V_4+V_3-V_2 \ , \tag{A1}$$

with [the dots indicating the scalar product of SU(2) spinors]

$$2V_2=\left[\sum_{r=1,2}\Lambda_r[(\varphi_L^{(r)})^\dagger\cdot\varphi_L^{(r)}+(\varphi_R^{(r)})^\dagger\cdot\varphi_R^{(r)}]+\Lambda\sigma^2\right]M^2 \ , \tag{A2}$$

$$2V_3=\sigma\left[\sum_{r=1,2}\rho_r[(\varphi_L^{(r)})^\dagger\cdot\varphi_L^{(r)}-(\varphi_R^{(r)})^\dagger\cdot\varphi_R^{(r)}]\right] \ , \tag{A3}$$

$$V_4=V_4^\varphi+V_4^\sigma+V_4^{\varphi\sigma} \ , \tag{A4}$$

with

$$\begin{aligned}4V_4^\varphi=\sum_{r=1,2}\Big(&\alpha_r[(\varphi_L^{(r)})^\dagger\cdot\varphi_L^{(r)}+(\varphi_R^{(r)})^\dagger\cdot\varphi_R^{(r)}]^2+\beta_r\{[(\varphi_L^{(r)})^\dagger\cdot\varphi_L^{(r)}]^2+[(\varphi_R^{(r)})^\dagger\cdot\varphi_R^{(r)}]^2\}\\&+2\mu[(\varphi_L^{(1)})^\dagger\cdot\varphi_L^{(1)}(\varphi_L^{(2)})^\dagger\cdot\varphi_L^{(2)}+(\varphi_R^{(1)})^\dagger\cdot\varphi_R^{(1)}(\varphi_R^{(2)})^\dagger\cdot\varphi_R^{(2)}]+2\nu[(\varphi_L^{(1)})^\dagger\cdot\varphi_L^{(1)}(\varphi_R^{(2)})^\dagger\cdot\varphi_R^{(2)}+(\varphi_L^{(2)})^\dagger\cdot\varphi_L^{(2)}(\varphi_R^{(1)})^\dagger\cdot\varphi_R^{(1)}]\\&+\gamma[(\varphi_L^{(1)})^\dagger\varphi_L^{(2)}(\varphi_R^{(1)})^\dagger\varphi_R^{(2)}]+\text{Hermitian conjugate}\Big) \ .\end{aligned} \tag{A5}$$

Later, we denote by Γ the terms with the coefficient γ:

$$\begin{aligned}&4V_4^\sigma=\lambda\sigma^4 \ ,\\&4V_4^{\varphi\sigma}=2\sigma^2\sum_{r=1,2}\{\lambda_r[(\varphi_L^{(r)})^\dagger\cdot\varphi_L^{(r)}+(\varphi_R^{(r)})^\dagger\cdot\varphi_R^{(r)}]\} \ .\end{aligned} \tag{A6}$$

Like the Higgs fields, the constant M has the dimension of mass; it is introduced for dimension homogeneity. As stated earlier, the Higgs potential $\mathcal{H}$ depends on 15 dimensionless real constants, which are

$$\alpha_r,\beta_r,\mu,\nu,\gamma,\lambda,\lambda_r,\rho_r,\Lambda,\Lambda_r \ . \tag{A7}$$

The existence of a local (i.e., with a strictly negative Hessian) maximum at the origin requires $\Lambda>0$, $\Lambda_r>0$. That $\mathcal{H}$ be bounded below requires that $V_4>0$ for all field values. This requires that the values of the ten parameters α, β, μ, ν, γ, λ, and λ_r be inside a ten-dimensional convex domain $\mathcal{D}$. We will need to know an open subset of it, which we will define below.

It is convenient to use some explicit notation. Let the values of the four Higgs doublet components be complex numbers a_r,b_r,c_r,d_r defined by

$$\varphi_L^{(1)}=\begin{bmatrix}a_1\\b_1\end{bmatrix},\quad \varphi_R^{(1)}=\begin{bmatrix}c_1\\d_1\end{bmatrix},\quad \varphi_L^{(2)}=\begin{bmatrix}a_2\\b_2\end{bmatrix},\quad \varphi_R^{(2)}=\begin{bmatrix}c_2\\d_2\end{bmatrix} . \tag{A8}$$

Then

$$\Gamma=(a_1^*a_2+b_1^*b_2)(c_1^*c_2+d_1^*d_2)+\text{complex conjugate}. \tag{A9}$$

By contracting each spinor with its conjugate, we obtain four of the five quadratic G_0 invariants of V_2. We denote them by

$$\begin{aligned}(\varphi_L^{(1)})^\dagger\cdot\varphi_L^{(1)}&=|a_1|^2+|b_1|^2=Z_1^2\ ,\\ (\varphi_R^{(1)})^\dagger\cdot\varphi_R^{(1)}&=|c_1|^2+|d_1|^2=Z_2^2\ ,\\ (\varphi_L^{(2)})^\dagger\cdot\varphi_L^{(2)}&=|a_2|^2+|b_2|^2=Z_3^2\ ,\\ (\varphi_R^{(2)})^\dagger\cdot\varphi_R^{(2)}&=|c_2|^2+|d_2|^2=Z_4^2\ .\end{aligned} \tag{A10}$$

With the notation $\sigma=Z_5$, we obtain, for the Higgs potential $(1\le i,j,k,\le 5)$,

$$\begin{aligned}V_4&=\frac{1}{4}\left[\sum_{ij}Z_i^2K_{ij}Z_j^2+\gamma\Gamma\right]\ ,\\ V_3&=\tfrac{1}{2}MZ_5\sum_k N_kZ_k^2,\quad V_2=\tfrac{1}{2}M^2\sum_k L_kZ_k^2\ ,\end{aligned} \tag{A11}$$

with

$$K=\begin{bmatrix}\alpha_1+\beta_1 & \alpha_1 & \mu & \nu & \lambda_1\\ \alpha_1 & \alpha_1+\beta_1 & \nu & \mu & \lambda_1\\ \mu & \nu & \alpha_2+\beta_2 & \alpha_2 & \lambda_2\\ \nu & \mu & \alpha_2 & \alpha_2+\beta_2 & \lambda_2\\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda\end{bmatrix},$$
$$N^T=\begin{bmatrix}\rho_1\\ -\rho_1\\ \rho_2\\ -\rho_2\\ 0\end{bmatrix},\quad L^T=\begin{bmatrix}\Lambda_1\\ \Lambda_1\\ \Lambda_2\\ \Lambda_2\\ \Lambda\end{bmatrix}. \tag{A12}$$

The boundedness below requires that $V_4>0$ for any value of the fields. We leave to the reader the proof of the lemma:

$$\begin{aligned}&x^2+y^2\neq 0,\quad \rho x^4+2\tau x^2y^2+\sigma y^4>0 \Longleftrightarrow \rho>0\ ,\\ &\sigma>0,\quad \tau>-\sqrt{\rho\sigma}\ .\end{aligned} \tag{A13}$$

Applied to V_4 in (A11) and (A12), this lemma yields the inequalities $(r=1,2)$

$$\begin{aligned}&\lambda>0,\quad \alpha_r+\beta_r>0,\quad 2\alpha_r+\beta_r>0\ ,\\ &\mu+\sqrt{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}>0\ ,\\ &\nu+\sqrt{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}>0\ ,\\ &\lambda_r+\sqrt{\lambda(\alpha_r+\beta_r)}>0\ .\end{aligned} \tag{A14}$$

For given relations between the field values, it is easy to find other conditions on the ten parameters of V_4 in order that this term is positive: for instance,

$$\sum_{i=1,2}(2\alpha_i+\beta_i)+2\mu+2\nu\pm\gamma>0\ . \tag{A15}$$

More complicated inequalities need to be added to define the domain $\mathcal{D}$. Those of (A14) and (A15) define only a convex domain $\mathcal{D}'\supset\mathcal{D}$, which is sufficient for our purposes.

We assume that the coefficient γ is small, so that we can treat the $\gamma\Gamma$ term in V_4^φ as a perturbation. Remarkably, the unperturbed Higgs potential, which we denote by $\mathcal{H}_0$, depends only on the five variables Z_i whose moduli at the equilibrium are the five VEV's of (4.13). The gradient and Hessian of $\mathcal{H}_0$ are

$$\frac{\partial\mathcal{H}_0}{\partial Z_i}=Z_iF_i+\tfrac{1}{2}\delta_{i5}M\sum_k N_kZ_k^2\ ,$$

with

$$F_i=\sum_k(K_{ik}Z_k^2)+MZ_5N_i-M^2L_i\ , \tag{A16}$$

$$H_{ij}=\frac{\partial^2\mathcal{H}_0}{\partial Z_i\partial Z_j}=\delta_{ij}F_i+2Z_iK_{ij}Z_j+M(\delta_{i5}N_jZ_j+N_iZ_i\delta_{j5})\ . \tag{A17}$$

Note that, at any extremum,

$$0=\sum_{k=1}^{5}Z_k\frac{\partial\mathcal{H}}{\partial Z_k}=4V_4+3V_3-2V_2\ ;$$

hence,

$$\mathcal{H}=\tfrac{1}{4}(V_3-2V_2)\ . \tag{A18}$$

2. Higgs lowest minima with vacuum expectation values satisfying Eq. (4.13)

The breaking induced by the Higgs potential $\mathcal{H}_0$ will yield the modulus of the five Z_i's, i.e., the five VEV's of (4.13); all values of their phases are equiprobable, and the perturbation term $\gamma\Gamma$ will introduce some changes in them without affecting the hierarchy.

To build the Higgs potential $\mathcal{H}_0$ given the physical VEV's, replace the Z_i's by the five desired VEV's in (4.13) in the five equations $\partial\mathcal{H}_0/\partial Z_i=0$ [see (A16)]. This imposes 5 linear relations among the 14 constants of (A7) other than γ. The 14 constants must also satisfy the 3 inequalities in (A7), the inequality $V_4>0$, and the positivity of the Hessian (A17). And one has also to verify that the other orbits of extrema either are not minima or they correspond to higher minima. It is sufficient to do all this in an open subset $\mathcal{O}$ of $\mathcal{D}$.

Let $\mathcal{O}$ be contained in a neighborhood of $\Delta\subset\mathcal{D}$:

$$\Delta:\{\alpha_r=\lambda_r=\mu=\nu=\gamma=0,\beta_r>0,\lambda>0\}\ . \tag{A19}$$

Δ has been chosen in order to make K_{ij} diagonal. The extremum corresponding to the five $Z_i\neq 0$ is

$$\begin{aligned}&Z_5^2=\lambda^{-1}M^2\left[\Lambda+\sum_{r=1,2}\rho_{Ir}^2\beta_r^{-1}\right],\quad i\le i\le 4\ ,\\ &r=\left[\frac{i+1}{2}\right],\quad Z_i^2=\beta_r^{-1}M^2[L_i-N_i(Z_5/M)]\ .\end{aligned} \tag{A20}$$

Indeed, the four Z_i^2, $1 \le i \le 4$, are solutions of $F_i = 0$ [see (A16)]; then,

$$\sum_{k=1}^{4} N_k Z_k^2 = -2MZ_5 \sum_{r=1,2} \rho_r^2 \beta_r^{-1} . \tag{A21}$$

Using (A18), one verifies that the values of the potential at these extrema is independent of the sign of Z_5 (it is the same for the 32 possible combination of signs of the Z_i's). At these extrema, the Hessian H_{ij} is not diagonal [because of the terms in δ_{i5} or δ_{5j} in (A17)]. We can write it as $H = D\tilde{H}D$, were D is a diagonal matrix with elements $\{Z_i, Z_2, Z_3, Z_4, M\}$ and the symmetric matrix $\tilde{H}$ depends only on the parameters $\beta_r > 0$, ρ_r, $\Lambda_r > 0$, $\Lambda > 0$. One verifies that its determinant and all its principal minors are positive; this proves both the positivity of $\tilde{H}$ and also that of H. So the 2^5 extrema of (A20) are minima with the same value.

For simplicity, we assume that the physical extremum is that with all $Z_i > 0$. According to (4.11) and (4.13), the coordinates of this minimum satisfy

$$v_L^{(1)} = Z_1 < v_L^{(2)} = Z_2 < v_R^{(2)} = Z_4 < v_R^{(1)} = Z_3 < \chi = Z_5 \tag{A22}$$

and $(Z_3/Z_5) < 10^{-2}$. We can assume

$$\beta_1 = \beta_2 = \lambda = 1, \quad \Lambda = \left[1 + \sum_{r=1,2} \rho_r^2 \beta_r^{-1}\right]^{-1} . \tag{A23}$$

All the other nonvanishing parameters of the Higgs potential are determined by [we add inequalities implied by (A22)]

$$\begin{aligned} &M = Z_5, \quad \Lambda_1 = \tfrac{1}{2}(Z_1^2 + Z_2^2) M^{-2} < 10^{-4} , \\ &\Lambda_2 = \tfrac{1}{2}(Z_3^2 + Z_4^2) M^{-2} < 10^{-4} , \\ &\rho_1 = \tfrac{1}{2}(Z_2^2 - Z_1^2) M^{-2} > 0 , \\ &\rho_2 = \tfrac{1}{2}(Z_4^2 - Z_3^2) M^{-2} < 0, \quad |\rho_r| < \Lambda_r . \end{aligned} \tag{A24}$$

We prove that for this Higgs potential any other extremum is not a minimum by showing that the corresponding Hessian has some negative diagonal term. Let Z_i' be the coordinates of such an extremum. Then at least one $Z_i' = 0$. If $Z_5' = 0$, then from (A16) and (A17), we find $H_{55} = -\Lambda M^2 < 0$. If $Z_5' \neq 0$, at least one $Z_i' = 0$ for $1 \le i \le 4$. Then $H_{ii} = -M^2(L_i - N_i Z_5' M^{-1})$, which can also be proved to be negative. For all the four values of i, one can show that the value of $Z_5' \neq 0$ can be replaced by Z_5 with an accuracy of 10^{-2}. Then, according to (A20) and (A23) ($\beta_r = 1$), $H_{ii} \sim -Z_i^2 < 0$.

[1] A. Davidson and K. C. Wali, Phys. Rev. Lett. **59**, 393 (1987); **60**, 1813 (1988). See also S. Rajpoot, Phys. Lett. B **191**, 122 (1987).

[2] A. Davidson and K. C. Wali, Phys. Rev. Lett. **58**, 2623 (1987).

[3] Henceforth we shall refer to these couplings as Y couplings.

[4] A. Davidson, S. Ranfone, and K. C. Wali, Phys. Rev. D **41**, 208 (1990); S. Ranfone, *ibid.* **42**, 3819 (1990). References related to our work and some earlier work are contained in these papers.

[5] M. Dine, W. Fischler, and M. Srednicki, Phys. Lett. **104B**, 199 (1981).

[6] For a discussion of this procedure, see, for instance, M. L. Sage and Mark S. Child, J. Chem. Phys. **90**, 7257 (1989).

[7] J. Gasser and H. Leutwyler, Phys. Rev. **87**, 77 (1982).

[8] A. Davidson and K. C. Wali, Phys. Rev. Lett. **46**, 691 (1987); **48**, 11 (1982); A. Davidson, P. Nair, and K. C. Wali, Phys. Rev. D **29**, 1504 (1984). For a recent discussion of the role of horizontal symmetry on producing mass hierarchies, see M. Leurer, Y. Nir, and N. Seiberg, Nucl. Phys. **B398**, 319 (1993).