ANALYSIS OF Δ_{1232} POLARIZATION

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Abstract

The observed polarization domain of a spin 3/2 Δ produced in a quasi two body or inclusive reaction is a sphere in three dimensions when the beam and the target are unpolarized and a sphere in five dimensions otherwise. In the strong ,electromagnetic or neutrino production of Δ's we study what information can be gained with the use of polarized beam or target and we compare the predictions of some models.

The Δ_{1232} is the most copiously produced baryonic resonnance. In all usual reactions it is created by the excitation with some beam of a nucleon target with a change in the spin state of its quarks. The produced Δ is observed through its decay products πN. Since it has spin 3/2 this observation gives some information on its polarization and therefore some insight on the mechanism of the production reaction. In this paper we shall study this point and investigate what can be learned in supplement if the reaction is performed with a polarized beam or target. We realistically assume that the polarization of the nucleon from the Δ decay is not observed. To simplify the discussion we also assume here[1] that the only observed polarization of the final particles is that of the Δ, as given from its decay. In part 1 we recall the general description of Δ polarization; we refer for more details to our previous publications[2,3]. The observed polarization of the Δ is described by five parameters whose possible set of values is the interior of a sphere in a five dimensional space. In part 2 we review the different possible reactions according to the nature of the beam: mesons, nucleons, electrons or muons, photons, neutrinos; the target is a nucleon.

1. The observed polarization domain of the Δ.

1.1 The intensity and polarization space of the reaction.

We denote by $\underline{p}_B, \underline{p}_T, \underline{p}_\Delta$ respectively the energy-momentum of the beam, the target, the Δ resonnance; by T the transition matrix of the reaction, by ρ_e the polarization density matrix of the initial state (beam and target), by ρ_Δ that of the Δ and by σ the differential cross-section. T is a function of $\underline{p}_B, \underline{p}_T, \underline{p}_\Delta$ and eventually of the other particle energy-momenta; when T is suitably normalized, all these quantities are related by the equation:

$$\sigma \rho_\Delta = \sum T \rho_e T^\dagger, \qquad (1,1)$$

where $\sum$ can represent a sum over the other final particle spins and eventually an integration over some or all other final particle momenta and even, in the case of an inclusive reaction, a summation over all possible reactions compatible with the observation.

This subsection and the next one are valid for any resonnance of spin j and it is only later that we shall specialize to j=3/2. So ρ_Δ is a Hermitean operator acting on $\mathcal{H}_j$, the (2j+1)-dimensional spin space of the resonnance. $\mathcal{H}_j$ is the Hilbert space carrier of the irreducible representation up to a phase $D_j(R)$ of SO(3). The set of Hermitean operators acting on $\mathcal{H}_j$ form a $(2j+1)^2$-dimensional real vector space $\mathcal{E}(\mathcal{H}_j)$ which carries a natural Euclidean scalar product $(A,B)=\mathrm{tr}AB$ and which is called the intensity and polarization space of the studied reaction. The action of SO(3) on $\mathcal{H}_j$ defines a reducible linear representation of this group on $\mathcal{E}(\mathcal{H}_j)$ which is equivalent to

$$D_j \otimes \bar{D}_j \sim \oplus_{L=0}^{2j} D_L \quad (1.2)$$

(where — is the complex conjugate). This corresponds to a decomposition of $\mathcal{E}(\mathcal{H}_j)$ into a direct sum of vector spaces:

$$\mathcal{E}(\mathcal{H}_j) = \oplus_{L=0}^{2j} \mathcal{E}^{(L)}, \qquad \dim \mathcal{E}^{(L)} = 2L+1 ; \quad (1.3)$$

and to the multipole decomposition of ρ:

$$\rho = \Sigma_{L=0}^{2j} \rho_L , \quad (1.3')$$

where $\rho_o = (2j+1)^{-1} I$ is the density matrix of the unpolarized state.

The matrix ρ must be positive (i.e. its eigenvalues must be non negative) so it must be inside $\mathcal{C}$, the convex self dual cone of positive matrices of $\mathcal{E}(\mathcal{H}_j)$.

1.2 The polarization domain $\mathcal{D}$.

The polarization space $\mathcal{E}$ is the $(2j+1)^2-1$ dimensional Euclidean vector space which can be identified either as the hyperplane tr =1 in $\mathcal{E}(\mathcal{H}_j)$ or with the summand $\mathcal{E}$ in:

$$\mathcal{E}(\mathcal{H}_j) = \mathcal{E}^{(0)} \oplus \mathcal{E} \quad \text{where } \mathcal{E} = \oplus_{L=1}^{2j} \mathcal{E}^{(L)}. \quad (1.4)$$

The set in $\mathcal{E}$ of the possible $\rho-\rho_o$ is the polarization domain $\mathcal{D}$; it is the intersection of the cone $\mathcal{C}$ with the hyperplane tr ρ =1 in $\mathcal{E}(\mathcal{H}_j)$. $\mathcal{C}$ is a convex self dual domain[2].

It is customary to normalize the polarization degree d_ρ between o, for the unpolarized state ρ_o, and 1, for a pure state of polarization $\rho=P$ where $P=|x\rangle\langle x|$, $x\rangle \in \mathcal{H}_j$, $\langle x|x\rangle =1$; i.e. P is a rank one Hermitean projector $P=P^2=P^*$, trP=1. Then for the state of density matrix ρ, the polarization degree is given by:

$$d_\rho^2 = \frac{2j+1}{2j}(\rho-\rho_o, \rho-\rho_o) = \frac{2j+1}{2j}(\mathrm{tr}\,\rho^2 - \frac{1}{2j+1}). \quad (1.5)$$

From now on, we will use on $\mathcal{E}$ the metric defined by the polarization degree; so the polarization domain is contained inside the unit sphere of $\mathcal{E}$.

1.3 The observed polarization domain $\mathcal{D}$.

We now consider the j=3/2 resonnace Δ_{1232}. Its main decay mode is πN. The angular distribution $\mathcal{J}(\theta,\varphi)$ of this parity conserving two body decay measures only the even multipoles (i.e, here the L=2 quadrupole) of ρ_Δ. To measure the odd multipoles L=1 and L=3 one would have to observe the angular distribution $\underset{\sim}{P}(\theta,\varphi)$ of the polarization of the decay nucleon. Since this is technically difficult we assume that it is not done.

Such a partial observation corresponds to an orthogonal projection g from the intensity and polarization space $\mathcal{E}_{16}(\mathcal{H}_{1/2})$ onto that of the even multipoles $\mathcal{E}^{(E)}$; here:

$$\mathcal{E}_{16}(\mathcal{H}_{\frac{1}{2}}) \xrightarrow{g} \mathcal{E}_6^{(E)} = \mathcal{E}_1^{(0)} \oplus \mathcal{E}_5^{(2)}. \qquad (1.6)$$

Correspondingly, the observed polarization domain is the projection of $\mathcal{D} \subset \mathcal{E}_{15}$ on $\mathcal{E}_5^{(2)}$. One shows that $\mathcal{E}_5^{(2)}$ is a symmetry plane of $\mathcal{D}$ so the <u>observed polarization domain $\mathcal{D}$</u> is the intersection:

$$\mathcal{D} = \mathcal{D} \cap \mathcal{E}_5^{(2)}. \qquad (1.7)$$

This implies that the observed $\rho = \rho_0 + \rho_2$ is a positive matrix. Similarly the projection of the cone $\mathcal{C} \subset \mathcal{E}_{16}(\mathcal{H}_{\frac{1}{2}})$ on $\mathcal{E}_6^{(E)}$ is :

$$g(\mathcal{C}) = \mathcal{C}^{(E)} = \mathcal{C} \cap \mathcal{E}_6^{(E)}. \qquad (1.8)$$

Finally, we show in the appendix that $\mathcal{D}$ is the ball in $\mathcal{E}_5^{(2)}$ bounded by the sphere S_4 centered at 0 and of radius $3^{-\frac{1}{2}}$. Therefore, experiemental data on polarization is repreaented by points in this ball $\mathcal{D}$.

<u>1.4 B-symmetric reactions.</u>

We call B-symmetric the following experimental reactions: they conserve parity and the observation conditions forget the dependence of σ and ρ on energy momenta other than p_B, p_T, p_Δ. Either the reaction is quasi two body or integration over phase space of the other final particles is performed or the reaction is inclusive. Let B be the reflection ,in energy momentum space, through the 3-plane spanned by p_B, p_T, p_Δ (In the case of forward or backward Δ there is a one parameterfamily of such planes). Since $B^2=1$,in any linear representation the eigen values of B are 1 and -1. The corresponding eigen spaces in $\mathcal{E}_5^{(2)}$ are respectively of dimension 3 and 2, so we denote them by $\mathcal{E}_3^{(2)+}$ and $\mathcal{E}_2^{(2)-}$. In table 2 we give explicitly the coordinates of these subspaces of $\mathcal{E}_5^{(2)}$ while in table 1 we have recapitulated the list of spaces and geometrical domains introduced in the paper.

For a B-symmetric reaction, when the initial polarization is invariant by B, the observed Λ polarization has the same property and in this case the polarization domain is reduced to

$$\mathcal{D}_+ = \mathcal{D} \cap \mathcal{E}_3^{(2)+}, \qquad (1.9)$$

whose boundary $\mathcal{S}_+$ is the sphere S_2 of center 0 and radius $3^{-\frac{1}{2}}$. For example, the observed Λ polarization produced in a B-symmetric reaction with unpolarized beam and target depends only on three parameters and the experimental data is represented by points in $\mathcal{D}_+$. In the general case, when the observed Λ polarization is represented by a point $M_* \in \mathcal{D}$, we will denote its projections on $\mathcal{E}_3^{(2)+}$ and

$\mathcal{E}_2^{(2)-}$ by M and M' respectively. M must be in $\mathcal{D}_+$ and M' is in

$$\mathcal{D}_- = \mathcal{D} \cap \mathcal{E}_2^{(2)-} , \qquad (1.10)$$

whose boudary $\partial\mathcal{D}_-$ is the circle S_1 of center O and radius $3^{-\frac{1}{2}}$.Remark that

$$|OM|^2 + |OM'|^2 \leq \frac{1}{3} , \qquad (1.11)$$

where the equality occurs when M_{ob} is on the boundary S_4 of $\mathcal{D}$.

1.5 The Ellipsoid of polarization transfer E_2.

Let us denote by ρ_e the density matrix of the initial particles In this section we study the linear map f,

$$f = g_o h , \qquad (1.12)$$

where h is defined by (1) and g is the orthogonal projection defined by (6).Each initial polarization is transformed into a point of the intensity and observed polarization space:

$$f(\rho_e) = g(\sigma \rho_\Delta) = \sigma\rho . \qquad (1.13)$$

Consider the simplest case when only one initial particle is polarized and it has spin 1/2 or it is a photon. In both cases all possible pure polarization states ρ_e form a sphere S_2. Its image by f is denoted by E_2 and is called the <u>ellipsoid of polarization transfer</u> . We study successively two cases:

i) <u>spin 1/2 beam or target</u>.Its polarization density matrix is $\rho_e = \frac{1}{2}(I + \vec{w}.\vec{\sigma})$. When $\vec{w}=0$, there is no polarization ,then $f(\rho_e)$ is the point $m \in \mathcal{V}^{(E)}$,center of E_2. In the general case $f(\rho_e) \in \eta E_2$ if we call η the polarization degree:$0 \leq \eta \leq 1$ (see 2.2 below). But to communicate the physical results, the degree of polarization of the beam and/or the target is irrelevant, so we assume that $\vec{w}^2=1$ i.e. $f(\rho_e) \in E_2$ .In the laboratory system, the three interesting directions of the polarization vector $\vec{w}$ of the target are:$\pm\vec{w}_\ell \propto \vec{p}_B$, $\pm\vec{w}_s \propto \vec{p}_\Delta - \vec{p}_B(\vec{p}_B.\vec{p}_\Delta)(\vec{p}_B^{\,2})^{-1}$ transverse in the reaction plane,$\pm\vec{w}_n \propto \vec{p}_B \times \vec{p}_\Delta$ normal to the reaction plane. The same formulae hold for the beam in its rest system[4] when one replaces $\vec{p}_B$ by $\vec{p}_T$; then $\pm\vec{w}_\ell$ corresponds to the beam longitudinal polarization. These three directions define three orthogonal diameters of S_2. They are mapped by f onto a set of three conjugated diameters of E_2 that we denote respectively by n_-n_+, s_-s_+, l_-l_+.Because $\vec{w}$ is an axial vector,$\vec{w}_n$ is invariant by B while $\vec{w}_s$ and $\vec{w}_\ell$ change sign. So for a B-symmetric reaction:

$$n_-mn_+ \in \mathcal{E}_1^{(0)} \oplus \mathcal{E}_3^{(2)+} \quad ; \quad s_-ms_+ ,\; l_-ml_+ \in \mathcal{E}_2^{(2)-} \qquad (1.14)$$

Moreover s_-s_+ and l_-l_+ are conjugated diameters of the ellipse $E_2 \cap \mathcal{E}_2^{(2)-}$ and n_-n_+ is a principal axis of the ellipsoid E_2 since it is orthogonal to $\mathcal{E}_2^{(2)-}$,its conjugated plane.

ii) Polarized photon beam. The set of pure polarization states is the Poincaré sphere S_2; we can apply the results of i) with the following translation dictionnary:

$\pm w_\ell$ describe the right and left circular polarizations, while the photon linear polarization is described by

$\pm w_n$ when the polarization planes are perpendicular to or in the reaction plane,

$\pm w_s$ when the polarization planes are at 45° of the reaction plane.

The results about B-symmetric reactions are just carried over.

iii) A particular case, very important in the applications, occurs when the point m representing the experimentally observed $\sigma\rho$ for unpolarized beam and target is on the surface of the cone $\mathcal{C}^{(E)}$. We can then apply a very general theorem:

If a bounded convex domain S is mapped linearly in a convex domain $\mathcal{C}$ and if the image $m=f(s)$ of a point s of the interior of S is on the boundary of $\mathcal{C}$, then $f(S)$ is included in $\partial\mathcal{C} \cap T_m(\mathcal{C})$, where $T_m(\mathcal{C})$ is the tangent plane of $\mathcal{C}$ at m.

Proof. The tangent hyperplane $T_m(\mathcal{C})$ defines two open half spaces. One of them has no common point with $\mathcal{C}$; the image $f(s_1 s s_2)$ of any segment in S containing s in its interior cannot have points in this half space, so it is in the complement and more precisely, because of the linearity of f and the convexity of $\mathcal{C}$, it is in $T_m(\mathcal{C})$.

Applied to our particular case, this theorem shows that the polarization transfer ellipsoid E_2 is degenerated into a segment of middle m and carried by the cone generatrix containing m. So the only effect of beam and/or target polarization is a variation of the differential cross section while the observed polarization remains constant.

1.6 The forward and backward B-symmetric Δ production.

For such conditions the reaction is invariant by a $O(2)$ group generated by the rotations about the common axis $\vec{p}_B // \vec{p}_\Delta$ (in the target or c.m. system) and the reflections through planes containing this axis. The irreducible linear representations of this group are 2-dimensional (we denote them by d_m, m integer >0) except two 1-dimensional ones: d_o^- and the trivial one d_o^+. The 5-dimensional representation on the observed polarization space $\mathcal{E}_5^{(2)}$ reduces to the direct sum:

$$d_o^+ \oplus d_1 \oplus d_2 \,. \qquad (1.15)$$

This decomposition and the B-symmetric decomposition $\mathcal{E}_5^{(2)} = \mathcal{E}_3^{(2)+} \oplus \mathcal{E}_2^{(2)-}$ define in $\mathcal{E}_5^{(2)}$ a complete basis of five orthogonal axes when the forward and backward B-symmetric Δ production is considered as a limiting case with fixed reaction plane. These axes cut the sphere $S_4 = \partial\mathcal{D}$ at the points:

Q_1, Q_2 defined by the carrier space of d_o^+,
T_1, T_2 defined by the intersection of the space of d_1 and $\mathcal{E}_3^{(2)+}$,
U_1, U_2 " " " and $\mathcal{E}_3^{(2)-}$,
R_1, R_2 " " " of d_2 and $\mathcal{E}_3^{(2)+}$,
V_1, V_2 " " " " and $\mathcal{E}_3^{(2)-}$.

The remaining sign ambiguities will be completly resolved in part 2 by very simple reaction models. We insist on the fact that this complete explicit description of the observed polarization domain is coordinate independent. Table 2 recalls all these information and also gives explicit coordinates in both helicity and transversity quantizations. Figure 1 represents the domain $\mathcal{D}$.

In a forward or backward B-symmetric reaction the mapping f commutes with the action of the O(2) group defined above. The initial polarizations normal to the collinear process, $\vec{w}_n$ and $\vec{w}_s$ transform as d_1 for a spin 1/2 particle and as d_2 for a photon beam. By Schur's lemma one predicts for the corresponding observed Δ polarization the vanishing of its projection on R_1R_2 and V_1V_2 for a spin 1/2 particle, and on T_1T_2 and U_1U_2 for a photon beam. Similarly, since an initial polarization $\vec{w}_\ell$ (longitudinal for a spin 1/2 beam, circularly polarized for a photon beam) transforms as d_o^-, one predicts that the observed polarization is independent[5] from $\vec{w}_\ell$.

The choice of helicity or transversity quantization (or of any quantization axis) is of pure convenience. However there is one choice to be made which is physically relevant: that of the channel (see e.g. ref.2a). Indeed a rotation of the coordinate frame around the normal to the beam and Δ momenta induces a rotation about the same angle around O in the plane of $\mathcal{D}_-$ and about the double angle around the vertical axis P_1P_2 of the ball $\mathcal{D}_+$. For instance, to pass from s-channel to t-channel quantization (e.g. Jacob and Wick have chosen s-channel helicity coordinates while Gottfried and Jackson have chosen t-channel helicity with some conflict in the sign conventions between the two groups) one has to do a "crossing rotation" around the normal. Its angle vanishes only in the forward or backward direction. We advise experimentalists to plot their data in the different channels: some features may be clearer in one of them. For instance helicity conservation in a given channel requires that the representative points M stay, for the different values of t, in the Q_1Q_2 diameter of $\mathcal{D}_+$ in the plot corresponding to this channel. In the case of one pion exchange M stays at the point Q_2.

TABLE 1 - List of Spaces and Geometrical domains

$\mathcal{E}_n$ denote a n dimensional real vector space with Euclidean metric.

$\mathcal{B}_n$ is a ball of $\mathcal{E}_n$ i.e. its surface $\partial\mathcal{B}_n$ is a sphere S_{n-1} centered at the origin of $\mathcal{E}_n$.

$\mathcal{H}_{3/2}$, the Δ spin space, is a 4-dimensional Hilbert space.

$\mathcal{E}_{16}(\mathcal{H}_{3/2})$ = space of Hermitian operators on $\mathcal{H}_{3/2}$, intensity and polarization space.

$\mathcal{C}$ = convex cone of positive matrices in $\mathcal{E}_{16}(\mathcal{H}_{3/2})$.

$\mathcal{E}_{16}(\mathcal{H}_{3/2}) = \mathcal{E}_1^{(0)} + \mathcal{E}_3^{(1)} + \mathcal{E}_5^{(2)} + \mathcal{E}_7^{(3)}$ multipole decomposition.

$\rho_\Delta = \rho_0 + \rho_1 + \rho_2 + \rho_3$, multipole decomposition of density matrix, $\rho_0 = \frac{1}{4} I$.

$\mathcal{E}_{16}(\mathcal{H}_{3/2}) = \mathcal{E}_1^{(0)} \oplus \mathcal{E}_{15}$, $\mathcal{E}_{15}$ is the polarization space $\rho_\Delta - \rho_0 \in \mathcal{E}_{15}$.

$\{H \in \mathcal{C}, \mathrm{tr}\, H=1\} = \rho_0 + \mathcal{D}$ where $\mathcal{D}$ is the polarization domain $=\{\rho_\Delta - \rho_0\}$.

$\mathcal{E}_6^{(E)} = \mathcal{E}_1^{(0)} + \mathcal{E}_5^{(2)}$, intensity and observed polarization space.

$\mathcal{C}^{(E)} = \mathcal{C} \cap \mathcal{E}_6^{(E)} = \{\sigma\rho\}$, ρ is the observed Δ-polarization density matrix.

$\mathcal{O} = \mathcal{D} \cap \mathcal{E}^{(2)} = \mathcal{B}_5$ observed polarization domain $=\{\rho-\rho_0\}$.

B , the B-symmetric operator, is defined in 1.4 ; $B^2 = I$.

$\mathcal{E}_3^{(2)+}$, $\mathcal{E}_2^{(2)-}$ eigen spaces of B acting in $\mathcal{E}_5^{(2)}$, with the eigen values ± 1 .

$\mathcal{O}_+ = \mathcal{O} \cap \mathcal{E}_3^{(2)+} = \mathcal{B}_3$, B-symmetric observed polarization domain .

$\mathcal{O}_- = \mathcal{O} \cap \mathcal{E}_2^{(2)-} = \mathcal{B}_2$, B-antisymmetric observed polarization domain.

The ball $\mathcal{O}$, and therefore $\mathcal{O}_+$, $\mathcal{O}_-$ have a radius $3^{-1/2}$.

$E_2 \subset \mathcal{C}^{(E)}$, ellipsoid of polarization transfer.

In part 2 we also define $\mathcal{E}_4^{(E)+} = \mathcal{E}_1^{(0)} \oplus \mathcal{E}_3^{(2)+}$ **and** $\mathcal{C}^{(E)+} = \mathcal{C} \cap \mathcal{E}^{(E)+}$

Table 2 : Explicit plot of the observed polarization domain $\mathcal{D}$.

When n Δ-resonnances are observed in an experiment, the moments of their decay angular distribution are : $\langle Y^L_M \rangle = \frac{1}{n} \sum_{i=1}^{n} Y^L_M(\theta_i,\varphi_i)$. Only the following can be non zero : $\langle Y^o_o \rangle = \frac{1}{\sqrt{4\pi}}$, $\langle Y^2_M \rangle = -\frac{1}{\sqrt{4\pi}} t^2_M$ where the t^2_M are the multipole parameters of the observed $\rho = \frac{1}{4}(1+ \sum_M t^2_M (T^2_M)^*)$. The matrix elements of $(T^L_M)^{\mu}{}_{\mu'}$ of the multipole operators are the C.G. coefficients $\langle \frac{3}{2}\mu | \frac{3}{2}\mu' LM \rangle$. We denote by ${}^H t^2_M$ and ${}^T t^2_M$ the multipoles parameters in Helicity or Transversity coordinates. B-symmetric observed polarization $\Longrightarrow$ ${}^H t^2_M$ real $\Longleftrightarrow$ ${}^T t^2_{\pm 1} = 0$.

Coordinates in $\mathcal{E}^{(2)+}_3$: x,y,z; $\qquad \mathcal{D}_+ : x^2 + y^2 + z^2 \leq \frac{1}{3}$;

$$x = -\sqrt{\frac{10}{3}}\,\mathrm{Im}\,{}^T t^2_2 = \frac{4}{\sqrt{3}}\,\mathrm{Im}\,{}^T\rho_{3-1} = -\sqrt{\frac{10}{3}}\,\mathrm{Re}\,{}^H t^2_1 = \frac{4}{\sqrt{3}}\,\mathrm{Re}\,{}^H\rho_{31} ,$$

$$y = \sqrt{\frac{10}{3}}\,\mathrm{Re}\,{}^T t^2_2 = \frac{4}{\sqrt{3}}\,\mathrm{Re}\,{}^T\rho_{3-1} = \sqrt{\frac{5}{6}}\,\mathrm{Re}\,{}^H t^2_2 - \frac{\sqrt{5}}{2}\,{}^H t^2_0 = \frac{2}{\sqrt{3}}\mathrm{Re}\,{}^H\rho_{3-1} - {}^H\rho_{33} + {}^H\rho_{11} ,$$

$$z = \sqrt{\frac{5}{3}}\,{}^T t^2_0 = \frac{2}{\sqrt{3}}({}^T\rho_{33} - {}^T\rho_{11}) = -\left(\sqrt{\frac{5}{12}}\,{}^H t^2_0 + \sqrt{\frac{5}{2}}\,\mathrm{Re}\,{}^H t^2_2\right) = -2\mathrm{Re}\,{}^H\rho_{3-1} - \frac{1}{\sqrt{3}}({}^H\rho_{33} - {}^H\rho_{11}) .$$

Coordinates in $\mathcal{E}^{(2)-}_2$: u,v; $\qquad \mathcal{D}_- : u^2 + v^2 \leq \frac{1}{3}$;

$$u = -\sqrt{\frac{10}{3}}\,\mathrm{Im}\,{}^T t^2_1 = -\frac{4}{\sqrt{3}}\,\mathrm{Im}\,{}^T\rho_{31} = \sqrt{\frac{10}{3}}\,\mathrm{Im}\,{}^H t^2_1 = \frac{4}{\sqrt{3}}\,\mathrm{Im}\,{}^H\rho_{31} ,$$

$$v = -\sqrt{\frac{10}{3}}\,\mathrm{Re}\,{}^T t^2_1 = \frac{4}{\sqrt{3}}\,\mathrm{Re}\,{}^T\rho_{31} = \sqrt{\frac{10}{3}}\,\mathrm{Im}\,{}^H t^2_2 = -\frac{4}{\sqrt{3}}\,\mathrm{Im}\,{}^H\rho_{3-1} .$$

As usual, to save writing we denote $\rho_{mm'}$ by $\rho_{2m,2m'}$. Note that the observed ρ is deduced from the actual ρ_Δ by $\rho_{mm'} = \frac{1}{2}((\rho_\Delta)_{mm'} + (-1)^{m-m'}(\rho_\Delta)_{-m',-m})$; see Appendix A(6), $(\Gamma_j)_{mm'} = (-1)^{j-m}\delta_{m,-m'}$.

Figure 1 represents $\mathcal{D}$ and its remarkable points.

Physical points of $\mathcal{D}$ in B-symmetric reactions

Points	x	y	z	u	v	d_m *)	$\mathcal{D}_\pm$	${}^T\rho$	${}^H\rho$
$Q_1 Q_2$	0	$\mp\frac{1}{2}$	$\mp\frac{1}{2\sqrt{3}}$	0	0	0	+	$Y_1 Y_2$	$X_1 X_2$
$T_1 T_2$	$\pm\frac{1}{\sqrt{3}}$	0	0	0	0	1	+	$K_+ K_-$	$V_+ V_-$
$U_1 U_2$	0	0	0	$\pm\frac{1}{\sqrt{3}}$	0	1	-	$U_- U_+$	$U_+ U_-$
$R_1 R_2$	0	$\pm\frac{1}{2\sqrt{3}}$	$\mp\frac{1}{2}$	0	0	2	+	$Z_+ Z_-$	$H_+ H_-$
$V_1 V_2$	0	0	0	0	$\pm\frac{1}{\sqrt{3}}$	2	-	$V_+ V_-$	$K_- K_+$
$P_1 P_2$	0	0	$\pm\frac{1}{\sqrt{3}}$	0	0		+	$X_1 X_2$	$Y_1 Y_2$

*) O(2) representation d_m (see 1.6) for forward and backward reactions.

$$X_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, X_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X_1 + X_2 = 2\rho_0 ,$$

$$Y_1 = \frac{1}{8}\begin{pmatrix} 1 & 0 & -\sqrt{3} & 0 \\ 0 & 3 & 0 & -\sqrt{3} \\ -\sqrt{3} & 0 & 3 & 0 \\ 0 & -\sqrt{3} & 0 & 1 \end{pmatrix}, Y_2 = \frac{1}{8}\begin{pmatrix} 3 & 0 & \sqrt{3} & 0 \\ 0 & 1 & 0 & \sqrt{3} \\ \sqrt{3} & 0 & 1 & 0 \\ 0 & \sqrt{3} & 0 & 3 \end{pmatrix}, Y_1 + Y_2 = 2\rho_0 ;$$

$$H_\pm = \frac{1}{4}\begin{pmatrix} 1 & 0 & \pm 1 & 0 \\ 0 & 1 & 0 & \pm 1 \\ \pm 1 & 0 & 1 & 0 \\ 0 & \pm 1 & 0 & 1 \end{pmatrix}, K_\pm = \frac{1}{4}\begin{pmatrix} 1 & 0 & \pm i & 0 \\ 0 & 1 & 0 & \pm i \\ \mp i & 0 & 1 & 0 \\ 0 & \mp i & 0 & 1 \end{pmatrix} \begin{matrix} H_+ + H_- = 2\rho_0 , \\ K_+ + K_- = 2\rho_0 , \end{matrix}$$

$$V_\pm = \frac{1}{4}\begin{pmatrix} 1 & \pm 1 & 0 & 0 \\ \pm 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mp 1 \\ 0 & 0 & \mp 1 & 1 \end{pmatrix}, U_\pm = \frac{1}{4}\begin{pmatrix} 1 & \pm i & 0 & 0 \\ \mp i & 1 & 0 & 0 \\ 0 & 0 & 1 & \mp i \\ 0 & 0 & \pm i & 1 \end{pmatrix}, \begin{matrix} U_+ + U_- = 2\rho_0 , \\ V_+ + V_- = 2\rho_0 , \end{matrix}$$

$$Z_\pm = \frac{1}{8}\begin{pmatrix} 2\mp\sqrt{3} & 0 & \pm 1 & 0 \\ 0 & 2\pm\sqrt{3} & 0 & \pm 1 \\ \pm 1 & 0 & 2\pm\sqrt{3} & 0 \\ 0 & \pm 1 & 0 & 2\mp\sqrt{3} \end{pmatrix}, \qquad Z_+ + Z_- = 2\rho_0 .$$

For forward or backward Δ production see 1.6 and 1.5;

$Q_1 Q_2 \ni M\, L_- L_+$ no, or $\vec{W}_\ell$, initial polarization ,

$T_1 T_2 \ni N_+ N_-$ initial spin $\frac{1}{2}$ $\vec{W}_n$ polarization ,

$U_1 U_2 \ni S_+ S_-$ initial spin $\frac{1}{2}$ $\vec{W}_s$ polarization ,

$R_1 R_2 \ni N_+ N_-$ photon beam $\vec{W}_n$ polarization ,

$V_1 V_2 \ni S_+ S_-$ photon beam $\vec{W}_s$ polarization;

Q_2 in s-channel is the Δ polarization in $\pi + N \to \Delta$ (resonnance formation),

Q_2 in t-channel in the Δ -polarization for one π exchange in $\pi N \to \rho\Delta$,

P_2 is the Δ -polarization for ρ exchange in $\pi + N \to \pi\Delta$ (magnetic dipole transition, $\Delta J = 1$, Stodolsky Sakurai model, quark model) .

2. Reactions producing Δ_{1232}.

In all the examples of reactions that we will study we assume that the target is a nucleon (in practice a proton). These reactions can be characterized by the nature of the beam and the particle multiplicity of the final state. We will consider

meso-production $\pi N \to \pi\Delta \quad KN \to K\Delta$ (2.1)

$\pi N \to \rho\Delta$, $KN \to K^*\Delta$; $\pi N \to \pi\pi\Delta$, $KN \to K\pi\Delta$, $\pi N \to \pi\pi\pi\Delta, \cdots$ (2.2 a;b)

production by nucleon $NN \to N\Delta$; $NN \to \pi N\Delta, \ldots$ (2.3 a;b)

lepto-production $eN \to e\Delta$, $\mu N \to \mu\Delta$; $eN \to e\pi\Delta$, $\mu N \to \mu\pi\Delta$,. (2.3' a;b)

photo-production $\gamma N \to \pi\Delta$; $\gamma N \to \pi\pi\Delta, \ldots$ (2.4 a;b)

neutrino production $\nu N \to \mu\Delta$, $\nu N \to \nu\Delta$; $\nu N \to \mu\pi\Delta, \ldots$ (2.5 a;b)

In addition to these exclusive reactions we will consider first the inclusive reaction

$$B N \to X \Delta \qquad (2.6)$$

where B is any beam of the reactions (2.1) to (2.5).

2.1 Strong or electromagnetic Δ production in inclusive reactions.

For each beam energy and each momentum $\vec{p}_\Delta$, the experimental data $\sigma\rho$ (differential cross section and observed polarization density matrix) can be represented by a point $m \in \mathcal{C}^{(E)}$ in the intensity and observed polarization space $\mathcal{E}_6^{(E)}$. Since the reaction is B-symmetric ,for unpolarized beam and target the corresponding Δ polarization is represented by $M \in \mathcal{D}_+$. Similarly $m \in \mathcal{C}^{(E)+} = \mathcal{C} \cap \mathcal{E}_4^{(E)+}$ where $\mathcal{E}_4^{(E)+} = \mathcal{E}_1^{(0)} + \mathcal{E}_3^{(1)+}$ .See figure 2; If the initial polarization is along the normal $\vec{w}_n$, the final state is still B symmetric.Let η be the polarization degree of the beam or target: $0 \leq \eta \leq 1$.In general one performs the experiment for both orientations of the polarization $\vec{w} = \pm\eta\vec{w}_n$.We denote by $\tilde{\sigma}_+$ and $\tilde{\sigma}_-$ the respective cross section and by $\tilde{\rho}_+$ and $\tilde{\rho}_-$ the respective observed Δ polarizations; these are represented by the points M_+ and M_- in $\mathcal{D}$ while $\tilde{\sigma}_+\tilde{\rho}_+$ and $\tilde{\sigma}_-\tilde{\rho}_-$ are represented respectively by the points m_+ and m_- in $\mathcal{C}^{(E)+}$. The linearity in ρ_e of equation (1.1) implies that m is the middle of the segment m_+m_- .The corresponding points M, M_+, M_- in $\mathcal{D}_+$ are obtained by conic projection; they are aligned but M is not the middle of M_+M_-. The observables σ, ρ corresponding to the unpolarized initial state are given by

$$\sigma = \frac{1}{2}(\tilde{\sigma}_+ + \tilde{\sigma}_-) \ , \quad \rho = \frac{\tilde{\sigma}_+}{2\sigma}\tilde{\rho}_+ + \frac{\tilde{\sigma}_-}{2\sigma}\tilde{\rho}_- . \qquad (2.7)$$

In order to communicate experimental results in a way independent from the polarization degree it is natural to compute the cross sections σ_+ and σ_- and the polarizations ρ_+ and ρ_- corresponding to the complete polarization $\eta=1$. From the linearity in ρ_e of equation (1.1) one obtains:

$$\sigma_\pm = \frac{\eta \pm 1}{2\eta}\tilde{\sigma}_+ + \frac{\eta \mp 1}{2\eta}\tilde{\sigma}_- , \qquad (2.8a)$$

$$\sigma_\pm \rho_\pm = \frac{\eta \pm 1}{2\eta}\tilde{\sigma}_+ \tilde{\rho}_+ + \frac{\eta \mp 1}{2\eta}\tilde{\sigma}_- \tilde{\rho}_- . \qquad (2.8b)$$

The corresponding points are denoted by n_+, n_- in $\mathcal{E}^{(E)+}$ and by N_+, N_- in $\mathcal{D}_+$. The points N_- M_- M M_+ N_+ are aligned and their distances satisfy:

$$\frac{N_+N_-}{M_+M_-} \quad \frac{MM_- - MM_+}{MN_- - MN_+} = \eta \;, \qquad \frac{MN_+}{MN_-} = \frac{\sigma_-}{\sigma_+} . \qquad (2.9\ a,b)$$

When the initial polarization $\vec{w}$ is in the reaction plane (see 1.5 for initial spin 1/2 or photon beam polarization),

$$\vec{w} = \eta\,(\cos\omega\ \vec{w}_\ell + \sin\omega\ \vec{w}_s)\,, \qquad (2.10)$$

the observed Δ polarization is B-antisymmetric and the cross section σ is independent of $\vec{w}$ since it is B symmetric. The polarization is now represented by a point in $\mathcal{D}$ whose projection on $\mathcal{D}_+$ is the point M describing the observed polarization for unpolarized beam and target; its projection $M'(\eta,\omega)$ on $\mathcal{D}_-$ sweeps the interior of the ellipse projection of E_2 on $\mathcal{D}_-$ when η varies from 0 to 1 and ω from 0 to 2π. As we saw in (1.11) this ellipse is inside the circle of radius

$$r = \left(\tfrac{1}{3} - |OM|^2\right)^{\frac{1}{2}} . \qquad (2.11)$$

To communicate a complete set of experimental polarization measurements one gives two conjugated diameters of this ellipse, for instance the longitudinal and side diameters L_+L_-, S_+S_-. These four points are $M'(\eta,\omega)$ for $\eta=1$ and $\omega = 0, \pi, \frac{\pi}{2}, \frac{3\pi}{2}$ respectively.

Models can be tested by the comparison between the predicted and the observed σ, M, σ_+, N_+, L_+, S_+ as functions of the invariants s, t and u or the missing mass. The number of independent observables is 1+3+1+3+2+2=12.

2.2 Meso production: $\pi N \to \pi\Delta$ or $KN \to K\Delta$.

These are the quasi elastic two body reactions of (2.1). No summation is involved in equation (1.1) for precise experiments so one can reconstruct the 2x4/2 = 4 complex amplitudes. Indeed a complete theoretical description of such a reaction is given by seven real functions of s and t and ,as we just saw, twelve linearly independent observables can be measured with a polarized target : there must be five constraints on these observables. They are due to the rank condition on ρ. For an initial pure polarization state ($\eta=1$), ρ_Δ is also a pure state (rank ρ_Δ =rank $\rho_e=1$). Then, as we showed in the appendix, the rank of the observed ρ is 2 and its representative point M_{ob} is on the sphere $S_4 = \partial\mathcal{D}$. Equivalently, the ellipsoid E_2 is on the surface $\partial\mathcal{C}^{(E)}$ of the cone $\mathcal{C}^{(E)}$. This ellipsoid is therefore axially symmetrical around its N_-N_+ diameter. $\mathcal{E}_2^{(E)-}$ is its equatorial plane.

With unpolarized target one observes $\sigma\rho$ i.e. four parameters represented by $m \in \mathcal{C}^{(E)+}$. The diameter n_-n_+ has to be in the three dimensional plane conjugated to om: it is therefore completly determined by the two parameters fixing its direction in this 3-plane. It is equivalent to say that in the ball $\mathcal{D}_+$ the orientation of the segment N_-MN_+ is arbitrary but the points N_- and N_+ are on the sphere S_2 boundary of $\mathcal{D}_+$. This is one constraint on N_+ and the other constraints between observables is (2.9b). In particular:

$$\text{if } \quad OM \perp N_-N_+ \quad \text{then} \quad \sigma_+=\sigma_-=\sigma. \qquad (2.12)$$

We remark that when $\rho_{\pm\eta}$ is observed, that is when M_+ and M_- are known, the position of N_+ and N_- does not depend on the polarization degree η so this quantity is "independently" measured by equation (2.9a); see figure 2.

The projection of E_2 on $\mathcal{D}_-$ is the circle of center 0 and of radius r given in (2.11). The seventh amplitude parameter to be measured is obtained by the azimuth of either L_+ or S_+ in $\partial\mathcal{D}_-$.

In the forward or backward direction there is at most one nonvanishing amplitude (ref. 2c). If it is different from zero, it makes a unique prediction: the point $Q_2 \in \partial\mathcal{D}_+$ (see fig. 1); therefore, from the general argument of 1.5iii, $\sigma_+=\sigma_-$ and there are no polarization effects from the target polarization (i.e. N_+ and N_- are in Q_2 and L_+, L_-, S_+, S_- are in 0).

A complete theory gives the amplitudes as functions of the invariants s and t. A model can depend on other arbitrary parameters In any case, any model with some predictive power for this reaction can be characterized by relations between the amplitudes and therefore by more than five constraints between the twelve observables at fixed s and t. To test the model one has to verify that these new constraints are also respected. For instance in a large class of models the helicity amplitude are relatively real (e.g. one particle exchange, here a ρ, or one Regge trajectory exchange); this implies that a B-symmetric polarization state has only even multipoles so $\rho = \rho_\Delta$; since for unpolarized target rank ρ_Δ=rank ρ_e=2, the rank of the observed ρ is also 2, so the point M is on $S_2 = \partial \mathcal{D}_+$ As we saw in 1.5iii the observed polarization of the Δ is independent from the target polarization, so in that case the four non vanising real amplitudes of the model are determined by σ, ρ, σ_+ (1,2,1). If the model imposes more relations among the real amplitudes it will predict a subdomain of $\partial \mathcal{D}_+ = S_2$ for M. For instance the Stodolsky Sakurai model, the naive quark model or the selection rule $\Delta J=1$ all predict that, for all values of t, M is P_2 (see fig. 1) and $\sigma_+ = \sigma_- = \sigma$.

2.3 Reactions $\pi N \to \pi\pi\Delta$ or $KN \to K\pi\Delta$.

If the invariant mass and angular momentum of the final mesons are fixed the reaction is a quasi two body reaction: the full section 2.2 applies here if the meson resonnace has zero spin; otherwise there is no rank condition imposed by angular momentum conservation, although such condition might be implied by the mechanism of the reaction (see next section for similar examples).

In the general case these reactions with more than two final particles or quasi particles are not B-symmetric. If precise measurements of the momenta of the Δ and final mesons are made, since these mesons have spin zero, the rank condition is still valid: with a polarized target the experimental points N_+ N_- L_+ L_- S_+ S_- must be on the sphere $S_4 = \partial\mathcal{D}$ where they form three perpendicular diameters of a sphere S_2. In less precise experiemnts with large bins in the particle momenta these experiemntal points will be the barycenters of the "precise experiment" points (with the differential cross section as weight) so the representative points of these less precise experiments will be somewhat at the interior of $\mathcal{D}$.

2.4 Other parity conserving reactions.

These include all reactions of (2.3) and (2.4). Because of the spin of the particles other than the target nucleon and the Δ, there is no rank condition on ρ_Δ. When there is only one other final particle besides the Δ, the reaction is B-symmetric. The number of complex amplitudes, for instance in reactions (2.3a) or (2.3'a) is $2x2x2x4/2 = 16$; so, for these reactions, the reaction mechanism may depend on 31 real observable functions of s and t. As we saw in 2.1, with the observation of the even polarization of the Δ with arbitrary beam or target polarization, there are only twelve observables to be measured. This is far enough for amplitude reconstruction but it gives a good insight on the reaction mechanism and it can confirm or contradict model predictions.

For instance "factorization" of the amplitudes implies that rank ρ_Δ =rank ρ_e so N_+, N_- are on $S_2 = \partial\mathcal{D}_+$ and S_+S_-, L_+L_- are two perpendicular diameters of the circle $\partial\mathcal{D}_-$. A single Regge trajectory model predicts factorization and reality of the amplitudes and parity conservation at each vertex (see e.g. ref.3); this implies that ρ_Δ is even and has rank two; so rank ρ =2 i.e. M_{ob} is on the sphere $S_2 = \partial\mathcal{D}_+$ .As we saw in 1.5iii and 2.2, whenever the point M_{ob}, representing the observed Δ polarization produced with unpolarized beam and target, is on the boundary of $\mathcal{D}$, the observed Δ polarization is independent from the beam or target polarization ; however the initial polarization $\pm\vec{w}_n$ yields the differential cross sections σ_+, σ_- generally different from σ. The prediction rank ρ=2 is still valid for multi-particle production in a multi-Regge model with single trajectory exchange[3].

Let us also recall that in the forward or backward limit, if the cross section does not vanish, the polarization goes to the kinematical limit studied in 1.6. However if the cross section vanishes, in this limit the representative point m in the intensity and polarization space goes to the cone vertex o (see figure 2) and there is no condition on the polarization limit since it depends on the orientation of the tangent to the trajectory of m at o.

2.5 Neutrino production: $\nu N \to \mu\Delta$, $\nu N \to \nu\Delta$.

In this case the best information is obtained from the two-body reactions; we leave to the reader the extension of the discussion to the multiparticle production. Even the two body reactions are not B-symmetric since they do not preserve parity. Furthermore the neutrino/antineutrino beam is left/right circularly polarized. However time reversal invariance for weak interaction implies that the representative point M_{ob} of the observed polarization produced with unpolarized target is still in $\mathcal{D}_+$ .Of course the five components of the Δ quadrupole polarization should be measured and the verification that those in $\mathcal{D}_-$ vanish is a check of time reversal invariance. The kinematical predictions of 1.6 for the forward and backward limit are not disturbed by the neutrino polarization and since they are only based on angular momentum conservation they are still valid for these parity violating transitions.

The V-A current-current structure of the interaction, well verified for charged currents, predicts that the degree of the μ-polarization is v/c; when the muon enrgy is very large compared to its mass, the muon is completely polarized with same helicity as that of the neutrino/antineutrino beam. From the point of view of the rank condition, in this high energy approximation, the situation is completly similar to that of 2.2: two body meso-production. Even the simplest measurement of the Δ-polarization in neutrino production can tell us important information on the nucleon form factors and principaly on the nature of the neutral currents[7]. There seems to be no experimental data yet available. In figure 3 we compare theoretical predictions with experimental data for the neutrino production of Δ via charged currents.

Conclusion.

It seems that many physicists are unfamiliar with the polarization of particles of spin larger than 1/2, except photons. Nature gives us the spin 3/2 Δ_{1232} in relative abundance and the partial observation of its polarization is so much easier than that of the nucleons. We hope that this paper has shown that this Δ polarization anlysis is not more difficult and yields as much information on physical models.

APPENDIX

The observed polarization domain is the interior of a sphere S_4 of radius $3^{-1/2}$.

One easily proves[2] that for a particle of spin j, the multipoles ρ_L defined in (3') satisfy:

$$\rho_L = (-1)^L \Gamma_j \bar{\rho}_L \Gamma_j^{-1} ; \qquad A(1)$$

where Γ_j is the Wigner matrix defined by

$$\overline{D}_j(R) = \Gamma_j D_j(R) \Gamma_j^{-1}, \quad \bar{\Gamma}_j = \Gamma_j = (-1)^{2j} \Gamma_j^T, \quad \Gamma_j^2 = (-1)^{2j} I. \qquad A(2)$$

Therefore, the projection ρ_E of ρ on the even multipole space

$$\mathcal{E}^{(E)} = \bigoplus_{0 \le L \text{ even} \le 2j} \mathcal{E}^{(L)} \qquad A(3)$$

is given by

$$\rho_E = \frac{1}{2}(\rho + \Gamma_j \bar{\rho} \Gamma_j^{-1}). \qquad A(4)$$

Incidently $\rho > 0$ implies $\Gamma_j \bar{\rho} \Gamma_j^{-1} > 0$ since both matrices are Hermitean and have same eigenvalues, so $\rho_E > 0$. Moreover from A(4) we deduce that:

$$\text{rank}\, \rho_E \le 2\, \text{rank}\, \rho . \qquad A(5)$$

Equation (1.5) applied to ρ_E gives us its polarization degree; let us compute it when ρ describes a pure polarization state $|x\rangle$ i.e. $\rho = |x\rangle\langle x|$ with $\langle x|x\rangle = 1$ and $\bar{\rho} = |\bar{x}\rangle\langle \bar{x}|$. Then from A(4)

$$\text{tr}\, \rho_E^2 = \frac{1}{4}(\text{tr}\, \rho^2 + \text{tr}\, \Gamma_j \bar{\rho}^2 \Gamma_j^{-1}) + \frac{1}{2} \langle x|\Gamma_j \bar{x}\rangle\langle \bar{x}|\Gamma_j^{-1} x\rangle . \qquad A(6)$$

When Γ_j^{-1} is antisymmetric $\langle \bar{x}|\Gamma_j^{-1} x\rangle$ vanishes identically, so when 2j is odd tr $\rho_E^2 = 1/2$ and

for 2j odd ; $d_E = \sqrt{\frac{2j-1}{4j}}$ $\left(=\frac{1}{\sqrt{3}}\right.$ for $\left. j = \frac{3}{2}\right)$. A(7)

To sumarize: when 2j is odd the pure polarization states are projected on the sphere of radius $((2j-1)/4j)^{1/2}$ centered at the origin in the intensity and even polarization space $\mathcal{E}^{(E)}$ of dimension $k=(2j-1)(j+1)$. Of course k=0 for j=1/2. The pure states of polarization form a manifold of dimension 4j in the polarization space $\mathcal{E}(\mathcal{H}_j)$. Its projection on $S_{k-1} \subset \mathcal{E}^{(E)}$ cannot cover completely this sphere if $4j-(k-1) < 0$ i.e. $j>3/2$. We now prove that for j=3/2 it does cover the sphere S_4.

The group SU(2j+1) acts on the 2j+1 dimensional Hilbert space $\mathcal{H}_j$ by its fundamental representation and it acts on the polarization space $\mathcal{E}$ of dimension $(2j+1)^2-1$ by its adjoint representation. Following Racah [6] we consider the chain of subgroups:

for 2j odd, $SU(2j+1) \supset Sp(2j+1) \supset SU(2)$, A(8)

where Sp (2j+1) is the compact symplectic group in 2j+1 dimensions and SU(2) is the universal covering of SO(3). The adjoint representation of SU(2j+1), when restricted to the subgroup Sp(2j+1), decomposes into two irreducible representations: one of them is carried by the space $\oplus_{0<L \text{ even} \leq 2j} \mathcal{E}^{(L)}$ of non-trivial even multipoles; its dimension is $(2j-1)(j+1)$. For j=3/2 this space is $\mathcal{E}^{(2)}$ and its dimension is 5. From the isomorphism of the Lie algebras $Sp(4) \sim SO(5)$ and the unicity of their five dimensional irreducible representation, the orbits of these groups on $\mathcal{E}^{(2)}$ are the sphere S_4 centered at the origin. Since the action $\rho \to u\rho u^*$, $u \in SU(4)$, preserves the eigen values of ρ, the domain $\mathcal{D}$ is invariant by SU(4) and its projection $\mathcal{D}$ on $\mathcal{E}^{(2)}$ is invariant by the subgroup Sp(4); so it is bounded by a sphere S_4 centered at the origin and more precisely by that of radius $3^{-1/2}$ as shown by (A7).

REFERENCES AND FOOT-NOTES

1. For instance a vector meson(ρ,K*) or another Δ could be also produced, so polarization correlation between the final particles could be observed. This rather increases the number of polarization observables. For instance the corresponding study for $\rho\Delta$ or $K^*\Delta$ production, where 19 polarization parameters can be measured, has been made in reference 2d.

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a) Matrice densité de polarization, Gif-sur-Yvette, Summer School 1970 (R. Salmeron editor, Ecole Polytechnique, Palaiseau).
b) Nucl. Phys. B38 (1972) 477.
c) Fortschr. Physik 24 (1976) 259.
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4. In a covariant treatment of polarization $\underline{w}$ is a normalized axial four vector orthogonal to the energy momentum $\underline{p}$. We use here only the particle rest frame, so the time component of $\underline{w}$ vanishes and the spin $\frac{1}{2}$ polarization is completly represented by $\vec{w}$, See e.g. L. MICHEL, N. Cim. Supp. 14 (1959) 95.

5. The action of the forward O(2) group in the space $\mathcal{E}^{(1)}$ of Δ dipole polarization decomposes into the irreducible representation $d_0^- \oplus d_1$. So the initial polarization $\vec{w}_\ell$ may yield some Δ dipole polarization but it is not observed in the decay.

6. G. RACAH, Phys. Rev. 76 (1949) 1353.

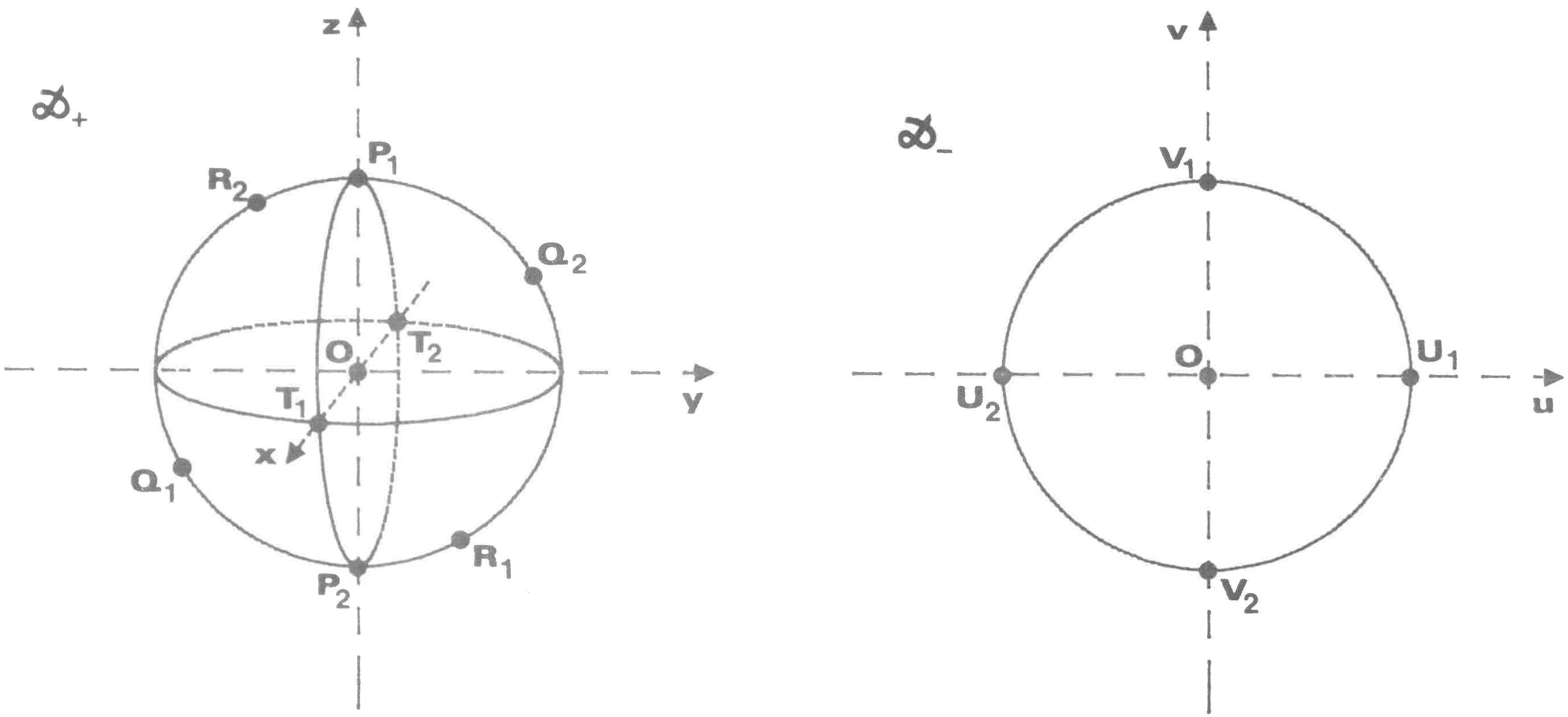
7. A. LE YAOUANC, L. OLIVER, O. PENE, J. C. RAYNAL, C. LONGUEMARE, Phys. Rev. D15 (1977) 2447.

Figures Captions

Fig.1- Projections $\mathcal{D}_+$ and $\mathcal{D}_-$ of the observed polarization domain $\mathcal{D}$ on its B-symmetric and B-antisymmetric spaces $\mathcal{E}_3^{(2)+}$ and $\mathcal{E}_2^{(2)-}$ (Cf. § 1.4). The remarkable points P,Q,R,T,U,V can be defined in an intrinsic way (Cf. § 1.6 and Table 2).

Fig.2- Points $m\, m_+ m_- n_+ n_-$ in the intensity and polarization domain $\mathcal{C}^{(E)}$, and their conic projection $M\, M_+ M_- N_+ N_-$ in the polarization domain $\mathcal{D}$ (cf. § 1.5, 2.1 and 2.2). The points in the segment $n_+ n_-$ satisfy the equalities $m\, m_+ = m\, m_- = \eta(m\, n_+) = \eta(m\, n_-)$. The points $n_+ n_-$ and the whole ellipsoid E_2 are drawn in the surface of the cone, as required in the case of meso-production (Cf. § 2.2).

Fig.3- Plot in $\mathcal{D}^+$ of some model predictions for the reaction $\nu p \longrightarrow \mu^- \Delta^{++}$. (▼) Ph. Salin, Nuovo Cimento 48A (1967) 506 ; (▲) S.L. Adler, Ann. Phys. 50 (1968) 189 ; (◀) J. Bijtebier, Nucl. Phys. B21 (1970) 158 ; (▶) P. Zucker, Phys. Rev. 40 (1971) 3350. The experimental result is taken from P.A. Schreiner and Frank von Hippel, Phys. Rev. Letters 30 (1973) 307.



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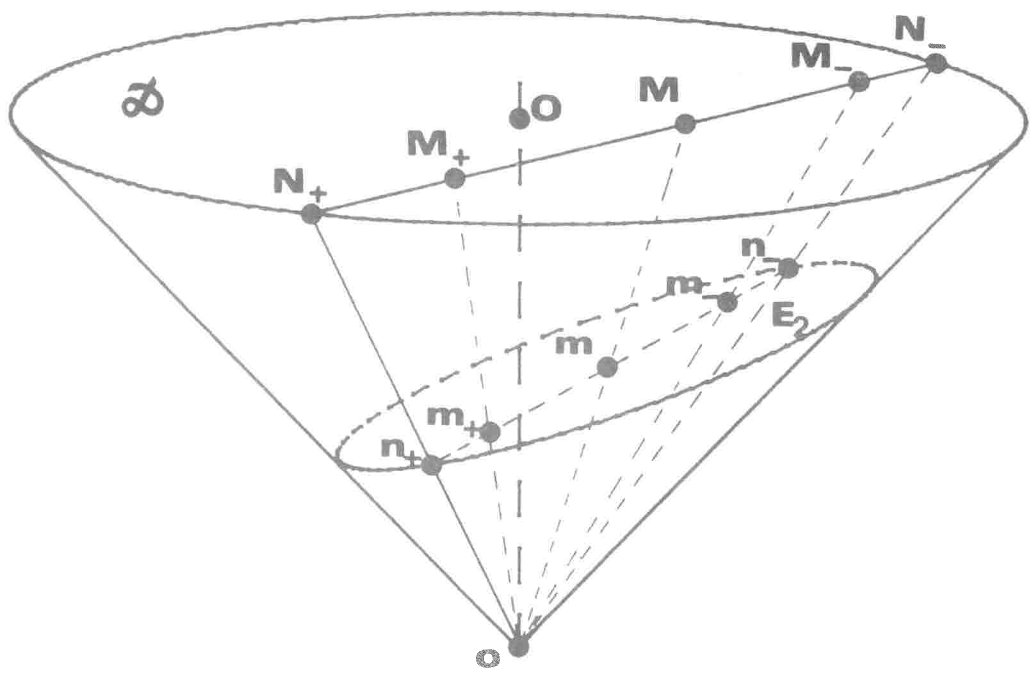

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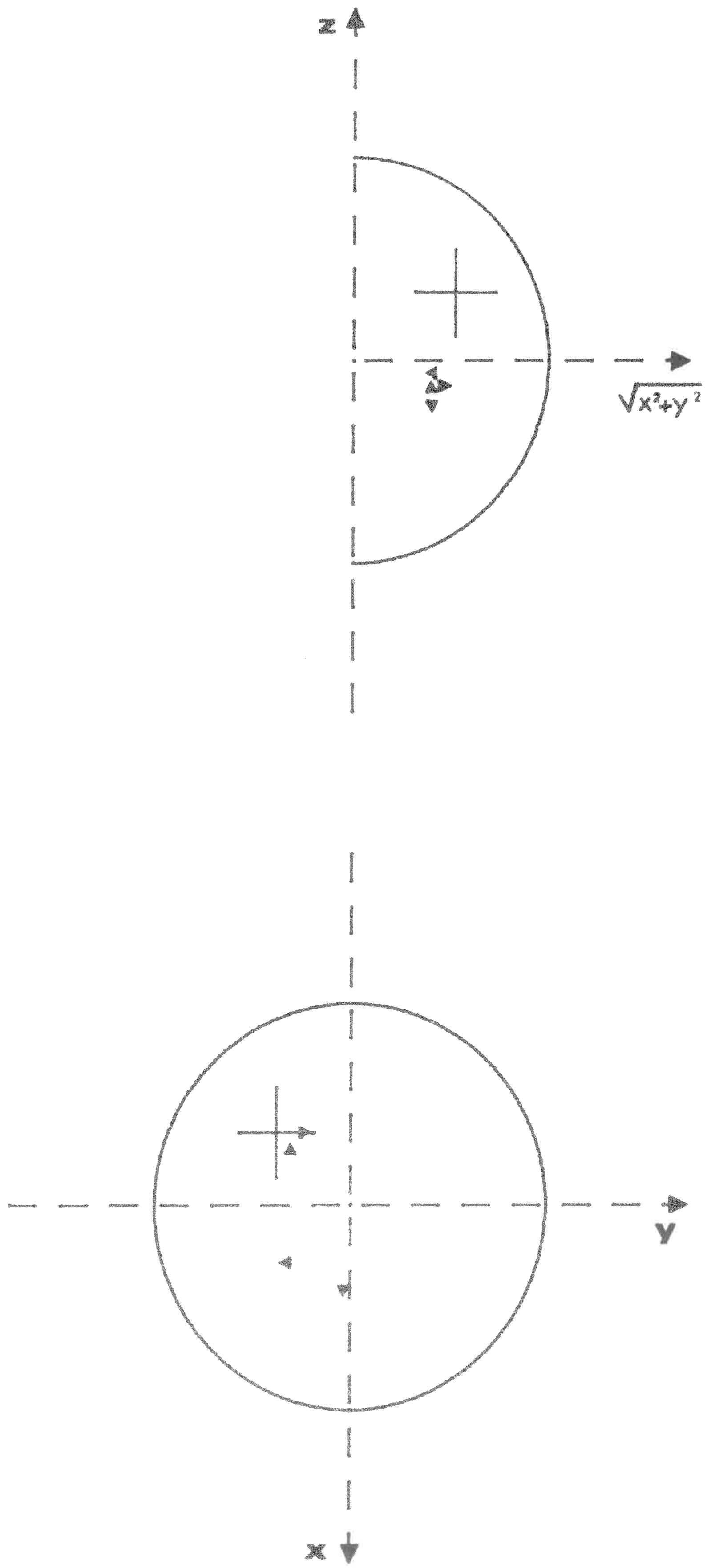

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