POLARIZATION DENSITY MATRIX

How to present its measurement

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"The first problem in data presentation is the realization that the problem of presentation exists. It is possible to compare models with experimental data using an insensitive variable such that agreement is obtained whereas if another variable had been used disagreement would have been found. The real problem is to find variables, or way of displaying the data such that the "physics" of the experiment is clearly visible".

D. R. O. Morrisson
(Proceedings of the Lund International Conference on elementary Particles, p. 261)

There is surely such a problem for the communication of polarization measurements. For several years we have been interested in this problem either in papers of ours, or lectures and mainly in discussions with experimentalists.

This preprint starts a series of preprints where the problem will be studied case by case, for the different values of spin and the different types of experiment. It will contain tables and diagrams directly useful to the experimentalists. We give our planned table of contents in next page. The mark 1 indicates the content of this issue. Following issues will contain a cumulative table.

As much as possible every part is self contained and can be read independently. For more details or for general study (valid for any spin value) we will refer the reader to the appendices, which are labelled A1 , A2 , ...

We hope to receive many comments from our colleagues and we intend to write an improved version before publication.

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0 Introduction.

Relativistic invariance requires that transition rates for particle decay or particle reactions are functions of the momenta of the involved particles only through the Lorentz invariants formed with their energy momentum vectors $\underline{p}$, i.e., $\underline{p}_i^2 = m_i^2$ and $\underline{p}_i \cdot \underline{p}_j$. The value of these invariants can be deduced from the energies and momenta in some particular frame (the center of mass frame, the laboratory frame, etc...). Useful experimental results must be given directly or indirectly in terms of these invariants. For instance for a three particle decay, this is made by using the Dalitz plot or a s, t, u plot†. What is physically relevant in such a decay is the position inside this geometrical plot of the point which represents an event. This plot has moreover the property that its natural measure $d\sigma$ is that of the phase space after integration on the kinematical group of symmetry of this decay, i.e., the rotation group in the rest frame of the decaying particle.

The dimension d of the diagram of invariants, i.e., the number of linearly independent products $\underline{p}_i \cdot \underline{p}_j$ with $i \neq j$ increases with the number n of involved particles

$$d = 3n - 10 \text{ for } n \geq 4, \quad d = 0 \text{ for } n \leq 3 \qquad (1)$$

(for $n = 4$, $d = 2$, indeed $s + t + u = \sum m_i^2$).

So for a many particle final state the high dimension of the diagram of invariants makes its representation more difficult. However some recent work (e.g. Van Hove - 69) have emphasized the usefulness of geometrical plots.

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† In the Dalitz plot of the energies E_i of 3 particles from the decay at rest of a particle with energy momentum $\underline{p}_0 = (m, \vec{0})$, the three invariants are $m E_i = \underline{p}_0 \cdot \underline{p}_i$ and as well known $s = (p_0 - p_1)^2$, $t = (p_0 - p_2)^2$, $u = (p_0 - p_3)^2$.

In the particle reactions made in high energy physics, some of the involved particles have a spin different from zero. Then a complete information on the physics of the reaction requires the measurement of particle polarizations and also, eventually, the use of polarized targets and/or polarized beams. Of course many experiments do measure polarization. This includes experiments producing resonances, because the same approximation which attributes a mass and a spin to a resonance also defines its polarization states.

It must be however recognized that for the description of polarization of particles with spin greater than 1/2, there are many different parametrizations in the literature. Too often the choice of the polarization parameters is made for the comparison with a fashionable theoretical model. But experimentalists who expect their experimental results to be useful for many years, should use a more intrinsic description of polarization, since the mean life of most of the theoretical models is not more than one year !

What is more disheartening is to see that the domain of value of these parameters seems to be unknown to some authors ; at least it is generally not indicated. And often some experimental points appear outside the allowed domain of values.

We see the situation as follows : for each point of the Dalitz plot (i.e., set of values of the p_i) of a given experiment, the measured particle polarization can be represented by a point in a domain $\mathcal{D}$ which is a convex domain in an Euclidian space of dimension N (Table 1 in I.0. gives the value of N as a function of the particle spins and Table 1 in II.0. gives N for one-particle states in given experimental conditions).

In this part, we give general properties of $\mathcal{D}$. In the other parts the domain $\mathcal{D}$ will be computed and described for each spin value and different types of experiments. Remark for instance that there is a natural metric on $\mathcal{D}$ and it is somewhat puzzling that several fashionable parametrizations correspond to a choice of non orthonormal coordinates for $\mathcal{D}$!

What is physically relevant is the geometrical position of each experimental point in $\mathcal{D}$: is it on the boundary ? is it near a remarkable point, etc... ? Indeed, with the discussion of each experimental situation we will indicate which part of the domain $\mathcal{D}$ is preferred by different theoretical models. Given in this intrinsic way, results of an experiment could be

TABLE 1

Number N of polarisation parameters as functions of the particle spin j (with non zero mass)

Single particle state						Two particle state			
j	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, 1)$	$(1, \frac{3}{2})$	$(2, \frac{3}{2})$
N	3	8	15	24	35	15	35	143	399

for one particle $N = (2j+1)^2 - 1$

for k-particles $N = (\prod_{i=1}^{k}(2j_i+1))^2 - 1$

} When all polarisation correlations are included.

more easily compared to any new theoretical model † as soon as it appears.

This method of presenting polarization measurements may make also less difficult the comparison of several identical experiments performed by different groups. The combined result will be given by the barycenter of the representative points corresponding to a same kinematics. Finally, by plotting the margin of errors on the same diagrams it would be worth to see how small there are, compared to the size of the domain ††.

As much as possible, part II has been written independently of this part I, so experimentalists do not need this part for using part II. The main body of this part I.1 to 4 has been written to explain part II and those to come. The 8 appendices to part I have been written(or planed) for further explanations. They also intend to prove every equation or statement used in the practical parts.

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† This would be particularly easy when our theoretical colleagues will take the habit to describe their predictions in the same intrinsic fashion. It is also useful for theoreticists to check whether their predictions are inside the allowed domain !

†† Sometimes published statistical errors are of the size (unknown to their authors) of the domain. One may then wonder to know what is measured.

1. What is Polarization.

Polarization is what one has to measure in order to characterize the kinematical state of a particle whose momentum is known. Indeed the polarization operators form a complete set of commuting observables with the energy momentum operators for the one-particle states.

The pure polarization states of a particle with a given momentum form a Hilbert space $\mathcal{H}_n$ of dimension

$n = 2j+1$ where j is the spin of the particle with positive mass

$n = 2$ for a photon

$n = 1$ for a two-component massless neutrino.

A pure polarization state can be described :

- either by a unit vector $|x\rangle \in \mathcal{H}_n$ (i.e. $\langle x|x\rangle = 1$) defined up to a phase,
- or by the rank one projector $P_x = |x\rangle\langle x|$ which is Hermitian $P^* = P$ and satisfies $\mathrm{tr}\, P_x = \langle x|x\rangle = 1$.

A general polarization state is a "mixture" and is described by a "polarization density operator" ρ , represented by a $n \times n$, trace one, Hermitian matrix (i.e. $\rho = \rho^*$, $\mathrm{tr}\,\rho = 1$) which is positive, i.e., all its eigenvalues λ_i are non negative. Indeed such an operator can be decomposed into (summation over repeated indices is implied)

$$\rho = \lambda_i P_i \tag{1}$$

where P_i are orthonormal, rank one, Hermitian projectors on an orthonormal set of eigenvectors of ρ

$$P_i P_j = \delta_{ij} P_j \tag{2}$$

Then the λ_i's which satisfy

$$\begin{aligned} \lambda_i &= \mathrm{tr}\,\rho P_i \\ \lambda_i &\geq 0 \\ \Sigma_i \lambda_i &= \mathrm{tr}\,\rho = 1 \end{aligned} \tag{3}$$

are probabilities to observe the particle whose polarization is described by ρ , into the pure polarization states P_i . The number of $\lambda_i \neq 0$ is called the rank of the matrix. When all λ_i's are different, the decomposition (1) is unique.

For a system of k particles (with non zero masses) whose momenta are known, the Hilbert space of polarization states has dimension $n = \prod_{i=1}^{k} (2j_i+1)$; since it is the tensor product of the $\mathcal{H}_{2j_i+1}$ for each particle. The polarization state of the system of k particles is described by a density operator ρ, represented by a $n \times n$, trace one, Hermitian matrix. So equations (1), (2) and (3) extend to this case : indeed the mathematical description of polarization is similar for the case of 1 or k particles. However the physical interpretation is different : in the latter case one has to measure the correlation between these polarizations in addition to the polarization of each particle. This is quite analogous to the measurement of momenta. Consider the example of a sample of π^o decaying at rest into two photons. The momentum distribution of each photon is isotropic (since π^o has spin 0) but momentum conservation requires a strict correlation between the momenta of the two photons : these momenta are opposite. Similarly the polarization distribution of each photon is isotropic, but angular momentum , P and T conservations require a strict correlation between the two photon polarizations (there are diametrically opposite points on the Poincaré sphere, i.e., orthogonal states in the two-dimensional Hilbert space of polarization, since π^o is pseudoscalar).

2. The domain $\mathcal{D}_n$ of polarization density matrices.

The $n \times n$ Hermitian matrices form a $n^2 = N + 1$ dimensional Euclidean space $\mathcal{E}_{N+1}$ whose scalar product is

$$(\rho_1 , \rho_2) = \mathrm{tr}\ \rho_1 \rho_2 \qquad (4)$$

(The integer N was given in Table I for some spin values).
Of course orthogonal states in $\mathcal{H}_n$ (i.e $\langle x | y \rangle = 0$) are orthogonal in $\mathcal{E}_{N+1}$ (indeed $P_x P_y = 0$), and the scalar product (4) has therefore an obvious physical meaning.
The condition $\mathrm{tr}\ \rho = 1$ defines a Euclidean subspace $\mathcal{E}_N$ of $\mathcal{E}_{N+1}$.
And the condition $\mathrm{tr}\ \rho^2 = 1$ defines a sphere $\mathcal{S}_N$ of unit vectors.
The positive matrices form a cone $\mathcal{C}_N$ whose vertex is 0, the center of $\mathcal{S}_N$.
This positivity domain is convex, i.e., if $\rho_i \in \mathcal{C}_n$ then $\sum_i \lambda_i \rho_i \in \mathcal{C}_n$ when $\lambda_i \geqslant 0$.

We call polarization domain $\mathcal{D}_n$ the set of polarization density operators; their characteristic properties : hermiticity, trace one, positivity imply , see Fig. 1

$$\mathcal{D}_n = \mathcal{E}_N \cap \mathcal{C}_n$$

Rank one Hermitian projectors which correspond to pure states (i.e., $\rho^* = \rho = \rho^2$, $\mathrm{tr}\ \rho = 1$) belong to the boundary $\partial \mathcal{D}_n$ of $\mathcal{D}_n$ and to the sphere $\mathcal{S}_{N-1}$ intersection of the plane $\mathcal{E}_N$ and the sphere $\mathcal{S}_N$.
The center of this sphere $\mathcal{S}_{N-1}$ is $\rho_o = \frac{1}{n} \mathbb{1}$ the density matrix of the unpolarized state ; its radius is $\sqrt{(n-1)/n}$. Except in the case of one particle of spin $\frac{1}{2}$ (then $N = 3$) the pure states of polarization cover only a small part of $\mathcal{S}_{N-1}$.

For each value of n we will change the scale of $\mathcal{E}_N$, so that the radius of $\mathcal{S}_{N-1}$ be 1 . Then the distance of the point ρ to ρ_o , the center of $\mathcal{S}_{N-1}$, will be the degree of polarization d_ρ of ρ :

$$d_\rho = \left[\frac{n}{n-1}\ \mathrm{tr} (\rho - \rho_o)^2 \right] \frac{1}{2} \qquad (5)$$

with

$$0 \leq d_\rho \leq 1 . \qquad (5')$$

We will call $\mathcal{E}'_N$ the space $\mathcal{E}_N$ with this new scale and ρ' the point of $\mathcal{E}'_N$ corresponding to ρ

$$\rho' = \sqrt{\frac{n}{n-1}}\,(\rho - \rho_0) \tag{6}$$

We have then the properties

$$\rho'^* = \rho' \,,\ \mathrm{tr}\,\rho' = 0 \,,\ 0 \leq \mathrm{tr}\,\rho'^2 \leq 1 \tag{7}$$

$$\rho' + \sqrt{\frac{n}{n-1}}\,\rho_0 > 0 \tag{7'}$$

$$\mathrm{tr}\,\rho'_1\,\rho'_2 = \frac{n}{n-1}\,\mathrm{tr}\,\rho_1\,\rho_2 - \frac{1}{n-1} = \frac{1}{n-1}\left[n(\rho_1\,,\rho_2)-1\right] \tag{8}.$$

In the appendices A6, A7, A8 we will continue the general study of the geometrical shape of $\mathcal{D}_n$.

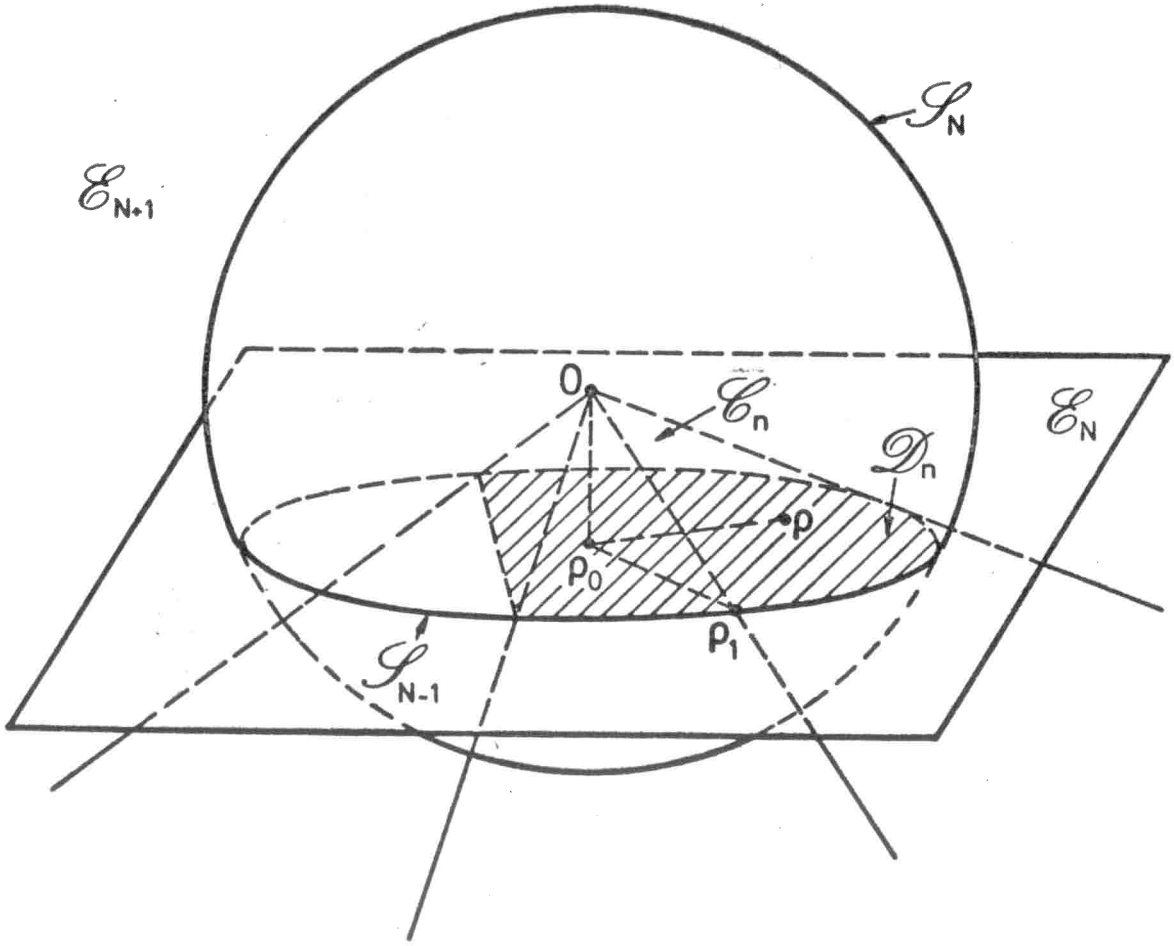

FIG.1

Fig. 1. The Euclidian space $\mathcal{E}_{N+1}$ of Hermitian operators on $\mathcal{H}_n$, the subspace $\mathcal{E}_N$ of trace one operators, and the unit sphere $\mathcal{S}_N$. The polarization domain $\mathcal{D}_n$ is the intersection of the convex cone $\mathcal{C}_n$ of positive operators and the subspace $\mathcal{E}_N$. Pure states are represented by the points of the boundary of $\mathcal{D}_n$ contained in the subsphere $\mathcal{S}_{N-1} = \mathcal{S}_N \cap \mathcal{E}_N$. By a dilatation of center O and factor $\sqrt{\frac{n}{n-1}}$ the radius of $\mathcal{S}_{N-1}$ is made unity, and then for any ρ the distance $d_\rho = |\rho - \rho_0|$ gives the degree of polarization of ρ.

3. The choice of reference frame and coordinate system.

It is of course useful to use a coordinate system in $\mathcal{E}_N$. This is even a necessity for the processing of physical information by computer; the density operator ρ has to be described by a set of real numbers.

This set will depend on two choices :

a) Choice of a reference frame for each particle.

To each particle we associate a tetrad, i.e., a set of four vectors $\underline{n}^{(\alpha)}$ $\alpha = 0, 1, 2, 3$, which are orthonormal, i.e. :

$$\underline{n}^{(\alpha)} \cdot \underline{n}^{(\beta)} = g^{\alpha\beta} \tag{9}$$

and whose time axis $\underline{n}^{(0)}$ is $\underline{p}/m$ where $\underline{p}$ is the energy momentum of the particle. Often this tetrad is specified by choosing the three (space-like) axes in the rest frame of the particle and transforming them by the boost (i.e., the pure Lorentz transformation) transforming the four-vector $(m, \vec{0})$ into $\underline{p} = (p^o, \vec{p})$. By convention the axis $\underline{n}^{(3)}$ is the "quantization axis". Up to ten years ago, the most usual choice used to be $\underline{n}^{(3)}$ along the normal to the scattering plane in the two-body collision $A + B \longrightarrow C + D$, i.e.,

$$\underline{n}^{(3)} \cdot \underline{p}_A = \underline{n}^{(3)} \cdot \underline{p}_B = \underline{n}^{(3)} \cdot \underline{p}_C = \underline{n}^{(3)} \cdot \underline{p}_D = 0 . \tag{10}$$

Then the quantization axis $\underline{n}^{(3)}$ is common to the four tetrads. This is called nowadays a transversity frame. The choice of $\underline{n}^{(1)}$ and $\underline{n}^{(2)}$ for each particle can be fixed for instance by choosing a "channel" e.g. s- or t- or u-transversity.

Helicity axis are also used; Jacob-Wick-59 introduced the s-helicity and Jackson the t-helicity, etc... In Appendix A 1 we will describe explicitly these different possible frames.

b) Choice of a coordinate system for $\mathcal{E}_N$.

By the choice of frame, the matrix which describes the density operator ρ is well defined and it can be parametrized by its matrix elements or by their real part and their imaginary part in order to use a set of real numbers. However, these numbers do not correspond to the value of the coordinates of ρ for an orthonormal basis of the Euclidean space $\mathcal{E}_N$. A multipole

expansion (for the rotation group) of the density matrix exhibits more physical properties. Let us first consider one particle states : then $n = 2j+1$.

The real and imaginary parts of the t^L_M , coefficients of the multipole expansion form a set of orthogonal coordinates. However there are different normalizations of these coefficients in the literature and most of them destroy the natural metric of $\mathcal{D}_n$. In Appendix A2 we define the multipole parameters that we shall use and denote by r^L_M . Then the degree of polarization d_ρ which is the distance in $\mathcal{E}'_N$ between ρ' and ρ'_o (see equation 6) is given by

$$d_\rho = \sum_{L,M} (r^L_M)^2 \qquad (1 \leqslant L \leqslant n-1 = 2j \ , \ -L \leqslant M \leqslant L)$$

The multipole expansion depends on the physical nature of $\mathcal{E}_N$.

As any linear operator on $\mathcal{H}_n$, ρ is transformed by the rotation † R into

$$\rho \rightsquigarrow D^{(j)}(R) \rho D^{(j)*}(R) \tag{11}$$

So the rotation group acts on the space $\mathcal{E}_{N+1}$ by the linear representation (up to an equivalence : $\sim$) :

$$D^{(j)} \otimes \bar{D}^{(j)} \sim D^{(j)} \otimes D^{(j)} \sim \oplus_{L=0}^{L=2j} D^{(L)} \tag{12}$$

where $\bar{D}^{(j)}$ is the complex conjugate of $D^{(j)}$.

Then ρ can be expanded into irreducible tensor operators :

$$\rho = \sum_{L=0}^{2j} \rho^{(L)}$$

with

$$\rho^{(0)} = \frac{1}{n} \mathbb{1} \ .$$

The linear combination of elements of ρ which transform as the irreducible representation $D^{(L)}$ of the rotation group form the L-multipole $\rho^{(L)}$ of the polarization state. The observation of the angular distribution of decay products of the polarized particle yields directly the value of some multipoles e.g., the even L-multipoles for the strong or electromagnetic (i.e. parity conserving) two-particle decays.

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† By "rotation" we mean rotation for the particle at rest and more generally transformation of the little group $\mathcal{L}_{\underline{p}}$ the subgroup of elements of the Lorentz group which leave the particle energy momentum $\underline{p}$ invariant.

For two particle states n is $(2j_1 + 1)(2j_2 + 1)$, and ρ is transformed by the "rotation" group according to

$$\rho \rightsquigarrow \{D^{(j_1)} \otimes D^{(j_2)}\}\ \rho\,(D^{(j_1)} \otimes D^{(j_2)})^* \qquad (13)$$

instead of (11). So ρ can be expanded into double-multipoles (L_1, L_2) with $0 \leq L_1 \leq 2j_1$, $0 \leq L_2 \leq 2j_2$, whose properties are straightforward generalization of single multipoles.

$$\rho = \sum_{L_1=0}^{2j_1} \sum_{L_2=0}^{2j_2} \rho^{(L_1, L_2)} \qquad (14)$$

Note that $\rho^{(L_1, 0)} = \rho_1^{(L_1)}$, $\rho^{(0, L_2)} = \rho_2^{(L_2)}$

where ρ_i is the polarization density matrix of the particle i when the polarization of the other particle is not observed (or not taken into account).
When $\rho - (\rho_1 \otimes \rho_2) \neq 0$, there is polarization correlation.

In the ideal case of very accurate polarization measurement, the position of the representative point of ρ in the domain $\mathcal{D}_n$ must be independent of the choice of the reference frame and polarization parameters (elements of ρ, multipoles parameters, etc...). However, when one varies the kinematics (considering for instance ρ as a function of s and t in a two-body collision), the transformation from a conventional frame of reference to another conventional frame may depend on the kinematics, so the trajectory (e.g. as function of s, t) of the representative point will look different in the two conventions. To give an example, in the two-body collision (see for details Appendix A1 §7 - 9) the transformation from an helicity frame to the transversity frame in the same channel is independent of the kinematics (it is a $-\frac{\pi}{2}$ "rotation" around the common $\underline{n}^{(1)}$ axis). But the transformation from the s-helicity frame (Jacob and Wick) to the t-helicity frame (Gottfried and Jackson) does depend on s and t. We suggest that it might be worthwhile to process the data in the computer through different coordinate systems and frames and see how the final results -grouped by bins- yield points in the domain $\mathcal{D}_n$ depending on the intermediate parametrization. If they do, it must be checked that it can be reasonably explained by a kinematics-dependent transformation.

Since such a transformation cannot be made on a published data grouped by bins, it might be worthwhile in some cases to publish the different diagrams corresponding to the different "physical" choice of frame (a s- or t- or u-channel frame is best suited for studying particle or trajectory exchanges in the corresponding channel).

For each value of spin and each type of experiment we propose a set of diagrams. Some of them average over these kinematical transformations, loosing physical information, but allowing a possible direct comparison between differently processed data.

We will not discuss difficult questions of physical interpretation which occur from two facts :

a) The accuracy of most experiments is limited by poor statistics. Of course such experiments yield only a limited physical information ; here we shall deal very little with the question : how to obtain all information contained inaa limited statistics, although we shall give a few hints.

b) The majority of the new particles are resonances, produced with a background. If the background is incoherent and unpolarized, its only effect is to decrease proportionally the degree of polarization defined in (5). However the situation is generally more complicated : for instance one may have interference between different orbital waves, and we will also study later such situations (see part V.). It is the art of physicists to know when a simple approximation cannot be usefully applied. But it is necessary to know first the simple approximation !

4. The domain $\hat{\mathcal{D}}_n$ which depends on the experiment.

We have discussed up to now the most general polarization density operator for the one or several particle states with given momenta. However, in an actual experiment the polarization density operator, may not be the most general one and we shall denote by $\hat{\mathcal{D}}_n$ its domain of value.

Angular momentum and parity conservation may imply constraints on ρ. Then $\hat{\mathcal{D}}_n$ is a subdomain of $\mathcal{D}_n$. We will consider essentially two types of conditions, which are studied in detail in Appendix A 3. One type is on the rank of ρ (i.e., the dimension of the space $\rho\mathcal{H}_n$) which has to be smaller than or equal to a given integer $k < n$. The other type of condition yields that $\hat{\mathcal{D}}_n$ is the intersection of $\mathcal{D}_n$ with a subspace $\hat{\mathcal{E}}_N$ of $\mathcal{E}_N$. In part II we shall treat explicitly, for each spin value, the following situation. The beam and the target are unpolarized; either it is a two-body reaction $A + B \rightarrow C + D$ or, when there are more final particles, one observes the polarization of only one of them and one averages over the momenta of all other final particles. In that case the experiment has a symmetry plane (three-dimensional plane in space-time). Parity conservation in such reaction implies that the polarization density operator is in a subspace $\mathcal{E}_B \subset \mathcal{E}_N$. The dimension of this subspace is :

$$\dim \mathcal{E}_B = \begin{cases} \frac{1}{2} n^2 = \frac{N+1}{2} & \text{if } n \text{ even} \\ \frac{1}{2}(n^2 - 1) = \frac{N}{2} & \text{if } n \text{ odd.} \end{cases} \qquad (15)$$

As we shall see in Appendix A 6, $\hat{\mathcal{D}}_n = \mathcal{E}_B \cap \mathcal{D}_n$ is also a convex domain.

In many experiments this whole domain $\hat{\mathcal{D}}_n$ cannot be observed or is not observed. That is, only a subset of the polarization parameters is observed while the value of the other is not known; for instance we have said that the observation of a two-body parity conserving decay allows to measure only the even L - multipoles. In all the physical situations we will study, this means that one observes only the orthogonal projection of ρ on a linear subspace H of $\hat{\mathcal{E}}_N$. For instance H can be the subspace $\mathcal{E}_{even}$ of even multipoles in the decomposition of $\mathcal{E}_N = \mathcal{E}_{even} \oplus \mathcal{E}_{odd}$ into a direct sum of orthogonal subspaces.

In general Γ, the projected image of $\hat{\mathcal{D}}_n$ on H (one also says, the apparent contour of $\hat{\mathcal{D}}_n$ on H) is strictly larger than the intersection $C = \hat{\mathcal{D}}_n \cap H$. (See Fig. 2). Of course $C = \Gamma$ when H is a symmetry r - plane of $\mathcal{D}_n$ (r = dim H). This is the general case met in part II and part III. In those parts we will study B - symmetric experiments (so $\rho \in \mathcal{E}_B$) and consider both cases : the odd part of the polarization is or is not measured.

The equality $C = \Gamma$ may still hold when H is not a symmetry plane of $\hat{\mathcal{D}}_n$ (e.g. : $\hat{\mathcal{D}}_n$ is an egg and H is its equatorial plane). This occurs for instance when one measures only the diagonal elements of ρ. When Γ is larger than C , the matrix which would be obtained by putting zero for the unobserved elements of a density matrix ρ would be non positive if its projection on Γ is outside C (see part V.).

In part II and part III, for each spin value , and for respectively one and two particle states, we will study explicitly the different domains $\hat{\mathcal{D}}_n$ of possible polarization density operators for different types of experiments ; we will give the physical signification of its different regions and eventually place the theoretical predictions of different models in $\hat{\mathcal{D}}_n$.

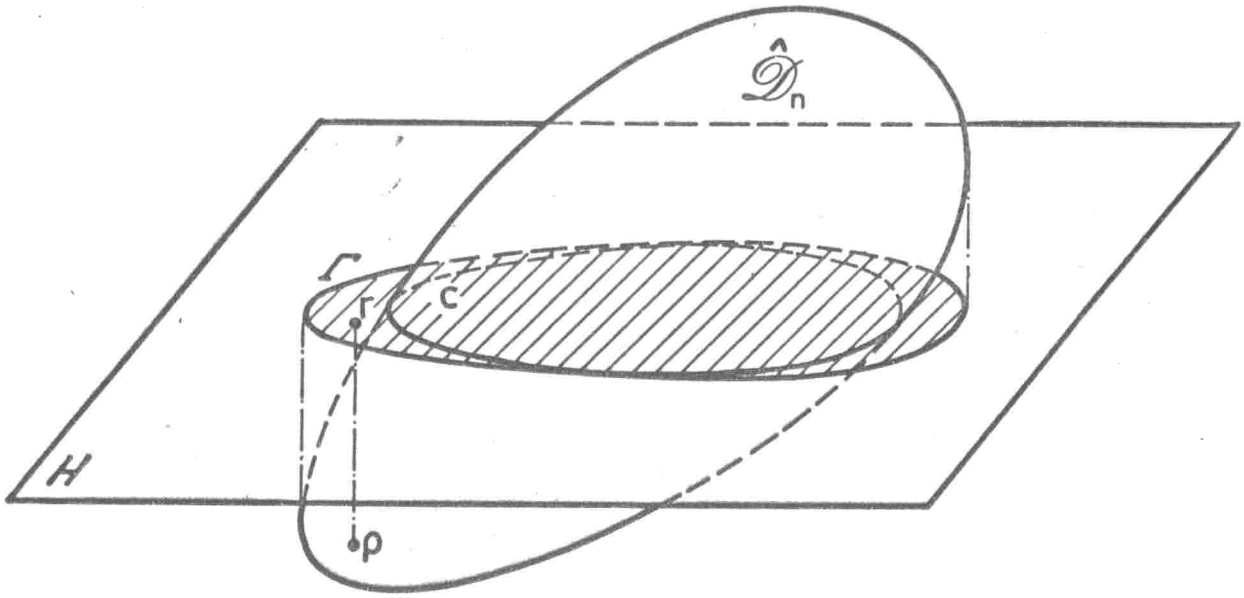

FIG. 2

Fig. 2. Γ is the vertical projection of $\hat{\mathcal{D}}_n$ on the horizontal two-plane H and $C = \hat{\mathcal{D}}_n \cap H$. Both C and Γ are convex and $C \subset \Gamma$.

Part I. APPENDIX 1

COVARIANT DESCRIPTION OF POLARIZATION AND COVARIANT FRAMES

1. THE PARTICLE AT REST

Consider a spin j , mass m particle at rest. Let $\mathcal{H}_{2j+1}$ the 2j+1 dimensional Hilbert space of its polarization pure states. It carries the irreducible unitary linear representation for short ("irrep") D^j of the rotation group SO(3) (more exactly one must use SU(2), the covering of SO(3), when j is half integer). Any operator on $\mathcal{H}_{2j+1}$ is transformed by the (covering of the) rotation R according to

$$A \rightsquigarrow D^j(R)\, A\, D^{*j}(R) = D^j(R)\, A\, D^j(R)^{-1}. \qquad (1)$$

The action (1) leaves invariant the Hermitian metric :

$$\langle A | B \rangle = \mathrm{tr}\, A^* B \ ; \qquad (2)$$

it also transforms the Hermitian operators into themselves. For the real vector space of the Hermitian operators of $\mathcal{L}(\mathcal{H})$, the scalar product (2) reduces to an Euclidian scalar product. We can also say that the rotation group acts on the space $\mathcal{L}(\mathcal{H})$ of linear operators on $\mathcal{H}$ according to the unitary representation :

$$D^j \otimes \overline{D}^j \sim D^j \otimes D^j \sim \oplus_{L=0}^{2j} D^L \qquad (3)$$

where $\sim$ means equivalent. The elements of the irreducible orthogonal subspaces of $\mathcal{L}(\mathcal{H})$ are called irreducible tensor operators or 2^L- multipole operators. So we can write the decomposition into multipole operators

$$\rho = \sum_{L=0}^{2j} \rho^{(L)}$$

$\rho^{(o)}$ is a multiple of $\mathbb{1}$. The orthogonality of the irreducible subspaces yields

$$\text{tr } \rho^{(o)} \rho^{(L)} = 0 = \text{tr } \rho^{(L)} \quad , \quad L \neq 0 \qquad (5)$$

and the condition $\text{tr } \rho = 1$ implies

$$\rho^{(0)} = \frac{\mathbb{1}}{2j+1} \mathbb{1} \qquad (6).$$

The unpolarized state has $\rho^{(0)}$ for density operator.

To represent operators by matrices one has to choose a frame 0xyz in our space (here always right-handed). By convention Oz is always chosen as the "quantization axis" i.e., the unit eigenvectors of the rotations around the axis Oz are taken for the orthonormal basis $|j\mu\rangle$

$$\langle j\mu' | j\mu \rangle = \delta^{\mu'}_{\mu} \qquad (7)$$

of $\mathcal{H}_{2j+1}$.

Then a vector $|x\rangle \in \mathcal{H}$ is

$$|x\rangle = \xi^{\mu} |j\mu\rangle \qquad (8).$$

It is transformed by R according to $|x'\rangle = \xi'^{\mu'} |j\mu'\rangle$ with

$$\xi'^{\mu'} = D^{j}(R)^{\mu'}{}_{\mu} \xi^{\mu} \qquad (9).$$

The corresponding density operator is

$$\rho = |x\rangle\langle x| = \rho^{\mu}{}_{\nu} |j\mu\rangle\langle j\nu| \qquad (10)$$

where its matrix elements are

$$\rho^{\mu}{}_{\nu} = \xi^{\mu} \bar{\xi}^{\nu} = \bar{\rho}^{\nu}{}_{\mu} = \langle j\mu|\rho|j\nu\rangle \qquad (10').$$

2. THE PARTICLE IN MOTION

All this is well-known to physicists. But when the particle is not at rest, too many physicists seem scared, although things are very simple and what to do was explained more than 30 years ago by E. P. Wigner in one of his many fundamental papers "On unitary representations of the inhomogeneous Lorentz group" Ann. Math. 40, 149 (1939).

We denote $\mathcal{H}_{2j+1}(p)$ the 2j+1 dimensional Hilbert space of pure polarization states of the particle of energy momentum $\underline{p} = (E, \vec{p})$ (with $\underline{p}^2 = m^2 = E^2 - \vec{p}^2$). The (covering of the) rotation group has to be replaced by the subgroup $\mathcal{L}_p$ of the (covering of the) homogeneous Lorentz group $\mathcal{L}$ which is the little group of $\underline{p}$.

$$\mathcal{L}_p = \text{set of } \Lambda \in \mathcal{L} \ , \ \Lambda \underline{p} = \underline{p} \ . \qquad (11)$$

This group is isomorphic to the rotation group :

indeed let Λ_p be the boost, i.e., the pure Lorentz transformation, along $\vec{p}$ and of velocity $\vec{p}/E$; then $\mathcal{L}_p$ is the conjugated of the rotation subgroup $SO(3)$ of $\mathcal{L}$ by Λ_p, i.e.

$$\mathcal{L}_p = \Lambda_p \cdot SO(3) \cdot \Lambda_p^{-1} \qquad (12) .$$

We will still call the elements of the little group $\mathcal{L}_p$ "rotations" but use this word between quotation marks. Given a Minkowski frame : time axis + Oxyz, its transformed by Λ_p is called a "tetrad" for the particle. It is a set of four orthonormal vectors

$$\underline{n}^{(0)} = \frac{\underline{p}}{m} , \quad \underline{n}^{(i)} \quad (i = 1, 2, 3) ; \qquad (13)$$

the set of $\underline{n}^{(i)}$ is denoted sometimes by $\vec{\underline{n}}$.

They satisfy

$$\underline{n}^{(\alpha)} \cdot \underline{n}^{(\beta)} = g^{\alpha\beta} \qquad (14) .$$

The right handed orientation is defined by

$$\det (n^{(0)}, n^{(1)}, n^{(2)}, n^{(3)}) = \frac{1}{4!} \epsilon^{\lambda\mu\nu\rho} \epsilon_{\alpha\beta\gamma\delta} n_\lambda^{(\alpha)} n_\mu^{(\beta)} n_\nu^{(\gamma)} n_\rho^{(\delta)} = 1 . \qquad (14')$$

By convention $\underline{n}^{(3)}$ is the quantization axis. To summarize, for each particle one has to choose a tetrad. It is often convenient but not compulsory to choose for the different particles the tetrads obtained by "boosting" the same reference frame. The use of any other element of $\mathcal{L}$ which transforms $(m, \vec{0}) = m\underline{t}$ into $\underline{p} = (E, \vec{p})$ is permissible. Such an element is a product $\Lambda_p R$, when R is a rotation.

3. WIGNER FORMALISM

We can now define the infinite dimensional Hilbert space $\mathcal{H}_{(m, j)}$ of all states of a particle of mass m and spin j. Indeed such a state is known when the value of the 2j+1 coordinate $\chi^\mu(\underline{p})$ are known for all $\underline{p}$ of the positive energy sheet Ω_m of the mass hyperboloid $\underline{p}^2 = m^2$. To summarize :

$\mathcal{H}_{(m, j)}$ is the Hilbert space of functions χ_μ defined on Ω_m with value in $\mathcal{H}_{2j+1}$, which have a finite norm for the scalar product $\ll \chi | \psi \gg$ in $\mathcal{H}_{(m, j)}$

$$\ll \chi | \psi \gg = \int_\Omega < \chi(p) | \psi(p) > \frac{d^3 \vec{p}}{2\sqrt{\vec{p}^2 + m^2}} \qquad (15)$$

where $\langle \chi | \psi \rangle = \sum_{\mu=-j}^{j} \bar{\chi}^{\mu}(p)\, \psi^{\mu}(p)$ is the scalar product in $\mathcal{H}_{2j+1}(\underline{p})$ and

$$\frac{1}{2}(\vec{p} + m^2)^{-\frac{1}{2}}\, d^3\vec{p} = \int_{p^o > 0} \delta(\underline{p}^2 - m^2)\, d^4\underline{p} \qquad (15')$$

is the invariant surface measure on Ω_m. This space $\mathcal{H}_{(m,j)}$ is called by mathematicians a direct integral of the $\mathcal{H}_{2j+1}(p)$

$$\mathcal{H}_{(m,j)} = \int_{\Omega_m}^{\oplus} \mathcal{H}_{2j+1}(\vec{p})\, \frac{d^3\vec{p}}{2\sqrt{\vec{p}^2 + m^2}} \qquad (16)$$

and $\chi^{\mu}(p)$ is called by physicists the wave-packet. A transformation of the (covering of the) Poincaré group $\mathcal{P}$ is the product of a homogeneous transformation $\Lambda \in \mathcal{L}$, the (covering of the) Lorentz group and of a translation in space and time $\underline{a}$. So an element of $\mathcal{P}$ is a pair $(\underline{a}, \Lambda)$. The group law is

$$(\underline{a}, \Lambda)(\underline{b}, \mathbf{M}) = (\underline{a} + \Lambda \underline{b},\ \Lambda \mathbf{M}) \qquad (17)$$

Wigner - 39 showed that the unitary representation $(\underline{a}, \Lambda) \rightsquigarrow U^{(m,j)}(\underline{a}, \Lambda)$ is defined by

$$\left[U^{(m,j)}(\underline{a}, \Lambda)\chi\right]^{\mu}(\underline{p}) = e^{i\underline{a}\cdot\underline{p}}\, D^{j}(R(\underline{p}, \Lambda))^{\mu}{}_{\mu'}\, \chi^{\mu'}(\Lambda^{-1}p) \qquad (18)\ .$$

When $\Lambda \underline{p} = \underline{p}$ i.e., $\Lambda \in \mathcal{L}_p$ then the rotation $R(\underline{p}, \Lambda)$ is that which appears in (12)

$$R(p, \Lambda) = \Lambda_p^{-1}\, \Lambda\, \Lambda_p \qquad (19)$$

If $\Lambda \underline{p} = p' \neq p$ then the rotation† $R(\underline{p}, \Lambda)$ is :

$$R(\underline{p}, \Lambda) = \Lambda_{p'}^{-1}\, \Lambda\, \Lambda_p = \Lambda_{\Lambda p}^{-1}\, \Lambda\, \Lambda_p \qquad (20)\ .$$

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† As we said this choice is convenient, but not compulsory. Wigner established the necessary and sufficient condition :

$$R(\underline{p}, \Lambda)\, R(\Lambda^{-1}\underline{p}, \mathbf{M}) = R(\underline{p}, \Lambda \mathbf{M})\ .$$

Of course in the particular case of a particle at rest $\underline{p} = m\underline{t}$, for a genuine rotation R, equation (18) reduces to

$$U^{(m,J)}(R)\,\psi^{\mu}(m\underline{t}) = D^{J}(R)^{\mu}_{\ \nu}\,\psi^{\nu}(m\underline{t})$$

which is just (9).

Given a tetrad $\underline{p}/m$, $\underline{n}^{(i)}$ it should become clear what is the "rotation" $R_i(\underline{p}, \theta)$ around the axis $n^{(i)}$ by an angle θ : points in the two-plane $\underline{p}, \underline{n}^{(i)}$ are unchanged, those in the two plane $n^{(j)}, n^{(k)}$ (i, j, k is a permutation of $1, 2, 3$) "rotate" by an angle θ.

Explicitly (see for instance Wigner [5])

$$D^{J}(R_3(\underline{p}, \varphi))^{\lambda}_{\lambda'} = e^{i\lambda\varphi}\,\delta^{\lambda}_{\lambda'}, \qquad -J \leqslant \lambda \leqslant J, \qquad -J \leqslant \lambda' \leqslant J \tag{21}$$

$$D^{J}(R_2(\underline{p}, \pi))^{\lambda}_{\lambda'} = e^{i\pi(J-\lambda)}\,\delta^{\lambda}_{-\lambda'} \tag{22}.$$

4. POLARIZATION OBSERVABLES

Relativistic invariance is expressed by the invariance under the Poincaré group $\mathcal{P}$. It implies the conservation of energy, momentum, and relativistic angular momentum. We denote by $E = P^0$, P^i, $M^{\mu\nu} = -M^{\nu\mu}$, the Hermitian operators on $\mathcal{H}$ the Hilbert space of physical states which correspond to the coordinates of these observables. These operators satisfy the commutation relations : ($\mu\nu\lambda\rho\sigma = 0, 1, 3, 4$, $i = 1, 2, 3$)

$$\left[P^{\mu}, P^{\nu}\right] = 0 \tag{23}$$

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = ig^{\mu\rho} M^{\nu\sigma} + ig^{\nu\sigma} M^{\mu\rho} - ig^{\mu\sigma} M^{\nu\rho} - ig^{\nu\rho} M^{\mu\sigma} \tag{23'}$$

$$\left[P^{\lambda}, M^{\mu\nu}\right] = ig^{\lambda\mu} P^{\nu} - ig^{\lambda\nu} P^{\mu} \tag{23"}$$

These operators form a representation on $\mathcal{H}$ of the Lie algebra of $\mathcal{P}$. The polynomials in P's and M's form a representation of $\tilde{\mathcal{P}}$, the "envelopping algebra" and they also are kinematical observables (when they are Hermitian). For example ($\epsilon_{0123} = 1$)

$$W_{\lambda} = \frac{1}{2}\,\epsilon_{\lambda\mu\nu\rho}\,P^{\mu}\,M^{\nu\rho} \tag{24}$$

$$\underline{P}^2 = P^{\mu} P_{\mu} = P^{\mu} P^{\nu} g_{\mu\nu} \tag{25}$$

$$\underline{W}^2 = W^{\mu} W_{\mu} = W_{\mu} W_{\nu} g^{\mu\nu} \tag{25'}$$

We remark that

$$\left[W^{\lambda}\ ,\ P^{\mu}\right] = 0 \qquad (26)$$

$$\underline{P}\ .\ \underline{W} = P^{\lambda}\ W_{\lambda} = 0 \qquad (27)$$

and also that $\underline{P}^2$, $\underline{W}^2$ commute with all P'_s and M'_s and therefore with all their polynomials. In other words, they are in the center of the envelopping algebra and they represent relativistic invariant observables.

For instance, if $\mathcal{H} = \mathcal{H}_{(m,\,j)}$ the Hilbert space of the states of a particle of non-vanishing mass m and spin j, the P'_s and M'_s form an irreducible representation of the Lie algebra of $\mathcal{P}$ and

$$\underline{P}^2 = m^2\ I \quad , \quad \underline{W}^2 = -m^2\ j(j+1)\ I \qquad (28)$$

where I is the identity operator on $\mathcal{H}_{(m,\,j)}$.

To completely characterize a one particle state, one must extract from the equations (23), (23'), (23") a complete set of commuting observables. When one includes the energy momentum P^{λ} in this set, the operators necessary to complete it are the polarization observables. As equation (26) shows we can choose functions of the W'_s . It must be noticed that the W'_s do not commute among each other; indeed from (24) and the (23)'s ,

$$\left[W_{\lambda}\ ,\ W_{\mu}\right] = i\,\epsilon_{\lambda\mu\nu\rho}\,P^{\nu}\,W^{\rho} \qquad (29)$$

One shows that a complete set of observables (one can also say : a maximal abelian subalgebra of $\hat{\mathcal{P}}$) including the P'_s is obtained by adding $\underline{W}^2$ and one component of W .

We can now use the direct integral decomposition of (16) for the operators on $\mathcal{H}_{(m,\,j)}$. For example

$$W^{\lambda} = \int_{\Omega_m}^{\oplus} W^{\lambda}(\underline{p})\ \frac{d^3\vec{p}}{2\sqrt{\vec{p}^2+m^2}} \qquad (30)$$

where $W^{\lambda}(p)$ is an operator on the 2j+1 dimensional Hilbert space $\mathcal{H}_{2j+1}(\underline{p})$.

Let $\underline{n}^{(\alpha)}(\underline{p})$ a tetrad for the energy momentum $\underline{p}$ (i.e., $n^{(0)} = \underline{p}/m$). We define for $m \neq 0$

$$S^{(\alpha)}(p) = \frac{1}{m}\ W(p).n^{(\alpha)} = \frac{1}{m}\ W^{\lambda}(p).n^{(\alpha)}_{\lambda} \qquad (31)$$

and we have equivalently (i, j, k = 1, 2, 3)

$$W(p) = -m\ \sum_i\ S^{(i)}(p)\ \underline{n}^{(i)}(p) \qquad (31')$$

Equation (27) means $S^{(0)} = 0$. We deduce from the commutation relations (29)

$$\left[S^{(i)}, S^{(j)} \right] = i \epsilon^{ij}_{k} S^{(k)} \tag{32}$$

This shows that the $S^{(i)}$ are the generators of the "rotations" of the little group $\mathcal{L}_p$ of $\underline{p}$. They can be called the covariant spin operators.

$\underline{W}^2$ and $S^{(3)}(\underline{p})$ complete the set of commuting observables containing P and, from our definition of polarization in I.1, $\underline{W}^2$ and $S^{(3)}(\underline{p})$, $\underline{p} \in \Omega_m$ are the polarization observables.

Of course

$$\sum_i (S^{(i)}(p))^2 = j(j+1)\, 1_{\underline{p}} \tag{33}$$

where I_p is the identity on $\mathcal{H}_{2j+1}(p)$.

Then the polarization density matrix of a particle of spin j, energy momentum $\underline{p}$ is (Michel - 59)

$$\rho(\underline{p}) = \frac{1}{2j+1} - s_\lambda \frac{W^\lambda}{m}(\underline{p}) + s_{\lambda\mu} \frac{W^\lambda}{m}(\underline{p}) \frac{W^\mu}{m}(\underline{p}) + \ldots + (-1)^{2j} s_{\lambda\mu\ldots\sigma} \overbrace{\frac{W^\lambda}{m}(\underline{p}) \frac{W^\mu}{m}(\underline{p}) \ldots \frac{W^\sigma}{m}(}^{2j \text{ factors}} \tag{34}$$

where the $s_{\lambda\mu\ldots}$ are respectively axial vector, second rank completely symmetric zero trace tensor, ..., zero trace ($s^\lambda_{\lambda\mu\ldots} = 0$), completely symmetric 2j-rank (pseudo if 2j odd) tensor. The conditions imposed on the $s^{\lambda\mu\ldots}$ by the positivity are partly given in Henry - de Rafael - 65.

If one replaces $\underline{W}(p)$ by its value in equation (31') where $S^{(i)}(p)$ are the three Hermitian $(2j+1)\times(2j+1)$ matrices satisfying (32), then $\rho(p)$ is the $(2j+1)\times(2j+1)$ matrices that we study in part I.

In § 3 we did not explain Wigner's treatement of the zero mass case. Let us consider it shortly here, from this operator algebra point of view. $\mathcal{H}$ is the Hilbert space spanned by the state-vectors of a particle with zero mass. So

$$P^2 = 0 \tag{35}$$

We choose a time axis $\underline{t} = \underline{n}^{(0)}$; let $n^{(\alpha)}$ be a tetrad with $\underline{p} = \alpha (\underline{n}^{(0)} + \underline{n}^{(3)})$ i.e., $\alpha = p^o$.

Equation (27) then implies

$$S^{(0)}(\underline{p}) + S^{(3)}(\underline{p}) = 0 \tag{36}$$

so $$\underline{W}^2(\underline{p}) = -S^{(1)}(\underline{p})^2 - S^{(2)}(\underline{p})^2 = -\xi^2 \leqslant 0 \qquad (37)$$

which is independent of $\underline{p}$.

The commutation rules of the $S^{(i)}$ are

$$\left[S^{(3)}, S^{(1)} \right] = i\alpha S^{(2)} \qquad (38)$$

$$\left[S^{(3)}, S^{(2)} \right] = -i\alpha S^{(1)} \qquad (38')$$

$$\left[S^{(1)}, S^{(2)} \right] = 0 \qquad (38'')$$

This is the Lie algebra of the two dimensional Euclidean group which is isomorphic to $\mathcal{L}_p$ when $\underline{p}^2 = 0$ and $\underline{p} \neq 0$.

Wigner showed that when $-\xi^2$ in equation (37) is different of zero dim $\mathcal{H}(p)$ is infinite, so he called this case zero mass, infinite spin representation $U_{(0,\infty)}$ of $\mathcal{P}$ (there are two inequivalent irrep. corresponding to the integer and half integer spin).

The zero mass finite spin case is given by $\xi = 0$, so

$$\underline{P}^2 = 0 \quad , \quad \underline{W}^2 = 0 \qquad (39)$$

which implies

$$\underline{W} = \lambda \underline{P} \qquad (40)$$

and the pseudoscalar λ (indeed W is an axial four-vector and P a polar four-vector) is the helicity. Then $\mathcal{H}(\underline{p})$ is one dimensional. When one adds space reflexions, we must double the dimension of $\mathcal{H}_{\underline{p}}$ in order to have $\pm\lambda$ for the helicity value. For the photon $\lambda = \pm 1$ (right and left circularly polarized photons). For neutrinos $\lambda = -\frac{1}{2}$ and for antineutrinos $\lambda = \frac{1}{2}$.

PARITY AND TIME REVERSAL (m ≠ 0 case)

We have spoken only of the "rotation" group. In order to study parity conservation, we have also to consider space inversion : $i_s(t, \vec{r}) = (t, -\vec{r})$ and space symmetries : the product of space inversion by an element of the connected Lorentz group.

Only relative parities are measured in quantum mechanics ; but a universally used convention gives a parity $\eta = \pm 1$ to each particle at rest ; and we denote $D^{(j,\eta)}$ the corresponding representation of the (covering of the) rotation group enlarged with space inversion i_s :

$$D^{(j,\eta)}(i_s) = \eta \mathbb{1}_{2j+1} \, . \qquad (41)$$

In the three-dimensional space a reflexion $S_{\vec{n}}$ through the plane orthogonal to $\vec{n}$ is the product of i_s and the rotation of π around $\vec{n}$. Covariantly one defines also a reflexion S_n through the three-dimensional plane orthogonal to the space like unit vector $\underline{n}$ with $\underline{n}^2 = -1$ and $\underline{n} \cdot \underline{p} = 0$ (See Fig. 1)

$$S_n \underline{a} = \underline{a} + 2(\underline{a} \cdot \underline{n})\, \underline{n} = i_s(\underline{p}) \cdot R_n(\underline{p}, \pi)\, a \tag{42}$$

where $R_n(\underline{p}, \theta)$ is the "rotation" of angle θ around $\underline{n}$ and

$$i_s(\underline{p})\, \underline{p} = \underline{p}\,, \quad i_s(\underline{p})\, \underline{n}^{(i)} = -\underline{n}^{(i)}\,. \tag{43}$$

So, with the use of (21), (22)

$$D^{(j,\eta)}(S_n(3))^{\lambda}_{\lambda'} = \eta\, e^{-i\pi\lambda}\, \delta^{\lambda}_{\lambda'} \tag{44}$$

$$D^{(j,\eta)}(S_n(2))^{\lambda}_{\lambda'} = \eta\, e^{-i\pi(j-\lambda)}\, \delta^{-\lambda}_{\lambda'} \tag{45}$$

The tensor product of the irreps $D^{(j,\eta)}$ of the orthogonal group is

$$D^{(j_1,\eta_1)} \otimes D^{(j_2,\eta_2)} \sim \bigoplus_{L=|j_1-j_2|}^{j_1+j_2} D^{(L,\eta_1\eta_2)} \tag{46}$$

Since $\eta = \pm 1$ is real, equation (1) shows that for every L, $\rho^{(L)}$ transforms under $D^{(L,+1)}$. For instance for $L = 1$, the dipole polarization is an <u>axial</u> vector.

For a particle at rest, one must also add to the invariance group the time inversion : $i_t(t, \vec{r}) = (-t, \vec{r})$ which acts on $\mathcal{H}_{2j+1}$ by an antiunitary transformation $V(i_t)$ defined by :

$$\left[V(i_t)\chi\right]^{\lambda} = D^{j}(i_t)^{\lambda}_{\lambda'}\, (\bar{\chi})^{\lambda'} = (-1)^{j-\lambda}\, \delta^{-\lambda}_{\lambda'} (\bar{\chi})^{\lambda'} \tag{47}$$

The covariant generalization is $i_t(\underline{p})$ defined on space-time. Since $V(i_t(\underline{p}))$ is an antiunitary operator, in the Fourier transformed space of energy momenta, E does not change sign and $\vec{p}$ does.

Explicitly we use the convention :

$$\left[V(i_t(p))\chi\right]^{\lambda}(\underline{p}) = (-1)^{j-\lambda}\, \delta^{-\lambda}_{\lambda'}\, (\bar{\chi})^{\lambda'}(\underline{p}') \tag{48}$$

where $\underline{p}' = (E, -\vec{p})$.

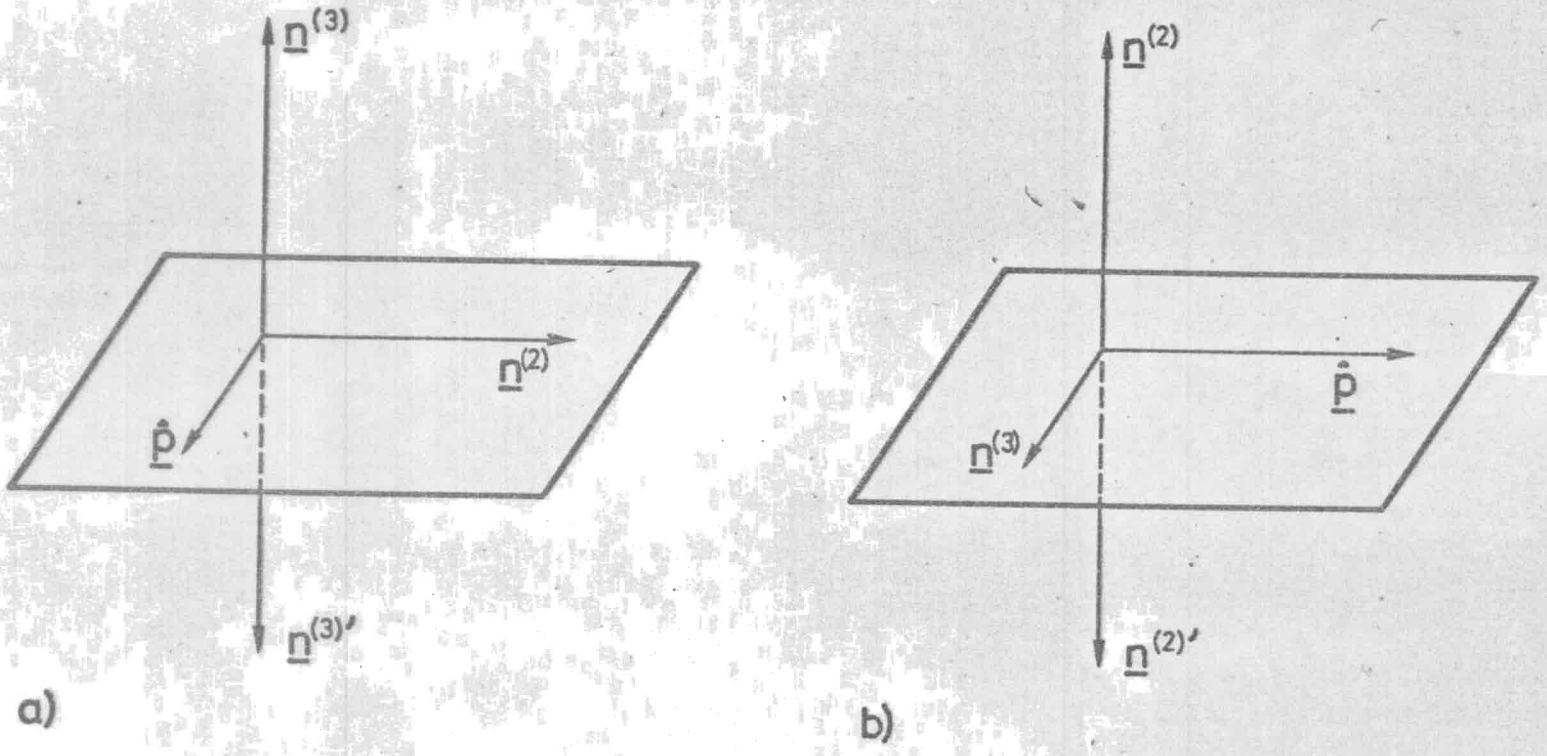

FIG. 1

Fig. 1. - The axis $n^{(1)}$ is not represented. a) Reflexion $S_{n^{(3)}}$ through the three-dimensional plane $\hat{p}$, $\underline{n}^{(1)}$, $\underline{n}^{(2)}$. b) Reflexion $S_{n^{(2)}}$ through the plane $\hat{p}$, $\underline{n}^{(1)}$, $\underline{n}^{(3)}$.

6. THE Γ MATRIX

We will have to study also charge conjugation and crossing symmetry. For these discrete operations and others we will have to use the Γ^j matrix which exhibits the equivalence of D^j and $\bar{D}^j$.

Indeed

$$\bar{D}^j(R) = \Gamma^j D^j(R)(\Gamma^j)^{-1} \tag{49}$$

where

$$(\Gamma^j)^{\lambda}_{\lambda'} = (-1)^{j-\lambda}\,\delta^{-\lambda}_{\lambda'} \tag{50}$$

Note that

$$(\Gamma^j)^2 = (-1)^{2j}\,\mathbb{1}_{2j+1} \tag{51}$$

and

$$(\Gamma^j)^T = (-1)^{2j}\,\Gamma^j \quad . \tag{51'}$$

(Remark that $(-1)^{j+\lambda} = (-1)^{-j-\lambda}$ and $(-1)^{4j} = 1$).

Equations (22) and (45) show that

$$D^j\left[R_2(\underline{p}, -\pi)\right] = \Gamma^j \quad , \tag{52}$$

$$D^{(j,\eta)}\left[S_n(2)\right] = \eta\,\Gamma^j \quad . \tag{53}$$

7. COVARIANT FRAMES

One tetrad must be chosen for each spinning particle. By convention this tetrad will be always right-handed (see 14'), i.e.,

$$\det(\underline{p}, \underline{n}^{(1)}, \underline{n}^{(2)}, \underline{n}^{(3)}) = m > 0 \quad , \tag{54}$$

and $\underline{n}^{(3)}$ will be chosen as quantization axis. Hence, to specify a tetrad, one has to choose only two of their three vectors $n^{(i)}$ orthogonal to $\underline{p}$, e.g., $\underline{n}^{(2)}$ and $\underline{n}^{(3)}$. One could publish with the experimental results a drawing of the laboratory representing also the chosen axis! It is more customary to fix these two unit vectors in a covariant way, i.e., as function of the four-momenta of the other particles in the same reaction. Let i be the label of the particle whose tetrad we want to fix. In order to define these unit vectors orthogonal to $\underline{p}_i$, we will consider pairs $\underline{p}_j, \underline{p}_j$, and triplets $\underline{p}_j, \underline{p}_k, \underline{p}_\ell$ of linearly independent four-momenta (the latter is only possible for reactions involving at least four particles!). We call a k-plane the linear subspace generated by k such linearly independent vectors.

A two-plane which contains a time like vector, $\underline{p}_i$, does also contain space like vectors; we will call it a hyperbolic two-plane. We can define such a two-plane $\underline{p}_i, \underline{p}_j$, by choosing another particle j (we assume that particles i and j are not relatively at rest). In this two-plane we can define the space like unit vector

$$\underline{q}_i(j) = \frac{1}{\mathrm{sh}\varphi_{ij}} (\hat{\underline{p}}_i \, \mathrm{ch}\varphi_{ij} - \hat{\underline{p}}_j) \ , \tag{55}$$

where

$$\hat{\underline{p}}_i = \underline{p}_i / m_i \ , \quad \mathrm{ch}\varphi_{ij} = \hat{\underline{p}}_i \cdot \hat{\underline{p}}_j \ , \quad \text{and} \quad \mathrm{sh}\varphi_{ij} > 0 \ . \tag{55'}$$

Then $\hat{\underline{p}}_i$ and $\underline{q}_i(j)$ form a orthonormal basis,

$$\hat{\underline{p}}_i \cdot \underline{q}_i(j) = 0 \ , \quad \hat{\underline{p}}_i^{\,2} = 1 \ , \quad \underline{q}_i(j)^2 = -1 \ . \tag{56}$$

We could consider two such orthonormal basis related by

$$\hat{\underline{p}}_i \cdot \hat{\underline{p}}_j = \underline{q}_i(j) \cdot \underline{q}_j(i) = \mathrm{ch}\varphi_{ij} \ , \tag{57}$$

$$\hat{\underline{p}}_i \cdot \underline{q}_j(i) = \underline{q}_i(j) \cdot \hat{\underline{p}}_j = \mathrm{sh}\varphi_{ij} \ . \tag{57'}$$

As function of the invariants

$$m_{ij}^2 = (\underline{p}_i + \underline{p}_j)^2 \tag{58}$$

we can write

$$\mathrm{ch}\varphi_{ij} = (m_{ij}^2 - m_i^2 - m_j^2)/2m_i m_j \ , \tag{59}$$

and

$$\mathrm{sh}\varphi_{ij} = \Delta(m_{ij}^2, m_i^2, m_j^2)^{1/2}/2m_i m_j \ , \tag{60}$$

with the notation

$$\Delta(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx \ . \tag{60'}$$

A three-plane containing a time like vector, $\underline{p}_i$, can be fixed by choosing two other four-momenta $\underline{p}_k$, $\underline{p}_\ell$ (we assume $\underline{p}_i$, $\underline{p}_k$, and $\underline{p}_\ell$ linearly independent, i.e., in the system of one of these three particles at rest, the other two are not colinear). One can define, as another space like unit vector, the normal $\underline{n}$ to such a three-plane $\underline{p}_i$, $\underline{p}_k$, $\underline{p}_\ell$, by

$$\underline{n} \cdot \underline{p}_i = 0 \ , \quad \underline{n} \cdot \underline{p}_k = 0 \ , \quad \underline{n} \cdot \underline{p}_\ell = 0 \ , \quad \underline{n}^2 = -1 \ . \tag{61}$$

The ambiguity of its sign can be removed by

$$\det(\underline{n}, \underline{p}_i, \underline{p}_k, \underline{p}_\ell) > 0 \ . \tag{61'}$$