Covariant Symmetric Non-associative Algebras on Group Representations

LOUIS MICHEL (*)

1. - Introduction

It is a great pleasure to dedicate this paper to Luigi, as token of our thirty year old friedship. Our two families have been several times neighbours; with our wives, we have seen growing the Radicati and Michel children. It was always a deep joy to meet. We also wrote many letters to each other. This Festschrift is a great circumstance to thank Luigi for all that I learned from him, from his approach to life and... from his approach to physics.

We have published together six papers [MR]. Most of these papers are related to spontaneous symmetry breaking. Most of them use or sharpen a mathematical tool which is the subject of this short paper. Jordan algebras are examples of "covariant symmetric non-associative algebras on group representations" (we shall simply call them "$\vee - algebras$"); they were invented $[JO1,2]$ for the need of physics sixty years ago. The second example of $\vee$-algebras explicitly used in physics was introduced by Gell-Mann [GE] on the octect, i.e. the adjoint representation of SU_3; he called it the "D-algebra". This algebra was introduced independently by Biedenharn [BI] for all SU_n adjoint representations. We made a systematic approach of these algebras and extended them to different representations of Lie groups. In physical applications the vectors of these algebras describe physical states; we emphasized that the idempotents of the algebras, or we can also say, the one dimensional subalgebras:

$$(1.1) \qquad v \vee v = \lambda v$$

are good candidates for states with spontaneous symmetry breaking.

(*)   I.H.E.S., F-91440 Bures-sur-Yvette.

2. - The ∨-algebras and their automorphisms

Consider a vector space $\mathcal{E}$ on the field $K = \mathbb{C}$ or $\mathbb{R}$. It becomes an algebra when we choose a homomorphism:

$$(2.1) \qquad \mathcal{E} \otimes \mathcal{E} \xrightarrow{\alpha} \mathcal{E}.$$

To have a commutative algebra, we consider only the symmetrized tensor product:

$$(2.2) \qquad \mathcal{E} \otimes_S \mathcal{E} \xrightarrow{\alpha} \mathcal{E}.$$

When there are no ambiguities about the algebra we consider, we shall often use the notation:

$$(2.3) \qquad x, y \in \mathcal{E}, \quad x \vee y \overset{\text{def}}{=} \alpha(x \otimes_S y).$$

Given $x \in \mathcal{E}$, the correspondence $\mathcal{E} \ni y \mapsto x \vee y$ is an endomorphism of $\mathcal{E}$ that we denote by:

$$(2.4) \qquad \forall y \in \mathcal{E}, \quad D_x y \overset{\text{def}}{=} x \vee y.$$

The symmetric algebras on $\mathcal{E}$ form a vector space, generally denoted by $\text{Hom}(\mathcal{E} \otimes_S \mathcal{E}, \mathcal{E})$ of dimension $n^2(n+1)/2$ where $n = \dim\mathcal{E}$.

The automorphism group of $\mathcal{E}$ is $GL_n(K)$. It acts naturally on $\text{Hom}(\mathcal{E} \otimes_S \mathcal{E}, \mathcal{E})$. Using the same letter for $g \in GL_n$ and the corresponding isomorphism $\mathcal{E} \xrightarrow{g} \mathcal{E}$, this action is:

$$(2.5) \qquad \alpha \mapsto g \circ \alpha \circ g^{-1} \otimes_S g^{-1}.$$

We denote by $(GL_n)_\alpha$ the stabilizer of α; it is the automorphism group of the algebra defined by α.

In physics, we often start from a symmetry group G and its representation on $\mathcal{E}$ of dimension n; it is given by a homomorphism $G \xrightarrow{T} GL_n$. We do not require the representation to be irreducible and we denote its character by:

$$(2.6) \qquad g \in G, \quad \chi(g) = \text{tr } T(g).$$

We recall that the determinant and the characters of the symmetrized tensor product representation satisfy:

$$(2.7) \qquad \begin{aligned} &\det(T(g) \otimes_S T(g)) = (\det T(g))^{n(n+1)/2}, \\ &tr(T(g) \otimes_S T(g)) = \chi_S(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right) \end{aligned}$$

If T is an irreducible representation of G and if the reduction of the symmetrized tensor product representation $T \otimes_S T$, contains the representation T without multiplicity, there exists a unique (up to a scale factor) G-equivariant map α (sometimes called an intertwining operator) which defines a $\vee$-algebra which has G as group of automorphisms. If T is contained in $T \otimes_S T$ with a multiplicity ν, this is also the dimension of the vector space of the covariant algebras. It may happen that G is a strict subgroup of Aut $\mathcal{A} \subset GL_n$; this occurs more often when the representation T is reducible. Indeed the dimension of the vector space of covariant $\vee$-algebras is then larger than one and for some directions of this vector space the automorphism group of the corresponding $\vee$-algebras might be larger (this is similar to the situation, well known to physicists, of accidental degeneracy accompanied with larger symmetry). A famous mathematical example of this phenomenon is told at the end of section 5.

It may happen that G is a strict subgroup of Aut $\mathcal{A} \subset GL_n$; this occurs more often when the representation T is reducible. That G is an automorphism group of the algebra, means:

$$(2.8) \qquad (T(g)x) \vee (T(g)y) = T(g)(x \vee y).$$

This implies for D_x defined in (2.4):

$$(2.9) \qquad T(g)D_x T(g)^{-1} = D_{T(g)x}.$$

If the representation T is irreducible or, more generally, if it does not contain the trivial representation, it leaves no non trivial linear form invariant. So, by taking the trace of the preceding equation:

$$(2.10) \qquad \text{No vector} \neq 0 \text{ invariant by } T \Rightarrow tr D_x = 0.$$

Unitary group representations are very important in physics: they leave invariant a Hermitian scalar product whose real, imaginary part is a symmetric (= orthogonal), antisymmetric (= symplectic) G-invariant bilinear form. In particular, the unitary representation might be real; if not it can always be considered as an orthogonal representation of double dimension.

We end this section by recalling definitions valid for all types of algebras. Given a $\vee$-algebra on $\mathcal{E}$ and two vector subspaces $\mathcal{B}_1, \mathcal{B}_2 \subset E$, we denote $\mathcal{B}_1 \vee \mathcal{B}_2$ the vector space $\{b_1 \vee b_2, \forall b_1 \in \mathcal{B}_1, \forall b_2 \in \mathcal{B}_2\}$. The vector subspace $\mathcal{B}$ is a subalgebra, an ideal of the $\vee$-algebra $\mathcal{A}$ when:

$$(2.11) \qquad \mathcal{B} \text{ subalgebra of } \mathcal{A} \Leftrightarrow \mathcal{B} \vee \mathcal{B} \subset \mathcal{B}, \quad \mathcal{B} \text{ ideal of } \mathcal{A} \Leftrightarrow \mathcal{B} \vee \mathcal{A} \subset \mathcal{B}.$$

The intersection of two subalgebras (respectively, ideals) is a subalgebra (an ideal). The sum of two ideals is an ideal.

3. - General methods of construction of ∨-algebras

In this section we will consider three general methods to construct ∨-algebras without starting from a group representation, as we have done in the previous section.

As we have seen, the n-dimensional ∨-algebras form a vector space of dimension $n(n + 1)/2$, and we could determine all automorphism groups of these algebras by finding the stabilizers of GL_n on $\mathrm{Hom}(\mathcal{E} \otimes_S \mathcal{E}, \mathcal{E})$ which is equivalent to the action on $(\mathcal{E}' \otimes_S \mathcal{E}') \otimes \mathcal{E}$ where $\mathcal{E}'$ is the dual space of $\mathcal{E}$. This representation of GL_n is the direct sum of two irreducible representations and the stabilizers are the intersections of the stabilizers of the two irreducible representations. This study could be done for each n.

Another approach is possible. The GL_2-representation $g \mapsto g \otimes g$ on the n^2 dimensional space can be realized on the space of $n \times n$ matrices by the action:

$$(3.1) \qquad\qquad m \mapsto gmg^\top.$$

There is a unique decomposition of m into a symmetric and an antisymmetric part:

$$(3.2) \qquad m = s + a, \quad s = \frac{1}{2}\left(m + m^\top\right), \quad a = \frac{1}{2}\left(m - m^\top\right)$$

which is invariant under the action (3.1). The action on the symmetric part realizes the representation $g \otimes_S g$. One must find the subgroups G of GL_n leaving stable an n-dimensional subspace of S, the set of $n \times n$ symmetrical matrices and such that the restriction of the G-representation on this invariant subspace be equivalent to the natural representation $G \subset GL_n$. We shall carry this program for $n = 2$ in the next section. Note that for $n = 1$, there is only (up to scaling) one ∨-algebra of dimension 1. The elements of $K^\times (= \mathbb{C}^\times, \mathbb{R}^\times)$ (the multiplicative group of the field) which are automorphisms of this algebra must satisfy $\lambda^2 = \lambda$, so $\lambda = 1$, the automorphism group is trivial.

The third approach starts from a *symmetric trilinear form* and a *non degenerate symmetric bilinear form*. We first recall how to build a completly symmetrical m linear form from $p_m(u)$ a homegenous polynomial of degree m defined on the n-dimensional vector space $\mathcal{E}_n$. The gradient of p_m at u along the vector $x \in \mathcal{E}_n$ is by definition:

$$(3.3) \qquad\qquad \mathcal{D}_{x.u}p_m(u) = \lim_{\theta \to 0}\ \left(p_m(u + \theta x) - p_m(u)\right)$$

For a fixed u the gradient is a linear form on $\mathcal{E}_n$, for a fixed x, it is a degree $m - 1$ homogeneous polynomial. Moreover, the homogeneity of p_m implies:

$$(3.4) \qquad\qquad p_m(\lambda u) = \lambda^m p_m(u) \Rightarrow \mathcal{D}_{u.u}p_m(u) = mp_m(u).$$

Remark also that:

$$(3.5) \qquad \mathcal{D}_{x.u}\mathcal{D}_{y.u}p_m(u) = \mathcal{D}_{y.u}\mathcal{D}_{x.u}p_m(u).$$

Then:

$$(3.6) \qquad \tilde{p}_m(x_1, x_2, \ldots, x_m) = (m!)^{-1}\mathcal{D}_{x_1.u}\mathcal{D}_{x_2.u}\ldots\mathcal{D}_{x_m.u}p_m(u).$$

is a completely symmetrical m-linear form such that:

$$(3.7) \qquad \tilde{p}_m(u, u, \ldots, u) = p(u).$$

A bilinear form $\tilde{q}(x, y)$ is non degerate if, and only if, $\forall x \in \mathcal{E}_n, \tilde{q}(x, y) = 0 \Rightarrow y = 0$. This is equivalent to say for the quadratic polynomial $q(u)$ (usually called a quadratic form) that the gradients $\mathcal{D}_{x_i.u}p_m(u)$ of n linearly independent vectors x_i are linearly independent. We call this quadratic form non degenerate (in a coordinate system $q(x) = q_{\alpha\beta}x^\alpha x^\beta$ is non degenerate $\leftrightarrow \det q_{\alpha\beta} \neq 0$). Given a homogeneous polynomial t of degree 3 and a non degenerate quadratic form q, one defines the $\vee$-algebra $\mathcal{A}$ on $\mathcal{E}_n$ by:

$$(3.8) \qquad \forall z \in \mathcal{E}_n, \quad \tilde{q}(x \vee y, z) = \tilde{t}(x, y, z).$$

This method allows to build a $\vee$-algebra with a given group of automorphisms G when the linear representation of G on $\mathcal{E}_n$ leaves invariant a non degenerate quadratic form (e.g., it is an orthogonal representation) and also a third degree polynomial. For instance, the permutation group $\mathcal{S}_n$ acting by permutation of coordinates of $\mathcal{E}_n$ is represented by real orthogonal matrices (with elements 1 or 0); it leaves invariant the one dimensional subspace $\mathcal{E}_1'$ of vectors with all components equal and the orthogonal subspace $\mathcal{E}_{n-1}'$ of vectors whose sum of their components vanishes. The polynomial $t = \sum_{\alpha=1}^{n}(x^\alpha)^3$ is evidently $\mathcal{S}_n$-invariant; with the invariant quadratic form $q = \sum_{\alpha=1}^{n}(x^\alpha)^2$, it defines a $\mathcal{S}_n$ covariant $\vee$-algebras. The vector subspaces $\mathcal{E}_1'$ and $\mathcal{E}_{n-1}'$ carry subalgebras.

This last method does not build the most general $\vee$-algebras. Indeed, given a $\vee$-algebra on $\mathcal{E}$, we say that it leaves invariant a symmetric bilinear form (denoted simply by (u, v), it can be degenerate), when:

$$(3.9) \qquad (x \vee y, z) = (x \vee z, y).$$

From the symmetry of the $\vee$ product and the symmetry of the bilinear form, (3.9) defines a completely symmetrical trilinear form that we have denoted $\{x, y, z\}$ in our papers (e.g. [MR5]). With (2.4) and (3.9) we see that D_x is a symmetric operator: $D_x = D_x^\top$. For any symmetric bilinear form on $\mathcal{A}$, we define:

$$(3.10) \qquad \mathcal{B}^\perp \stackrel{\text{def}}{=} \{x \in \mathcal{A}, \forall b \in \mathcal{B}, (x, b) = 0\},$$

LEMMA 1. *If a* $\vee$*-algebra* $\mathcal{A}$ *carries an invariant symmetric bilinear form,* $\mathcal{B}$ *ideal of* $\mathcal{A}$ $\Rightarrow$ $\mathcal{B}^{\perp}$ *ideal of* $\mathcal{A}$. *If the form is non degenerate,* $\mathcal{B} \cap \mathcal{B}^{\perp}$ *is a trivial* $\vee$*-algebra.*

Indeed, $\forall a \in \mathcal{A}$, $\forall b \in \mathcal{B}$, $\forall b' \in \mathcal{B}^{\perp}$, $0 = (b', b \vee a) = (b, b' \vee a) = (b \vee b', a)$. If the form is degenerate, $\mathcal{A}^{\perp}$ is a non trivial ideal.

4. - The 2 dimensional $\vee$-algebras and their automorphisms

It is a classic result on the diagonalization of quadratic forms on $\mathbb{C}$ that in the symmetric representation of $GL_n(C)$:

$$(4.1) \qquad\qquad g \in GL_n(C), \quad s^{\top} = s \mapsto gsg^{\top}$$

there are $n + 1$ orbits; they are characterized by the rank of s. One orbit is open dense, that of the invertible matrices; I belongs to it. Its stabilizer is the group of matrices which satisfy $gIg^{\top} = I$; this is the (complex) orthogonal group O_n. The group which leaves invariant the one dimensional subspace $\{\lambda I\}$ of the $n(n + 1)/2$-dimensional space S is generated by O_n and the dilations:

$$(4.2) \qquad\qquad gg^{\top} = \lambda I, \quad \text{with } \lambda^n = (\det g)^2.$$

In dimension $n = 2$ we use the three Pauli matrices:

$$(4.3) \qquad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

An easy computation yields for the matrices g which satisfy (4.2) for $n = 2$:

$$(4.4) \qquad \lambda = \det g, \; g = aI - bi\tau_2, \quad \lambda = -\det g, \; g = a\tau_3 + b\tau_1,$$
$$\text{with } a^2 + b^2 = \lambda.$$

The group O_2 and the dilations leave also invariant the 2-dimensional subspace of S spanned by the symmetric matrices $\alpha\tau_3 + \beta\tau_1$. Its representation on this space is:

$$(4.5) \qquad \lambda = \det g, \; s \mapsto gsg^{\top} \equiv (aI - bi\tau_2)(\alpha\tau_3 + \beta\tau_1)(a + bi\tau_2)$$
$$\Leftrightarrow g \mapsto (a^2 - b^2)I - 2abi\tau_2;$$

$$(4.5') \qquad \lambda = -\det g, \; s \mapsto gsg^{\top} \equiv (a\tau_3 + b\tau_1)(\alpha\tau_3 + \beta\tau_1)(a\tau_3 + b\tau_1)$$
$$\Leftrightarrow g \mapsto (a^2 - b^2)\tau_3 + 2ab\tau_1.$$

The conditions for this representation to be equivalent to that of (4.4) are:

$$(4.6) \qquad\qquad (a^2 - b^2) = a, \; 2ab = \epsilon b, \; \epsilon = \pm 1.$$

This implies either $b = 0$, so $a = 1$, $g = I$, $g = \tau_3$ or $a = \epsilon/2 \; \epsilon$; but $\epsilon = 1$ is ruled out, because it implies $\det g = 0$. So we obtain finally for the elements g of the automorphism group G of the algebra:

$$(4.7) \qquad I, \quad \frac{1}{2} \left(I \pm \sqrt{3} i \tau_2 \right), \quad \tau_3, \quad \frac{1}{2} \left(\tau_3 \pm \sqrt{3} \tau_1 \right).$$

This is the 6-element group C_{3v}, which is generated by two reflections and is subgroup of the real orthogonal group O_2; it is isomorphic to the permutation group S_3. By polarization equation (4.6) gives the algebra law; if we denote by $x, y; x', y'; \ldots$ the vector coordinates:

$$(4.8) \qquad \begin{pmatrix} x \\ y \end{pmatrix} \vee \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} xx' - yy' \\ -xy' - yx' \end{pmatrix}.$$

It is well known [MI3] that the invariant polynomials of the representation (3.9) of S_3 are polynomials in the two invariants $q = x^2 + y^2$, $t = \left(x^3 - 3xy^2 \right)/3$. So we have obtained an example of application of the third general method of the previous section:

$$(4.9) \quad u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad (u, u) = q, \quad u \vee u = \frac{1}{3} \operatorname{grad} t, \quad (u \vee u, u) = t, \quad u \vee u \vee u = (u, u)u.$$

Remark that this algebra is simple (no ideal except itself and 0). Note also that $(u \vee u) \vee u = u \vee (u \vee u)$ since the algebra is symmetric and the polarization of the third degree invariant yields an invariant symmetric triple product of vectors ($=$ symmetric trilinear form):

$$(4.10) \qquad \left(\begin{pmatrix} x \\ y \end{pmatrix} \vee \begin{pmatrix} x' \\ y' \end{pmatrix}, \; \begin{pmatrix} x'' \\ y'' \end{pmatrix} \right) = xx'x'' - yx'x'' - xy'x'' - xx'y''.$$

We are left to study the second non trivial orbit of $GL_2(C)$ on the symmetric quadratic forms, the orbit of those of rank one. The subgroup of $GL_2(C)$ which leaves invariant the one dimensional subspace $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ is the subgroup G of upper triangular matrices $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Indeed:

$$(4.11) \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} = \begin{pmatrix} a^2\alpha + 2ab\gamma + bd\beta & d(a\gamma + b\beta) \\ d(a\gamma + b\beta) & d^2\beta \end{pmatrix}.$$

The verification is done for $\beta = \gamma = 0$. But the reducible representation of G is not decomposable: it does not leave invariant a two dimensional subspace. For this we have to impose $b = 0$. Then the group of diagonal matrices $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ is represented on the subspace of quadratic forms of coordinates γ, β by $\begin{pmatrix} ad & 0 \\ 0 & d^2 \end{pmatrix}$. The two representations are equivalent if, and only if, $d = 1$.

So the symmetry group is isomorphic to $\mathbb{C}^\times$ and its representation is reducible. The corresponding algebra law is:

$$(4.12) \qquad \begin{pmatrix} x \\ y \end{pmatrix} \vee \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ yy' \end{pmatrix}.$$

The linear forms invariant by the group are, including a factor β:

$$(4.13) \qquad \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}, \text{ i.e. } \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right) = \beta yy'.$$

while the quadratic forms invariant by the algebra depend on two parameters:

$$(4.14) \qquad \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \text{ i.e. } \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right) = \alpha xx' + \beta yy'.$$

So (4.13) shows that the one dimensional subspace of $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is an ideal, while (4.14) shows, for $\beta = 0$ that the one dimensional subspace of $\begin{pmatrix} 0 \\ y \end{pmatrix}$ is also an ideal.

Any two dimensional subalgebra of a $\vee$-algebra is a (sub)algebra of either of the two forms found here.

5. - Other examples of $\vee$-algebras

The SU_n covariant $\vee$-algebra carried by the SU_n adjoint representation (of dimension $n^2 - 1$) [BI] [GE] [MR5] has been used in physics. The vector space of the SU_n Lie algebra can be represented by the $n \times n$ traceless Hermitean matrices $x = x^*$; the invariant orthogonal scalar product, the Lie and $\vee$ algebra laws are normalized to:

$$(5.1) \qquad (x, y) = \mathrm{tr}\, xy, \quad x \wedge y = -\frac{i}{2} [x, y], \quad x \vee y = \frac{1}{2} \{x, y\} - \frac{1}{n} (x, y) I$$

where $[\ ]$ and $\{\ \}$ indicate the commutator and the anticommutator. The $\vee$-algebra is trivial for $n = 2$. The roots r of the algebra have square length 2 and satisfy the characteristic equation:

$$(5.2) \qquad (r, r) = 2, \quad r^n - \frac{1}{2} (r, r) r^{n-2} = 0.$$

So:

$$(5.3) \qquad r \vee r \vee r = \frac{n-2}{n} r$$

Radicati and I have introduced the pseudo roots q:

$$(5.4) \qquad q \stackrel{\text{def}}{=} r \vee r, \quad (q, q) = \frac{2(n-2)}{n}, \quad q \vee q = \frac{n-4}{n} q,$$

$$(q, r) \equiv (r \vee r, r) = 0, \quad q \vee r \equiv r \vee r \vee r = \frac{n-2}{n} r$$

A Cartan subalgebra $\mathcal{C}$ is a maximal Abelian Lie subalgebra: it has dimension $n - 1$. It contains $n(n-1)$ roots that we label r_k, $1 \leq k \leq n(n-1)$. The Cartan subspace $\mathcal{C}$ is also a $\vee$ subalgebra, isomorphic to the $\vee$-algebra on the space $\mathcal{E}'_{n-1}$ we have studied in 3. after (3.8); it has $\mathcal{S}_n$ as automorphism group: this is here the Weyl group of SU_n. For $a \in \mathcal{C}$, the spectra of the operators ad a and D_a on the $n(n-1)$ dimensional space $\mathcal{C}^\perp$ orthogonal to the Cartan subspace are respectively:

(5.5) Spectra on $\mathcal{C}^\perp$:   ad $a = \{i(a, r_k)\}$, $D_a = \{(a, q_k)\}$ with $q_k = r_k \vee r_k$

Remark that with the interplay of these two algebras one makes a Z_2-graded SU_n Lie algebra. Another example of $\vee$-algebra for the 18-dimensional real irreducible representation of $SU_3 \times SU_3$ is given in [MR1].

When we restrict SU_n to SO_n the $n^2 - 1$ dimensional adjoint representation of SU_n decomposes into the direct sum of the adjoint representation of SO_n and its symmetric traceless rank 2 tensor representation of dimension $n(n+1)/2$. The latter carries a SO_n covariant $\vee$ algebra (see e.g. [MI2]). This is also the case of the SO_n representation of symmetric traceless tensors of rank $2k$; (e.g. for $k = 2$ [MI6,8,9]).

Let us give now an example of $\vee$ algebra from the mathematical literature. In 1955, in a famous paper [CH], Chevalley found that all known simple (non Abelian) finite groups are simple Lie groups on finite fields except the five Mathieu groups discovered between 1861 and 1873. From 1965 to 1975, 21 other so-called sporadic groups were discovered and it is now a theorem that there are no more. The largest sporadic group is called indifferently "Friendly giant" or "Monster". It was defined [GR] as the automorphism group of a $\vee$-algebra of dimension 196883. The construction of this $\vee$-algebra is clarified in [TI]. In a n-dimensional orthogonal space, a lattice is said to be even if the norm (x, x) of each vector is even. The Grammian Γ (= matrix of the scalar products of basis vectors) of an even lattice is a $n \times n$ matrix with integer elements. The lattice is self-dual if $|\det \Gamma| = 1$. Even self-dual lattices exist only in dimension multiple of 8; one for $n = 8$, the root lattice E_8; two for $n = 16$, $E_8 \oplus E_8$ and a lattice of D_{16} (well known in string theory); twenty four for $n = 24$, among them there is a unique one with shorter vectors of norm 4: the Leech lattice (found in 1965). The quotient of its symmetry group G, divided by its center $\{I, -I\}$ is a sporadic group Co_1 found by J. Conway (1968). The norm 4 and 6 vectors form two orbits of G whose stabilizers are two other sporadic groups, respectively Co_2 (of index 98280) and Co_3. The smallest irreducible representation of G are orthogonal and of dimension $24, 276, 299, \ldots$.

One builds naturally a larger group $C \supset G$, but I cannot give here the details; on its representation (direct sum of 3 irreducible representations) of dimension $299 + 98280 + 98304$, the C covariant $\vee$ algebras form a 6 dimensional vector space. One (up to a scale factor) of these $\vee$-algebras has an exceptionally large automorphism group: the Friendly Giant.

6. - The meaning of idempotent of $\vee$-algebra in physics

Radicati and I showed that in the space of internal symmetry (essentially flavours at that time) of the fundamental interactions, the direction of spontaneous symmetry breaking are idempotents of $\vee$ algebras [MR1,2,3,4]. This was extended by some of our students [PE] [DA]. It is true that from these algebras one can build G-invariant degree four polynomial (bounded below) similar to those proposed sixty years ago by Landau as mathematical model for second order phase transitions, and, more recently by Higgs (in the Lagrangian of the scalar Higgs field) for the spontaneous breaking of symmetry in gauge theories [MI1,2,4,5,7]. There is a difference with the Landau polynomial; for $\vee$-algebras with an invariant polynomial $(x \vee x, x)$ of degree three, the "Landau" polynomial has a degree three term which excludes second order phase transition, but describes first order phase transitions "not far from second order" as they occur sometimes in crystals and often in liquid crystals. It is true that for an orthogonal representation of a symmetry group G without fixed vectors $\neq 0$, the invariant polynomial of degree 3 are of the form $p(x) = (x \vee x, x) - \frac{3}{2} \lambda (x, x)$, so their extremas satisfy: $x \vee x = \lambda x$. However, for the dimension $n \geq 1$, they are all saddle points: indeed the Hessian is $2D_x - \lambda I$; at an extremum x, $(x, H_x x) = \lambda (x, x)$, $\operatorname{tr} H_x = -n\lambda$ (see 2.10).

The situation is different if we consider a general bifurcation problem with a symmetry group G. At a bifurcation point, the solutions are tangent to the space of an irreducible (orthogonal, if the problem is on the real) representation of G; with some analyticity hypothesis D. Sattinger [SA] (and his papers quoted there) has shown that bifurcations generally occur for irreducible representations with a G-covariant $\vee$-algebra, in the direction of idempotents.

I had the occasion to verify these properties [MI6,8,9] for the renormalization of the Landau-Higgs model. The symmetry group G acts on an orthogonal n-dimensional representation. The physics must be independent from the basis chosen for this representation; indeed the renormalization equation is O_n covariant and the critical exponents are O_n invariants. At that time the bifurcation equation was written [BR] in the $\varepsilon = d - 4$ expansion. It was convenient to consider all Lagrangians with quartic polynomials and to study the renormalization flow as a vector field u in the vector space $\mathcal{P}$ of quartic polynomials; the dimension of $\mathcal{P}$ is $1 + ((n + 6)(n + 1)n(n - 1)/24)$ since we assume that the n-dimensional representation of G is irreducible on the real. There is a O_n covariant $\vee$-algebra on $\mathcal{P}$. The renormalization flow u of a G invariant Lagrangian $\mathcal{L}$ stays tangent to the space $\mathcal{P}^G$ of G invariant quartic

polynomials. In the neighbourhood of small ϵ the number and the type of stability of the renormalization fixed point u^* depend only on the leading term of the renormalization equation, i.e. $u^* \vee u^* = \frac{2}{3} \varepsilon u^*$ if no further degeneracy appears (this is not the case for $n = 4$; this exceptional case has been completely treated in [TO]). Let $\tilde{G}$ the stabilizer of $\mathcal{L}$ in the O_n action; it might be strictly larger than G and it is the true symmetry group of the problem. If there are solutions u_i^*, we can form others by action of the stabilizer in O_n of $\mathcal{P}^G$. It can be shown that the latter is equal to the normalizer $N_{O_n}(\tilde{G})$. I could prove the theorem: if there is a stable fixed point, it is unique. This shows that the symmetry of a stable fixed point satisfies $\tilde{G} = N_{O_n}(\tilde{G})$. With J.C. Toledano, we have studied the physical implications of these results [MI10].

However there is much more to say on the use of covariant $\vee$-algebras in physics; I hope one day to work again on this subject with Luigi!

REFERENCES

[BI]   L.C. Biedenharn, *On the representations of the semi-simple Lie groups. I. The explicit construction of invariants for the unimodular unitary group in N-dimensions.* J. Math. Phys., **4** (1963) 436-445.

[BR]   E. Brezin - J.C. Le Guillou - J. Zinn Justin, *Discussion of critical phenomena for the n-vector model.* Phys. Rev. B10 (1974) 892-900.

[CH]   C. Chevalley, *Sur certains groupes simples.* Tohoku Math. J. (2) **7** (1955) 44-66.

[DA]   C. Darzens, *On chiral symmetry breaking* II. Ann. Phys., **76** (1973) 236-249.

[GE]   M. Gell-Mann, *Symmetries of Baryons and Mesons.* Phys. Rev., **125** (1962) 1067-1084.

[GR]   R.L. Griess Jr., *The Friendly Giant*, Inventiones Math., **69** (1982) 1-102.

[JO]   P. Jordan

1.   *Über Verallgemeinerungsmöglichkeiten der Formalismus der Quantenmechanik,* Gött Nach. (1933) 209-214.

2.   *Über die Multiplikation quantenmechanischer Grössen,* Z. Phys. **80** (1933) 209-214.

[MI]   L. Michel

1.   *Simple mathematical models of symmetry breaking. Application to particle physics.* p. 251-262 in "*Mathematical Physics and Physical Mathematics*", R. Maurin and R. Rączka edit., D. Reidel Publ. Co, Dordrecht 1976.

2.   *Les brisures spontanées de symétrie en physique.* p. 234-245 Colloque C7, supp. J. Physique **36** (1975) C7 41-51.

3.   *Invariants Polynomiaux des groupes de symétrie moléculaire et crystallographique.* p. 75-91 in Proc. 5th Intern. Colloq. Group Theoretical Methods in Physics, Academic Press 1977.

4.   *Minima of Higgs-Landau potentials.* p. 157-203 in *Regards sur la physique contemporaine,* Editions CNRS, Paris 1979.

5.   *The description of symmetry of physical states and spontaneous symmetry breaking.* p. 21-28 in "*Symmetries and broken symmetries in condensed matter physics*" Editor N. Boccara, IDSET Paris 1981.

6.   *Symmetry and the renormalization group fixed points of quartic Hamiltonians*, p. 63-82 in *Symmetry in particle physics*, I. Bars et al. edit. Plenum Press, New-York 1984.

7.   *Landau theory of second order phase transition and invariant theory*. p. 763-773, in *"Group Theoretical method in Physics"*, edit Markov, Man'ko, Shabad, Harwood Academic Pub. 1987.

8.   *Renormalization group fixed points of general n-vector model*. Phys. Rev. B29 (1984) 2777-2783.

9.   *Covariant vector fields and renormalization groups*. p. 162-184 in Proc. 13th International Colloquium in Group Theoretical Methods in Physics. World Scientific, Singapore, 1984.

10.   with J.C. TOLEDANO, *Symmetry criterion to the lack of a stable fixed point in the renormalization group recursion relations*. Phys. Rev. Lett. **54** (1985) 1832-1833.

[MR] L. MICHEL - L. RADICATI

1.   *On the dynamical breaking of $SU(3)$*. p. 19-34 in *"Symmetry principles at high energy"*. (Proc. 5th Coral Gables conf.), Benjamin, New-York 1968.

2.   *Breaking of the $SU(3) \times SU(3)$ Symmetry in Hadronic Physics*, p. 191-203 in *"Evolution of Particle Physics"* Academic Press, New-York 1970.

3.   *Geometrical properties of the fundamental interactions* in *"Mendeleev Symposium"*, Atti Accad. Sci. Torino, Sci. Fis. Mat. Natur., (1971) 377-389.

4.   *Properties of the breaking of Hadronic internal symmetry*. Ann. of Phys., **66** (1971) 758-783.

5.   *The geometry of the octet*. Ann. Inst. Henri Poincaré, **18** (1973) 185-214.

6.   with D. CONSTANTINESCU, *Spontaneous symmetry breaking and bifurcation from the Mac Laurin and Jacobi sequence*. J. Physique, **40** (1979) 147-159.

[PE] F. PEGORARO - J. SUBBA RAO, *Effect of weak interaction in the breaking of hadronic internal symmetry*. Nucl. Phys., B44 (1972) 221-235.

[SA] D.H. SATTINGER, *Bifurcation and symmetry breaking in applied mathematics*. Bull. Am. Math. Soc., **3** (1980) 779-819.

[TI] J. TITS, *Le Monstre*, Séminaire Bourbaki, 620 (1983).

[TO] J.C. TOLEDANO - L. MICHEL - P. TOLEDANO - E. BREZIN, *Fixed points and stability for anisotropic system with 4-component order parameters. Phys. Rev. B31 (1985) 7171-7196.*