We could now make a general convention for fixing the tetrad of any final particle i, produced in a collision between a beam particle b and a target particle t. For instance : we choose the two unit vectors $\underline{q}_i(b)$ and $\underline{n}$, the normal to the three-plane $\underline{p}_i, \underline{p}_b, \underline{p}_t$ with $\det(\underline{n}, \underline{p}_i, \underline{p}_b, \underline{p}_t) > 0$, as the vectors $\underline{n}^{(2)}$ and $\underline{n}^{(3)}$ of the tetrad. However this convention is not the most natural for some frequent type of reactions, which we study in the next paragrap.

8. TRANSVERSITY AND HELICITY FRAMES IN s-, t-, and u-channels

We will study here in detail only the case of two-body reactions with four-momenta

$$\underline{p}_1 + \underline{p}_2 = \underline{p}_3 + \underline{p}_4 , \qquad (62)$$

in which a natural correspondence can be established between one initial and one final particle,

$$1 \longleftrightarrow 3 \quad , \quad \text{and} \quad 2 \longleftrightarrow 4 \quad ,$$

having the same baryonic, and/or leptonic, and/or electric charge.

These reactions define (except in the colinear case of forward or backward scattering) one three-plane, and six two-planes, three of them containing a given particle. The two-planes have the following names :

$$\begin{array}{llll} \text{s-channel two-planes} & \underline{p}_1, \underline{p}_2 & \text{and} & \underline{p}_3, \underline{p}_4 \\ \text{t-channel two-planes} & \underline{p}_1, \underline{p}_3 & \text{and} & \underline{p}_2, \underline{p}_4 \\ \text{u-channel two-planes} & \underline{p}_1, \underline{p}_4 & \text{and} & \underline{p}_2, \underline{p}_3 . \end{array} \qquad (63)$$

We will use the letters $a, b \ldots$ as a symbol for any of these three channels, and call "ai" the particle associated to particle i in channel a. Explicitely, for

$$i = 1 \quad 2 \quad 3 \quad 4 \qquad (64)$$

we have

$$\begin{array}{lllll} \mathbf{si} = & 2 & 1 & 4 & 3 \\ ti = & 3 & 4 & 1 & 2 \\ ui = & 4 & 3 & 2 & 1 . \end{array} \qquad (64')$$

As usually, we will call also s, t, and u the kinematical invariants corresponding to these three channels. They can be written for any i :

$$\begin{aligned} s &= (\underline{p}_i + \underline{p}_{si})^2 \\ t &= (\underline{p}_i - \underline{p}_{ti})^2 \\ u &= (\underline{p}_i - \underline{p}_{ui})^2 \end{aligned} \qquad (65)$$

One sign convention must be made for the normal $\underline{n}$ to the reaction three-plane (see expression 61'). For this kind of reactions we shall always use the nearly standard "Basel convention" (Basel-61) :

$$\det(\underline{p}_1 + \underline{p}_2 , \underline{p}_1 , \underline{p}_3 , \underline{n}) = \det(\underline{p}_1 + \underline{p}_2 , \underline{p}_2 , \underline{p}_4 , \underline{n}) > 0 , \quad (66)$$

i.e.,

$$\det(\underline{n} , \underline{p}_1 , \underline{p}_2 , \underline{p}_3) > 0 . \quad (66')$$

In the center of mass system, i.e., in the rest frame of $\underline{p}_1 + \underline{p}_2 = \underline{p}_3 + \underline{p}_4$, expression (66) means that $\vec{p}_1 , \vec{p}_3 , \vec{n}$ and $\vec{p}_2 , \vec{p}_4 , \vec{n}$ are right handed triedra :

$$\vec{p}_1 \times \vec{p}_3 \cdot \vec{n} = \vec{p}_2 \times \vec{p}_4 \cdot \vec{n} > 0 \quad (66'')$$

We insist again that conventions are arbitrary but must be specified in each self-contained publication ! Let us review here some possible conventions.

The transversity tetrad, associated to a particle i and a channel a , is defined by choosing the quantization vector $\underline{n}^{(3)}$, up to a sign η, along the normal $\underline{n}$:

$$^{T}_{a}\underline{n}^{(3)}_{i} = {}^{T}_{a}\eta_{i} \cdot \underline{n} . \quad (67)$$

The vector $\underline{n}^{(2)}$ is chosen in the corresponding a-channel two-plane, and fixed by another sign ϵ :

$$^{T}_{a}\underline{n}^{(2)}_{i} = {}^{T}_{a}\epsilon_{i} \cdot \underline{q}_{i}(ai) . \quad (67')$$

The helicity tetrad associated to the particle i and the channel a is defined on the contrary with the quantization vector $\underline{n}^{(3)}$ in the corresponding a-channel two-plane, and the vector $\underline{n}^{(2)}$ along the normal $\underline{n}$. These vectors are fixed by two new signs η and ϵ :

$$^{H}_{a}\underline{n}^{(2)}_{i} = {}^{H}_{a}\eta_{i} \cdot \underline{n} , \quad (68)$$

$$^{H}_{a}\underline{n}^{(3)}_{i} = {}^{H}_{a}\epsilon_{i} \cdot \underline{q}_{i}(ai) . \quad (68')$$

It might be useful to prepare a table of the choice of the forty-eight signs ${}^{T}_{a}\eta_{i}$, ${}^{H}_{a}\eta_{i}$, ${}^{T}_{a}\epsilon_{i}$, ${}^{H}_{a}\epsilon_{i}$ made in different papers by the same or by different authors. We intend to do it. Let us just remark here that the most frequently used conventions for s-helicity (Jacob-Wick-59) and t-helicity (Gottfried-Jackson-64) correspond to opposite choices for the ${}^{H}\epsilon$'s . For Jacob and

Wick ${}^{H}_{s}n_i^{(3)}$ is along p_i in the center of mass frame, i.e., for $\vec{p}_i + \vec{p}_{si} = 0$; as equation (55) shows, this means ${}^{H}_{s}\epsilon = +1$. For Gottfried and Jackson, ${}^{H}_{t}n_i^{(3)}$ is along $\vec{p}_{ti}$ in the rest frame of particle i , i.e., for $\vec{p}_i = 0$; this means ${}^{H}_{t}\epsilon = -1$.

In this work we make the following conventions :

a) The signs ${}^{T}\eta$, ${}^{H}\eta$, ${}^{T}\epsilon$, and ${}^{H}\epsilon$ are independent of the particle i and of the channel a .

b) We want the two tetrads T and H of a given particle and a given channel to have the same vector $\underline{n}^{(1)}$, i.e.,

$$^{T}\underline{n}^{(1)} = {}^{H}\underline{n}^{(1)} . \tag{69}$$

This is equivalent to

$$^{T}\eta \cdot {}^{H}\eta \cdot {}^{T}\epsilon \cdot {}^{H}\epsilon = -1 . \tag{69'}$$

c) We extend the Basel convention to the basis vector along the normal in each tetrad, i.e.,

$$^{T}\eta = {}^{H}\eta = +1 . \tag{70}$$

Remark that conventions a), b), and c) leave still one arbitrary sign :

$$\epsilon = {}^{H}\epsilon = -{}^{T}\epsilon \tag{71}$$

Conventions a) and b) are natural and happely the most common in the literature. The Basel convention c) is more and more adopted. Thus for instance, Cohen-Tannoudji - Morel -Navelet - 68 adopt the same conventions and fix $\epsilon = +1$. However, as already pointed out, widespread conventions differ by the sign of ϵ . So we will keep this sign not fix, and include ϵ in our formulae when necessary. We took this decision because, as a matter of fact, most of our formulae are independent of ϵ . Let us see two examples :

A transversity tetrad is transformed in the corresponding helicity tetrad by a "rotation" of $+\frac{\pi}{2}$ around their common $\underline{n}^{(1)}$ vector. This is a consequence of conventions b) and c), since the sign of this rotation angle is given by :

$$\frac{1}{m}\det(\underline{p}, \underline{n}^{(1)}, {}^{T}\underline{n}^{(2)}, {}^{H}\underline{n}^{(2)}) = \frac{1}{m}\det(\underline{p}, \underline{n}^{(1)}, {}^{T}\underline{n}^{(3)}, {}^{H}\underline{n}^{(3)}) = {}^{H}\eta\, {}^{T}\eta = -{}^{H}\epsilon\, {}^{T}\epsilon = +1 \tag{72}$$

The crossing angle, to be studied in the next paragraph, is also independent of ϵ . This can be seen in equations (75) and (76) which contain two vectors $\underline{T}_{\underline{n}}(2)$ or $\underline{H}_{\underline{n}}(3)$ corresponding to different channels. Of course we suppose, according to convention a), the same sign ϵ for all channels.

To summarize, for the four particles and the three channels we fix the transversity tetrad by

$$\underset{a-i}{T_{\underline{n}}}(3) = \underline{n} \quad , \quad \underset{a-i}{T_{\underline{n}}}(2) = -\epsilon\, \underline{q}_i\,(ai) \quad , \tag{73}$$

and the helicity tetrad by

$$\underset{a-i}{H_{\underline{n}}}(2) = \underline{n} \quad , \quad \underset{a-i}{H_{\underline{n}}}(3) = \epsilon\, \underline{q}_i\,(ai) \quad , \tag{74}$$

where $\underline{n}$ is the "Basel normal" (see equations (66) and $\underline{q}_i\,(ai)$ is defined in (55) and (64'). The sign ϵ will generally not appear in our formulæ, but it should be fixed in every experimental paper measuring odd multipole polarization.

9. THE CROSSING ANGLE

The transformation between two transversity (or helicity) frames corresponding to the same particle i and different channels a and b, will be a "rotation" around the normal $\underline{n}$ which brings the a-channel two-plane into the b-channel two-plane. Let us call $\Psi_{i,ba}$ this crossing angle. From our sign conventions a) and c), one can use indifferently the T or the H tetrad for its definition. Indeed

$$\cos \Psi_{i,ba} = -\underset{a-i}{T_{\underline{n}}}(2) \cdot \underset{b-i}{T_{\underline{n}}}(2) = -\underset{a-i}{H_{\underline{n}}}(3) \cdot \underset{b-i}{H_{\underline{n}}}(3) \quad , \tag{75}$$

$$\sin \Psi_{i,ba} = \frac{1}{m_i} \det(\underline{p}_i, \underline{n}, \underset{a-i}{T_{\underline{n}}}(2), \underset{b-i}{T_{\underline{n}}}(2)) = \frac{1}{m_i} \det(\underline{p}_i, \underline{n}, \underset{a-i}{H_{\underline{n}}}(3), \underset{b-i}{H_{\underline{n}}}(3)) \tag{76}$$

(we recall that $\underline{n} = \underset{c-i}{T_{\underline{n}}}(3) = \underset{c-i}{H_{\underline{n}}}(2)$ for all i , all c).

From our conventions in (73) or (74) and the definition of the vectors $\underline{q}_i(j)$ in (55) to (60') one obtains :

$$\cos \Psi_{i,ba} = -q_i(ai) \cdot q_i(bi) = \left[\mathrm{ch}\varphi_{i,ai}\, \mathrm{ch}\varphi_{i,bi} - \mathrm{ch}\varphi_{ai,bi} \right] / \mathrm{sh}\varphi_{i,ai}\, \mathrm{sh}\varphi_{i,bi} =$$

$$= \frac{(m_{i,ai}^2 - m_i^2 - m_{ai}^2)(m_{i,bi}^2 - m_i^2 - m_{bi}^2) - 2m_i^2(m_{ai,bi}^2 - m_{ai}^2 - m_{bi}^2)}{\Delta(m_{i,ai}^2, m_i^2, m_{ai}^2)^{1/2}\, \Delta(m_{i,bi}^2, m_i^2, m_{bi}^2)^{1/2}}$$

But the invariants used in this expression are, according to equation (58),

$$m_{i,ai}^2 = (\underline{p}_i + \underline{p}_{ai})^2 \qquad (78)$$

If we want to use the standard invariants s, t, u defined in (65), we can write equation (62) in the form

$$\underline{p}_1 + \underline{p}_2 + \underline{p}_{\bar{3}} + \underline{p}_{\bar{4}} = 0 \quad , \qquad (79)$$

by changing the sign of the final state four-momenta. Equations (55) to (60') are also valid for time-like four-momenta $\underline{p}_i$, $\underline{p}_j$ with negative energy. But the sign of $\underline{q}_i(j)$ will be changed whenever j is a final state particle. And the sign of $\underline{q}_i(ai) . \underline{q}_i(bi)$ will be changed unless ai and bi be both initial or both final particles, i.e., (a, b) are (t, u) or (u, t) .

The right hand side of equation (77) can be written in a more symmetrical way. Calling $\Gamma_{i,ai,bi}$ its numerator and c = ba , we have from (79) :

$$\Gamma_{i,ai,bi} + \Gamma_{i,ai,ci} + \Delta(m_{i,ai}^2, m_i^2, m_{ai}^2) = 0 . \qquad (80)$$

This expression and the symmetry properties of Γ and Δ give the new form of $\Gamma_{i,ai,bi}$:

$$\Gamma_{i,ai,bi} = -\frac{1}{2}\left[\Delta(m_{i,ai}^2, m_i^2, m_{ai}^2) + \Delta(m_{i,bi}^2, m_i^2, m_{bi}^2) - \Delta(m_{i,ci}^2, m_i^2, m_{ci}^2 \right. \qquad (81)$$

Thus, cos $\psi_{i,ba}$ is given by the expressions (a_1) and (b_1) in Table 1 .

The explicit value of sin $\psi_{i,ba}$ defined in (76) is given by the expressions (a_2) and (b_2) in the same Table. The signs in (b_2) are easily obtained from (66) and (62).

Table 1. - Rotations around the normal which relate the frames associated with s-, t-, and u-channels

Crossing angle $\psi_{i,ba}$ which transform for particle i the frames associated to channel a into the frames associated to channel b :

(a$_1$) $$\cos \psi_{i,ba} = \epsilon_{ba}\left[(a-m_i^2-m_{ai}^2)(b-m_i^2-m_{bi}^2)-2m_i^2(c-m_i^2-m_{ci}^2)\right]/\left[\Delta_{i,a}\cdot\Delta_{i,b}\right]^{1/2} =$$

$$= -\epsilon_{ba}\left[\Delta_{i,a}+\Delta_{i,b}-\Delta_{i,c}\right]/\,2\left[\Delta_{i,a}\cdot\Delta_{i,b}\right]^{1/2}$$

(a$_2$) $$\sin \psi_{i,ba} = \epsilon_{i,ba}\,(1-\cos^2\psi_{i,ba})^{1/2} =$$

$$= \epsilon_{i,ba}\left[-\Delta(\Delta_{i,a},\Delta_{i,b},\Delta_{i,c})\right]^{1/2}/2\left[\Delta_{i,a}\cdot\Delta_{i,b}\right]^{1/2}$$

(b$_1$) Signs of the expressions in (a$_1$), for the same sign conventions in channels a and b ,

ϵ_{ba} = +1 for (b, a) = (u, t) , (t, u)

ϵ_{ba} = -1 in any other cases.

(b$_2$) Signs of the expressions in (a$_2$), for the Basel convention (66),

$\epsilon_{i,ba}$ = +1 for i = initial and (b, a) = (s, t), (u, t), (u, s)

for i = final and (b, a) = (t, s), (t, u), (s, u)

$\epsilon_{i,ba}$ = -1 in any other cases

(c) Terminology used in (a$_1$) and (a$_2$) :

a, b, c = label for the three different channels s, t, u, and for their corresponding invariants

ai = label for the particle associated to particle i in channel a (see 64')

m_i = mass of particle i .

$\Delta_{i,a} = \Delta(a, m_i^2, m_{ai}^2)$

$\Delta(x,y,z) = x^2+y^2+z^2-2xy-2yz-2zx$

Part I. APPENDIX 2

POLARIZATION DENSITY MATRIX - MULTIPOLE EXPANSION

1. THE DENSITY MATRIX ELEMENTS

The dynamical state of a sample of particles with fixed energy-momentum and fixed spin j is a statistical mixture of pure polarization states. It is described by the polarization density operator ρ acting on the vectors of a (2j+1)-dimensional Hilbert space $\mathcal{H}_{2j+1}$. If we choose a reference frame, the 2j+1 vectors $|jm\rangle$ eigenvectors of $\vec{J}^2$ and J_3 :

$$\vec{J}^2 \, |jm\rangle = j(j+1) \, |jm\rangle \qquad (1)$$
$$J_3 \, |jm\rangle = m \, |jm\rangle \ ,$$

form an orthonormal basis of the Hilbert space $\mathcal{H}_{2j+1}$:

$$\langle jm \, | \, jm' \rangle = \delta^m_{m'} \qquad (2)$$

In this basis the polarization density operator is represented by a (2j+1) $\times$ (2j+1), trace 1, Hermitian, semi-positive matrix. (See Appendix 6).
Its elements are :

$$\rho^m_{\ n} = \langle jm | \rho | jn \rangle \qquad (3)$$

with

$$\sum_m \rho^m_{\ m} = 1 \ . \qquad (4)$$

The $N = (2j+1)^2 - 1$ independent elements $\rho^m_{\ n}$ may be chosen as coordinates of the density matrix in the space $\mathcal{E}_N$. However, as it has been emphasized in I. 3, this coordinate system is not an orthogonal coordinate system for $\mathcal{E}_N$.

Furthermore if we perform a rotation of the frame of reference, the basis vectors $|jm\rangle$ undergo a unitary transformation :

$$|jm\rangle \rightsquigarrow U(R)|jm\rangle = |jm'\rangle \ D^j(R)^{m'}_{\ m} \qquad (5)$$

where $D^j(R)$ is the (2j+1)-dimensional representation of the rotation group. The corresponding unitary transformation of operators is :

$$\rho \rightsquigarrow \rho'_{\ n'} = U(R) \, \rho \, U^{-1}(R) \ , \qquad (6)$$

and the density matrix elements undergo the transformation :

$$\rho^m_{\ n} \rightsquigarrow \rho'^{m'}_{\ n} = D^j(R)^{m'}_{\ m} \, \rho^m_{\ n} \, D^j(R^{-1})^n_{\ n'} \ , \qquad (7)$$

i.e., the N independent coordinates $\rho^m_{\ n}$ transform into each other when the reference frame is rotated but the transformation law involves the reducible tensorial product of two representations D^j.

For these reasons we shall consider another set of coordinates for the density matrix, called the multipole parameters. Let us before review some of the properties of the Wigner's 3j-symbols and of the representations of the rotation group.

2. THE 3j-SYMBOLS. THE TENSOR $C_j^{mm'}$

The 3j-symbol is defined by the relation :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-)^{j_1-j_2-m_3}(2j_3+1)^{-\frac{1}{2}}(j_1m_1j_2m_2|j_3-m_3) \quad (8)$$

where $(j_1m_1j_2m_2|j_3-m_3)$ is the Clebsch-Gordan coefficient of Condon and Shortley (Condon - Shortley - 35).

The value of this 3j-symbol has been given by Wigner and Racah (Wigner-59), (Racah-42), (Rose-57) :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-)^{j_1-j_2-m_3}\left[\frac{(j_1+j_2-j_3)!\,(j_1-j_2+j_3)!\,(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!}\right]^{1/2}$$

$$\times\ (j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(j_3+m_3)!\,(j_3-m_3)!$$

$$\times \sum_r (-)^r\left[r!\,(j_1+j_2-j_3-r)!\,(j_1-m_1-r)!\,(j_2+m_2-r)!\,(j_3-j_2+m_1+r)!\,(j_3-j_1-m_2+r)!\right]^{-1} \quad (9)$$

These coefficients are nul unless the magnetic-quantum-number indices m_i satisfy the relation :

$$m_1 + m_2 + m_3 = 0$$

and the spin indices j_i satisfy the usual triangular relations.

The 3j-symbols are invariant under any even permutation of the columns. For any odd permutation we have :

$$\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = (-)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (10)$$

Another property of the 3j-symbols is :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (11)$$

Wigner (Wigner-59) has introduced the very convenient concept of covariant and contravariant 3j-symbols. The metric tensor which allows one to raise and lower the magnetic-quantum-number indices m is defined by :

$$C_j^{mm'} = (-)^{j-m}\,\delta_{m,-m'} = (-)^{j+m'}\,\delta_{m,-m'} \qquad (12a)$$

$$C^j_{mm'} = (-)^{j+m}\,\delta_{m,-m'} = (-)^{j-m'}\,\delta_{m,-m'} \qquad (12b)$$

The symmetry character of this tensor depends on the parity of 2j :

$$C_j^{m'm} = (-)^{2j}\,C_j^{mm'} \qquad (13a)$$

$$C^j_{m'm} = (-)^{2j}\,C^j_{mm'} \qquad (13b)$$

Since the tensor is not symmetric we must precise the place of the summation index. We shall adopt the convention that to raise or to lower an index we sum over the second index of the tensors C. For instance, the contravariant 3j-symbol is :

$$\begin{pmatrix} j_1 & j_2 & m_3 \\ m_1 & m_2 & j_3 \end{pmatrix} = C_{j_3}^{m_3 m'_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m'_3 \end{pmatrix} = (-)^{j_3-m_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \qquad (14)$$

Throughout this appendix, we adopt the convention that repeated magnetic-quantum-number indices are to be summed over $(m = -j, \ldots, +j)$, whilst summations over j, when they occur, are always indicated.

From (9), (11) and (14) one deduces the relation :

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \qquad (15)$$

The orthogonality of the Clebsch-Gordan coefficients leads to the orthogonality relations for the 3j-symbols :

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} m_1 & m_2 & m'_3 \\ j_1 & j_2 & j'_3 \end{pmatrix} = \frac{1}{2j_3+1}\,\delta_{j_3 j'_3}\,\delta^{m'_3}_{m_3} \qquad (16a)$$

$$\sum_{j_3} (2j_3+1)\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} m'_1 & m'_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = \delta^{m'_1}_{m_1}\,\delta^{m'_2}_{m_2} \qquad (16b)$$

Remark : The symbol : $\begin{pmatrix} j & j & 0 \\ m & m' & 0 \end{pmatrix} = \dfrac{(-)^{j-m}}{\sqrt{2j+1}}\,\delta_{m,-m'}$ (17)

has the properties of the tensor $C^j_{mm'}$. In fact, we have :

$$C^j_{mm'} = \sqrt{2j+1}\begin{pmatrix} j & j & 0 \\ m & m' & 0 \end{pmatrix} \qquad (18)$$

For this reason the metric tensor $C^j_{mm'}$ is sometimes called the 1j-symbol.

3 . THE UNITARY IRREDUCIBLE REPRESENTATIONS OF THE ROTATION GROUP

a) The homomorphism SU(2) ⟶ SO(3).

There exists a 2 : 1 homomorphism of the group SU(2) onto the rotation group SO(3). Explicitly, if $R(\vec{m}, \theta)$ is the rotation by an angle θ around the direction $\vec{m}$ ($\vec{m}^2 = 1$), the two matrices of the group SU(2) corresponding to this rotation are $\pm U(\vec{m}, \theta)$, with :

$$U(\vec{m}, \theta) = \cos\frac{\theta}{2}\,\mathbb{1} - i\sin\frac{\theta}{2}\;\vec{\tau}\cdot\vec{m} \tag{19}$$

where $\mathbb{1}$ is the unit 2 x 2 matrix and $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$ are the usual Pauli matrices.

The rotation around Ox is represented by :

$$U(\vec{Ox}, \theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \tag{20a}$$

The rotation around Oy is represented by the real matrix :

$$U(\vec{Oy}, \theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ +\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \tag{20b}$$

The rotation around Oz is represented by the diagonal matrix :

$$U(\vec{Oz}, \theta) = \begin{bmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \tag{20c}$$

From (20b) and (20c) one deduces that the matrix $U(\alpha, \beta, \gamma)$ corresponding to the rotation parametrized by the Euler angles α, β, γ is :

$$U(\alpha, \beta, \gamma) = \begin{bmatrix} e^{+\frac{i}{2}(-\alpha-\gamma)}\cos\frac{\beta}{2} & -e^{\frac{i}{2}(-\alpha+\gamma)}\sin\frac{\beta}{2} \\ e^{\frac{i}{2}(\alpha-\gamma)}\sin\frac{\beta}{2} & e^{\frac{i}{2}(\alpha+\gamma)}\cos\frac{\beta}{2} \end{bmatrix} \tag{21}$$

Note that the matrix $\Gamma \in SU(2)$ corresponding to the rotation by an angle $-\pi$ around Oy is :

$$\Gamma \equiv U(\vec{Oy}, -\pi) = i\tau_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{22}$$

The elements of this matrix are : $\Gamma^m_{\ m'} = (-)^{\frac{1}{2}-m}\, \delta^m_{\ -m'}$, (23)

i.e., the elements of the matrix Γ are identical to the components of the metric tensor $C^{mm'}_{1/2}$:

$$\Gamma^m_{\ m'} = C^{mm'}_{1/2} \tag{24}$$

The matrix Γ satisfies the following properties :

$$\Gamma^2 = -1 \tag{25a}$$

$$\Gamma^T = -\Gamma = \Gamma^{-1} \tag{25b}$$

The interest of the matrix Γ lies in the fact that it exhibits the important property of the group SU(2), namely that the matrices U are equivalent to their complex conjugate $\overline{U}$. In fact one verifies the relation

$$\overline{U} = \Gamma U \Gamma^{-1} \tag{26}$$

b) <u>The unitary irreducible representations of SU(2).</u>

The unitary irreducible representations of the group SU(2) are characterized by one number j (2j = non negative integer) and the dimension of the representation D^j is 2j+1. If the elements of the matrix U are :

$$U = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \qquad |a|^2 + |b|^2 = 1 \tag{27}$$

the matrix elements of the representation D^j are (m, m' = -j, ..., +j) :

$$\left[D^j(U)\right]^m_{\ m'} = \sum_r (-)^r \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j-r+m')!\,(j-r-m)!\,(r+m-m')!\,r!}\, a^{j-r+m'}\,\overline{a}^{j-r-m}\, b^{r+m-m'}\,\overline{b}^{r} \tag{28}$$

the summation being performed over all possible integer values of r such that none of the arguments of the factorials of the denominator be negative.

Let us consider three particular U matrices : U = $\mathbb{1}$ (i.e., a = 1, b = 0), U = Γ (i.e., a = 0, b = 1), and U = $\Gamma^T = \Gamma^{-1}$ (i.e., a = 0, b = 1).

$$D^j(\mathbb{1})^m_{\ m'} = \delta^m_{\ m'} \tag{29a}$$

$$(\Gamma^j)^m_{\ m'} \equiv D^j(\Gamma)^m_{\ m'} = (-)^{j-m}\, \delta^m_{\ -m'} \tag{29b}$$

$$\left[(\Gamma^j)^{-1}\right]^m_{\ m'} = (\Gamma^{jT})^m_{\ m'} \equiv D^j(\Gamma^T)^m_{\ m'} = (-)^{j+m}\,\delta^m_{\ -m'} = (-)^{2j}(\Gamma^j)^m_{\ m'} \tag{29c}$$

We can also consider the representation of the relation (26). We obtain :

$$D^j(\overline{U})^m_{\ m'} = D^j(\Gamma)^m_{\ n}\, D^j(U)^n_{\ n'}\, D^j(\Gamma^{-1})^{n'}_{\ n} \tag{30a}$$

The matrix elements of the matrix $\overline{D^j(U)}$ are :

$$D^j(\bar{U})^m_{\ m'} = \overline{D^j(U)^m_{\ m'}} = (-)^{m-m'} D^j(U)^{-m}_{\ -m'} \tag{30b}$$

The matrix elements of the matrix $D^j(U)^*$ are :

$$D^j(U^*)^m_{\ m'} = \left[D^j(U)^*\right]^m_{\ m'} = \overline{D^j(U)^{m'}_{\ m}} = (-)^{m-m'} D^j(U)^{-m'}_{\ -m} \tag{30c}$$

c) <u>The unitary irreducible representations of the rotation group.</u> (Rose -57)

Using Eq. (23) which gives explicitly the values of a and b as functions of the Euler angles α, β, γ, the representations of the rotation $R(\alpha,\beta,\gamma)$ are :

$$D^j(\alpha,\beta,\gamma)^m_{\ m'} = e^{-im\alpha}\, d^j(\beta)^m_{\ m'}\, e^{-im'\alpha} \tag{31a}$$

with

$$d^j(\beta)^m_{\ m'} = \sum_r (-)^r \frac{\sqrt{(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!}}{(j-r+m')!\,(j-r-m)!\,(r+m-m')!\,r!} \left(\cos\frac{\beta}{2}\right)^{2j-2r+m'-m} \left(-\sin\frac{\beta}{2}\right)^{2r+m-m'} \tag{31b}$$

The matrices $d^j(\beta)^m_{\ m'}$ are real and they have some interesting symmetry properties which can be directly verified on Eq. (31b) :

$$d^j(-\beta)^m_{\ m'} = d^j(\beta)^{m'}_{\ m} \tag{32}$$

$$d^j(\beta)^m_{\ m'} = (-)^{m-m'} d^j(\beta)^{m'}_{\ m} \tag{33}$$

$$d^j(\beta)^m_{\ m'} = (-)^{m-m'} d^j(\beta)^{-m}_{\ -m'} \tag{34}$$

$$d^j(\beta)^m_{\ m'} = d^j(\beta)^{-m'}_{\ -m} \tag{35}$$

Using (31b) one may also compute the matrix elements of $d^j(\beta)$ for some characteristicsvalues of β :

$$d^j(0)^m_{\ m'} = \delta^m_{\ m'} \tag{36}$$

$$d^j(-\pi)^m_{\ m'} = (-)^{j-m}\,\delta^m_{\ -m'} = (-)^{j+m'}\,\delta^m_{\ -m'} \tag{37}$$

$$d^j(+\pi)^m_{\ m'} = (-)^{j+m}\,\delta^m_{\ -m'} = (-)^{j-m'}\,\delta^m_{\ -m'} \tag{38}$$

From (37) and (29b) and (38) and (29c) we note that :

$$d^j(-\pi) = \Gamma^j \tag{39a}$$

$$d^j(\pi) = \Gamma^{jT} = (\Gamma^j)^{-1} = (-)^{2j}\,\Gamma^j \tag{39b}$$

From (30b), the matrix $\overline{D^j(\alpha,\beta,\gamma)}$ is related to $D^j(\alpha,\beta,\gamma)$ by the relation :

$$\overline{D^j(\alpha,\beta,\gamma)^m_{\ m'}} = (\Gamma^j)^m_{\ n}\, D^j(\alpha,\beta,\gamma)^n_{\ n'}\, (\Gamma^{j-1})^{n'}_{\ n} \tag{40}$$

and we have :

$$\overline{D^j(\alpha,\beta,\gamma)^m_{m'}} = (-)^{m-m'} D^j(\alpha,\beta,\gamma)^{-m}_{-m'} \tag{41}$$

This formula can be directly verified, using (31a), the reality of the d^j's and formula (34).

For integer j, m and m' can take the value 0. One has :

$$D^j(\alpha,\beta,\gamma)^m_0 = (-)^m \sqrt{\frac{4\pi}{2j+1}}\, Y^j_{-m}(\beta,+\alpha) \tag{42a}$$

$$D^j(\alpha,\beta,\gamma)^0_m = \sqrt{\frac{4\pi}{2j+1}}\, Y^j_{-m}(\beta,+\gamma) \tag{42b}$$

where the Y^j_m's are the usual spherical harmonics.

Note that we can define the contravariant spherical harmonics by the formula :

$$Y^m_j = C^{mm'}_j Y^j_{m'} = (-)^{j-m} Y^j_{-m} \tag{43}$$

Furthermore from (41) we have the relation :

$$\overline{Y^j_m} = (-)^m Y^j_{-m} \tag{44}$$

thus :

$$Y^m_j = (-)^j Y^j_m \tag{45}$$

d) The reduction of the tensorial product of two irreducible representations of the rotation group.

The tensorial product of two irreducible representations of the group SU(2) is completely reducible. The reduction formula can be symbolically written :

$$D^{j_1} \otimes D^{j_2} = \bigoplus_{L=|j_1-j_2|}^{j_1+j_2} D^L \tag{46}$$

Explicitly, the matrix elements of the matrix which reduces the product are the 3j-symbols of Wigner, and one has :

$$D^{j_1}(R)^{m_1}_{n_1} D^{j_2}(R)^{m_2}_{n_2} = \sum_{L=j_1-j_2}^{j_1+j_2} (-)(2L+1)\begin{pmatrix} m_1 & m_2 & L \\ j_1 & j_2 & M \end{pmatrix} D^L(R)^M_N \begin{pmatrix} N & j_1 & j_2 \\ L & n_1 & n_2 \end{pmatrix} \tag{47}$$

Using formula (30c) and the orthogonality property (16b) of the 3j-symbols, one shows the following relations :

$$D^{j_1}(R)^{m_1}_{n_1} \begin{pmatrix} n_1 & L & j_2 \\ j_1 & M & m_2 \end{pmatrix} D^{j_2}(R^{-1})^{m_2}_{n_2} = \begin{pmatrix} m_1 & L & j_2 \\ j_1 & M' & n_2 \end{pmatrix} D^L(R)^{M'}_M \tag{47'}$$

$$D^{j_1}(R)^{m_1}_{n_1} \begin{pmatrix} n_1 & M & j_2 \\ j_1 & L & m_2 \end{pmatrix} D^{j_2}(R^{-1})^{m_2}_{n_2} = D^L(R^{-1})^M_{M'} \begin{pmatrix} m_1 & M' & j_2 \\ j_1 & L & m_2 \end{pmatrix} \tag{47''}$$

The corresponding relation for the reduction of the product of two spherical harmonics is :

$$Y^{j_1}_{n_1} \; Y^{j_2}_{n_2} = \sum_{j,n} \sqrt{\frac{(2j_1+1)(2j_2+1)(2j+1)}{4\pi}} \begin{pmatrix} j_1 & j_2 & j \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j \\ n_1 & n_2 & n \end{pmatrix} \overline{Y^{j}_{n}} \tag{48}$$

which can be written, using (44)(45) :

$$Y^{j_1}_{n_1} \; Y^{j_2}_{n_2} = \sum_{j} \sqrt{\frac{(2j_1+1)(2j_2+1)(2j+1)}{4\pi}} \begin{pmatrix} j_1 & j_2 & 0 \\ 0 & 0 & j \end{pmatrix} \begin{pmatrix} j_1 & j_2 & n \\ n_1 & n_2 & j \end{pmatrix} Y^{j}_{n} \tag{49}$$

4 . THE NON-HERMITIAN MATRICES T^L_M

Let us consider the $(2j+1)^2$ operators

$$T^L_M \qquad (L = 0, \ldots\ 2j\ ; \quad M = -L, \ldots, +L,) \tag{50}$$

whose matrix elements in the basis $|\,jm\rangle$ of the Hilbert space $\mathcal{H}_{2j+1}$ are :

$$(T^L_M)^m_{\ n} \equiv \langle jm \,|\, T^L_M \,|\, jn\rangle = (jm \,|\, LM\,jn) \tag{51}$$

$$(T^L_M)^m_{\ n} = \sqrt{2j+1} \begin{pmatrix} m & L & j \\ j & M & n \end{pmatrix} \tag{51'}$$

where we have used the contravariant 3j-symbols of Wigner.Let us review the properties of these matrices :

a) The 3j-symbols are real, thus the matrix elements are real.

b) The elements of the matrix $T^{(0)}_0$ are, using (17) :

$$(T^{(0)}_0)^m_{\ n} = \sqrt{2j+1} \begin{pmatrix} m & 0 & j \\ j & 0 & n \end{pmatrix} = \delta^m_{\ n} \tag{52}$$

c) The matrices $T^{(L)}_0$ are diagonal, thus they are Hermitian, however the matrices T^L_M $M \neq 0$ are not Hermitian. Instead, using the properties of 3j-symbols one obtains :

$$(T^L_M)^T = (T^L_M)^* = (-)^M \; T^L_{-M} \tag{53}$$

d) Using the orthogonality relation (16a) of the 3j-symbols, one may also verify the following properties :

$$\mathrm{Tr}\,(T^{L\,*}_M \; T^{L'}_{M'}) = \frac{2j+1}{2L+1} \; \delta_{LL'} \; \delta_{MM'} \tag{54a}$$

and, using (52) :

$$Tr(T^L_M) = (2j+1)\, \delta_{L0}\, \delta_{M0} \qquad (54b)$$

e) The matrix T^M_L (with on upper index M), is related to the matrix T^L_M (with a lower index) by the relation :

$$T^M_L = C^{MM'}_L\, T^L_{M'} = (-)^{L-M}\, T^L_{-M} \qquad (55)$$

$$T^M_L = (-)^L\, T^{L\,*}_M \qquad (55')$$

f) If we perform a rotation of the frame of reference, the operators T^L_M transform according to Eq. (6) :

$$T^L_M \rightsquigarrow T'^L_M = U(R)\, T^L_M\, U(R^{-1})$$

and their matrix elements undergo the transformation :

$$(T^L_M)^m{}_n \rightsquigarrow (T'^L_M)^m{}_n = D^j(R)^m{}_{m'}\, (T^L_M)^{m'}{}_{n'}\, D^j(R^{-1})^{n'}{}_n$$

Using the definition (51') of the T^L_M and formula (47') for the reduction of the tensorial product of D^j, one obtains :

$$(T^L_M)^m{}_n \rightsquigarrow (T'^L_M)^m{}_n = (T^L_{M'})^m{}_n\, D^L(R)^{M'}{}_M ,$$

or

$$T^L_M \rightsquigarrow T'^L_M = T^L_{M'}\, D^L(R)^{M'}{}_M , \qquad (56)$$

i. e., for each L , the 2L+1 operators T^L_M transform into each other as the components of an irreducible tensorial operator when the reference frame is rotated.

In **appendix A6** we shall have to consider a particular case of formula(56), namely the case where the rotation R is the rotation by an angle $(-\pi)$ around the Oy axis. Then one has :

$$D^j(R) = D^j(0,-\pi,0) = \Gamma^j ,$$

$$D^L(R)^{M'}{}_M = D^L(0,-\pi,0)^{M'}{}_M = (\Gamma^L)^{M'}{}_M = (-)^{L-M'}\, \delta_{M',-M} ,$$

and one obtains :

$$\Gamma^j (T^L_M)(\Gamma^j)^{-1} = (-)^{L-M}\, T^L_{-M}$$

or, using (53)

$$\Gamma^j (T^L_M)^T (\Gamma^j)^{-1} = (-)^L\, T^L_M \qquad (57)$$

5 . THE MULTIPOLE PARAMETERS t^L_M

If ρ is a $(2j+1)\times(2j+1)$ polarization density matrix, let us consider the quantities t^L_M defined by :

$$t^L_M = \langle \rho \; T^L_M \rangle = \mathrm{Tr}\,(\rho\, T^L_M) \tag{58}$$

$$t^L_M = \sqrt{2j+1} \begin{pmatrix} n & L & j \\ j & M & m \end{pmatrix} \rho^m_n \tag{58'}$$

The t^L_M are called the multipole parameters of the density matrix ρ. They form a set of coordinates for the density matrix ρ. The properties of the density matrix can easily be expressed on these parameters.

a) From equation (52) and from $\mathrm{Tr}\,\rho = 1$ we obtain

$$t^0_0 = \mathrm{Tr}\,(\rho\, T^0_0) = 1 \tag{59}$$

b) The parameters t^L_0 are real, but the parameters t^L_M $M \neq 0$ are complex.

c) From the hermiticity of ρ and from (53) we obtain :

$$\overline{t^L_M} = \overline{\mathrm{Tr}\,(\rho\, T^L_M)} = \tag{60}$$

$$= \mathrm{Tr}\,(\rho^*\, T^{L\,*}_M) = (-)^M\, \mathrm{Tr}\,(\rho\, T^L_{-M})$$

$$\overline{t^L_M} = (-)^M\, t^L_{-M} \tag{61}$$

d) Using the orthogonality property (16b) of the 3j-symbols, the definition (58') may be easily inverted. One obtains the multipole expansion of the density matrix :

$$\rho^m_n = \sum_{L=0}^{2j} \frac{2L+1}{2j+1} (-)^L \,(T^M_L)^m_n\; t^L_M \tag{62}$$

or, using (55') :

$$\rho^m_n = \sum_{L=0}^{2j} \sum_{M=-L}^{+L} \frac{2L+1}{2j+1} \,(T^{L\,*}_M)^m_n\; t^L_M \tag{62'}$$

e) The degree of polarization d_ρ of the system described by the density matrix ρ is defined by

$$(d_\rho)^2 = \frac{2j+1}{2j} \left[\mathrm{Tr}\,\rho^2 - \frac{1}{2j+1} \right] \tag{63}$$

The quantity $\mathrm{Tr}\,\rho^2$ is easily calculated in terms of the multipole parameters. Using (62) and (54a) we obtain :

$$\mathrm{Tr}\,\rho^2 = \sum_{L=0}^{2j} \sum_{M=-L}^{+L} \frac{2L+1}{2j+1} \left| t^L_M \right|^2 \qquad (64)$$

The degree of polarization is :

$$(d_\rho)^2 = \sum_{L=1}^{2j} \sum_{M=-L}^{+L} \frac{2L+1}{2j} \left| t^L_M \right|^2 \qquad (65)$$

or

$$(d_\rho)^2 = \sum_{L=1}^{2j} \frac{2L+1}{2j} (t^L_0)^2 + \sum_{L=1}^{2j} \sum_{M=+1}^{L} \frac{2(2L+1)}{2j} (\mathrm{Re}\, t^L_M)^2 + \sum_{L=1}^{2j} \sum_{M=+1}^{L} \frac{2(2L+1)}{2j} (\mathrm{Im} t^L_M)^2 \qquad (65')$$

f) The multipole parameters t^L_M are very interesting because their transformation law under a rotation of the coordinate system is very simple. In fact, using formula (47') for the reduction of the tensor product of representations, equation (7) which expresses the transformation law of the matrix elements ρ^m_n can be written :

$$\begin{pmatrix} n & L & j \\ j & M & m \end{pmatrix} \rho^m_n \rightsquigarrow \begin{pmatrix} n & L & j \\ j & M & m \end{pmatrix} \rho'^m_n = \begin{pmatrix} n' & L & j \\ j & M' & m' \end{pmatrix} \rho^{m'}_{n'} D^L(R)^{M'}_M \qquad (66)$$

or, using the definition (58') one obtains :

$$t^L_M \longrightarrow t'^L_M = t^L_{M'} D^L(R)^{M'}_M \quad , \qquad (66')$$

i.e., for each L, the 2L+1 parameters t^L_M (M= -L, . . . , +L) transform into each other as the components of an irreducible tensor when the frame of reference is rotated.

Thus the density operator may be expanded into a sum of irreducible tensorial operators :

$$\rho = \sum_{L=0}^{2j} \rho^L \qquad (67)$$

The operator

$$\rho^L = \frac{2L+1}{2j+1} \sum_{M=-L}^{+L} (T^L_M)^* \, t^L_M \qquad (67')$$

which transforms as the representation D^L of the rotation group is called a 2^L - multipole.

Because of their transformation law (66'), the multipole parameters are the more convenient ones for the study of angular correlation. The general formulae for the angular distribution and for the angular distribution of polarization take a more compact form when they are written with the multipole parameters t^L_M. However the multipole parameters t^L_M present the practical inconvenient that they represent the Hermitian matrix ρ in a non Hermitian basis. In fact, let us consider the $(2j+1)^2$ - dimensional vector space of $(2j+1) \times (2j+1)$ complex matrices. This vector space can be considered as a Hilbert space $\mathcal{H}_{(2j+1)^2}$ if we put on the complex matrices a Hermitian scalar product. If A and B are two (2j+1)-dimensional complex matrices their scalar product is defined as :

$$\langle A, B \rangle = \mathrm{Tr}(A^\dagger B) \qquad (68)$$

On formula (54a) we see that the $(2j+1)^2$ matrices T^L_M form a real, orthogonal basis of the space $\mathcal{H}_{(2j+1)^2}$ and on formula (62) we see that the multipole parameters t^L_M are the complex coefficients of the expansion of the Hermitian density matrix ρ on this non-Hermitian basis.

In the following we shall consider the real coefficients of the expansion of the Hermitian, trace one, density matrix ρ, on a set of $(2j+1)^2-1$ Hermitian matrices.

Remark :

The multipole parameters defined by (58') or by (62) are the parameters used by Byers and Fenster (Byers-Fenster-63), by Dalitz (Dalitz-66) and by Jackson (Jackson-65).

Several other definitions of these parameters can be found in the literature. If we call t^L_M (B. F) the parameters defined by (58'), we have the following relations between these parameters and the parameters defined by de Rafael (de Rafael-66), by Kotanski and Zalewski (Kotanski-Zalewski-68) and by Ademollo, Gatto and Preparata (Ademollo-Gatto-Preparata-65) :

$$t^L_M \text{ (de Rafael)} = \overline{t^L_M \text{ (B. F)}} \qquad (69a)$$

$$t^L_M \text{ (Kotanski-Zalewski)} = \sqrt{\frac{2L+1}{2j+1}}\; t^L_M \text{ (B. F)} \qquad (69b)$$

$$t^L_M \text{ (Ademollo-Gatto-Preparata)} = \sqrt{\frac{2L+1}{2j+1}}\; \overline{t^L_M \text{ (B. F)}} \qquad (69c)$$

6. THE REAL MULTIPOLE PARAMETERS r^L_M AND THEIR CORRESPONDING OBSERVABLES Q^L_M

We look for a set of real, normalized multipole parameters such that, used as Euclidean coordinates for $\mathcal{E}_N$, the unpolarized state is represented by the origin, and pure states by points on a unit sphere. If we define (for $M > 0$)

$$r^L_0 = \sqrt{\frac{2L+1}{2j}}\; t^L_0 \tag{70a}$$

$$r^L_M = (-)^M \sqrt{\frac{2(2L+1)}{2j}}\; \mathrm{Re}\; t^L_M \tag{70b}$$

$$r^L_{-M} = (-)^M \sqrt{\frac{2(2L+1)}{2j}}\; \mathrm{Im}\; t^L_M \tag{70c}$$

the degree of polarization (65') can be written

$$(d_\rho)^2 = \sum_{L=1}^{2j} \sum_{M=-L}^{L} (r^L_M)^2 \quad , \tag{71}$$

and these multipole parameters r^L_M satisfy the desired conditions.

These measurable quantities are mean values of some polarization observables Q^L_M, by means of which they can be obtained from the polarization density operator ρ,

$$r^L_M = \langle Q^L_M \rangle = \mathrm{Tr}\,(Q^L_M \rho)\;. \tag{72}$$

According to (62), (54a) and (70) the N observables Q^L_M are represented in our $|jm\rangle$ basis by the Hermitian matrices (for $M > 0$)

$$Q^L_0 = \sqrt{\frac{2L+1}{2j}}\; T^L_0 \tag{73a}$$

$$Q^L_M = (-)^M \sqrt{\frac{2L+1}{2j}}\; \frac{1}{\sqrt{2}}\, (T^L_M + (-)^M\, T^L_{-M}) \tag{73b}$$

$$Q^L_{-M} = (-)^M \sqrt{\frac{2L+1}{2j}}\; \frac{-i}{\sqrt{2}}\, (T^L_M - (-)^M\, T^L_{-M}) \tag{73c}$$

Their orthonormality relation is obtained from (54a) :

$$\mathrm{Tr}\,(Q^L_M \,,\, Q^{L'}_{M'}) = \frac{2j+1}{2j}\, \delta_{LL'}\, \delta_{MM'} \tag{74}$$

Remark that these N operators Q^L_M (with $L > 0$) are traceless, according to (54b). They supply a basis for the space $\mathcal{E}'_N$, the translated from the $\mathcal{E}_N$ of our trace one density operators. Thus, in the multipole expansion of the density operators we have to add the trace. From (72) and (74) we obtain

$$\rho = \frac{\mathbb{1}}{2j+1} + \frac{2j}{2j+1} \sum_{L=1}^{2j} \sum_{M=-L}^{+L} r^L_M \, Q^L_M \tag{75}$$

These real multipole parameters r^L_M and their corresponding Hermitian matrices Q^L_M are a simple generalization of the Stokes polarization vector $\vec{P}$ and the Pauli matrices $\vec{\tau}$. In fact, for $j = 1/2$ and $L = 1$ equations (70) to (75) yield

$$r^1_0 = P_3 \tag{70a'}$$

$$r^1_1 = P_1 \tag{70b'}$$

$$r^1_{-1} = P_2 \tag{70c'}$$

$$d_\rho = |\vec{P}| \tag{71'}$$

$$\vec{P} = \langle \vec{\tau} \rangle = \mathrm{tr}(\vec{\tau} . \rho) \tag{72'}$$

$$Q^1_0 = \tau_3 \tag{73a'}$$

$$Q^1_1 = \tau_1 \tag{73b'}$$

$$Q^1_{-1} = \tau_2 \tag{73c'}$$

$$\mathrm{tr}(\tau_i . \tau_j) = 2\,\delta_{ij} \tag{74'}$$

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{P} . \vec{\tau}) \tag{75'}$$

Table 1.- Representation of the observables Q^L_M for $j = \frac{1}{2}, 1, \frac{3}{2}, 2.$

(a) $j = \frac{1}{2}$

$L = 1$

$$Q^1_o = \begin{bmatrix} 1 & \cdot \\ \cdot & -1 \end{bmatrix} \qquad Q^1_1 = \begin{bmatrix} \cdot & 1 \\ 1 & \cdot \end{bmatrix} \qquad Q^1_{-1} = i \begin{bmatrix} \cdot & -1 \\ 1 & \cdot \end{bmatrix}$$

(b) $j = 1$

$L = 1, 2$

$$Q^1_o = \sqrt{\frac{3}{4}} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 0 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix} \qquad Q^1_1 = \sqrt{\frac{3}{8}} \begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & 1 \\ \cdot & 1 & \cdot \end{bmatrix} \qquad Q^1_{-1} = i\sqrt{\frac{3}{8}} \begin{bmatrix} \cdot & -1 & \cdot \\ 1 & \cdot & -1 \\ \cdot & 1 & \cdot \end{bmatrix}$$

$$Q^2_o = \sqrt{\frac{1}{4}} \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & -2 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \qquad Q^2_1 = \sqrt{\frac{3}{8}} \begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & -1 \\ \cdot & -1 & \cdot \end{bmatrix} \qquad Q^2_{-1} = i\sqrt{\frac{3}{8}} \begin{bmatrix} \cdot & -1 & \cdot \\ 1 & \cdot & 1 \\ \cdot & -1 & \cdot \end{bmatrix}$$

$$Q^2_2 = \sqrt{\frac{3}{4}} \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix} \qquad Q^2_{-2} = i\sqrt{\frac{3}{4}} \begin{bmatrix} \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix}$$

(c) $j = \frac{3}{2}$

$L = 1, 2, 3$

$$Q^1_0 = \sqrt{\frac{1}{15}} \begin{bmatrix} 3 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -3 \end{bmatrix} \qquad Q^1_1 = \sqrt{\frac{1}{15}} \begin{bmatrix} \cdot & \sqrt{3} & \cdot & \cdot \\ \sqrt{3} & \cdot & 2 & \cdot \\ \cdot & 2 & \cdot & \sqrt{3} \\ \cdot & \cdot & \sqrt{3} & \cdot \end{bmatrix} \qquad Q^1_{-1} = i\sqrt{\frac{1}{15}} \begin{bmatrix} \cdot & -\sqrt{3} & \cdot & \cdot \\ \sqrt{3} & \cdot & -2 & \cdot \\ \cdot & 2 & \cdot & -\sqrt{3} \\ \cdot & \cdot & \sqrt{3} & \cdot \end{bmatrix}$$

$$Q^2_0 = \sqrt{\frac{1}{3}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} \qquad Q^2_1 = \sqrt{\frac{1}{3}} \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & 0 & \cdot \\ \cdot & 0 & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot \end{bmatrix} \qquad Q^2_{-1} = i\sqrt{\frac{1}{3}} \begin{bmatrix} \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & 0 & \cdot \\ \cdot & 0 & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \end{bmatrix}$$

$$Q^2_2 = \sqrt{\frac{1}{3}} \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} \qquad Q^2_{-2} = i\sqrt{\frac{1}{3}} \begin{bmatrix} \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix}$$

$$Q^3_0 = \sqrt{\frac{1}{15}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -3 & \cdot & \cdot \\ \cdot & \cdot & 3 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{bmatrix} \qquad Q^3_1 = \sqrt{\frac{2}{15}} \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & -\sqrt{3} & \cdot \\ \cdot & -\sqrt{3} & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \qquad Q^3_{-1} = i\sqrt{\frac{2}{15}} \begin{bmatrix} \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \sqrt{3} & \cdot \\ \cdot & -\sqrt{3} & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix}$$

$$Q^3_2 = \sqrt{\frac{1}{3}} \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & -1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \end{bmatrix} \qquad Q^3_{-2} = i\sqrt{\frac{1}{3}} \begin{bmatrix} \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

$$Q^3_3 = \sqrt{\frac{2}{3}} \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix} \qquad Q^3_{-3} = i\sqrt{\frac{2}{3}} \begin{bmatrix} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

(d_1) $j = 2$

$L = 1, 2, 3$

$$Q^1_0 = \sqrt{\tfrac{1}{8}}\begin{bmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 \end{bmatrix} \quad Q^1_1 = \sqrt{\tfrac{1}{16}}\begin{bmatrix} \cdot & \sqrt{2} & \cdot & \cdot & \cdot \\ \sqrt{2} & \cdot & \sqrt{3} & \cdot & \cdot \\ \cdot & \sqrt{3} & \cdot & \sqrt{3} & \cdot \\ \cdot & \cdot & \sqrt{3} & \cdot & \sqrt{2} \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot \end{bmatrix} \quad Q^1_{-1} = i\sqrt{\tfrac{1}{16}}\begin{bmatrix} \cdot & -\sqrt{2} & \cdot & \cdot & \cdot \\ \sqrt{2} & \cdot & -\sqrt{3} & \cdot & \cdot \\ \cdot & \sqrt{3} & \cdot & -\sqrt{3} & \cdot \\ \cdot & \cdot & \sqrt{3} & \cdot & -\sqrt{2} \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot \end{bmatrix}$$

$$Q^2_0 = \sqrt{\tfrac{5}{56}}\begin{bmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 \end{bmatrix} \quad Q^2_1 = \sqrt{\tfrac{5}{112}}\begin{bmatrix} \cdot & \sqrt{6} & \cdot & \cdot & \cdot \\ \sqrt{6} & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & -\sqrt{6} \\ \cdot & \cdot & \cdot & -\sqrt{6} & \cdot \end{bmatrix} \quad Q^2_{-1} = i\sqrt{\tfrac{5}{112}}\begin{bmatrix} \cdot & -\sqrt{6} & \cdot & \cdot & \cdot \\ \sqrt{6} & \cdot & -1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \sqrt{6} \\ \cdot & \cdot & \cdot & -\sqrt{6} & \cdot \end{bmatrix}$$

$$Q^2_2 = \sqrt{\tfrac{5}{56}}\begin{bmatrix} \cdot & \cdot & \sqrt{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{3} & \cdot \\ \sqrt{2} & \cdot & \cdot & \cdot & \sqrt{2} \\ \cdot & \sqrt{3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{2} & \cdot & \cdot \end{bmatrix} \quad Q^2_{-2} = i\sqrt{\tfrac{5}{56}}\begin{bmatrix} \cdot & \cdot & -\sqrt{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\sqrt{3} & \cdot \\ \sqrt{2} & \cdot & \cdot & \cdot & -\sqrt{2} \\ \cdot & \sqrt{3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{2} & \cdot & \cdot \end{bmatrix}$$

$$Q^3_0 = \sqrt{\tfrac{1}{8}}\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \end{bmatrix} \quad Q^3_1 = \sqrt{\tfrac{1}{16}}\begin{bmatrix} \cdot & \sqrt{3} & \cdot & \cdot & \cdot \\ \sqrt{3} & \cdot & -\sqrt{2} & \cdot & \cdot \\ \cdot & -\sqrt{2} & \cdot & -\sqrt{2} & \cdot \\ \cdot & \cdot & -\sqrt{2} & \cdot & \sqrt{3} \\ \cdot & \cdot & \cdot & \sqrt{3} & \cdot \end{bmatrix} \quad Q^3_{-1} = i\sqrt{\tfrac{1}{16}}\begin{bmatrix} \cdot & -\sqrt{3} & \cdot & \cdot & \cdot \\ \sqrt{3} & \cdot & \sqrt{2} & \cdot & \cdot \\ \cdot & -\sqrt{2} & \cdot & \sqrt{2} & \cdot \\ \cdot & \cdot & -\sqrt{2} & \cdot & -\sqrt{3} \\ \cdot & \cdot & \cdot & \sqrt{3} & \cdot \end{bmatrix}$$

$$Q^3_2 = \sqrt{\tfrac{5}{16}}\begin{bmatrix} \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot \\ 1 & \cdot & \cdot & \cdot & -1 \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix} \quad Q^3_{-2} = i\sqrt{\tfrac{5}{16}}\begin{bmatrix} \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot \end{bmatrix}$$

$$Q^3_3 = \sqrt{\tfrac{5}{16}}\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix} \quad Q^3_{-3} = i\sqrt{\tfrac{5}{16}}\begin{bmatrix} \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{bmatrix}$$

(a_2) $j = 2$

$L = 4$

$$Q^4_0 = \sqrt{\frac{1}{56}} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$Q^4_1 = \sqrt{\frac{5}{112}} \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & \cdot & -\sqrt{6} & \cdot & \cdot \\ \cdot & -\sqrt{6} & \cdot & \sqrt{6} & \cdot \\ \cdot & \cdot & \sqrt{6} & \cdot & -1 \\ \cdot & \cdot & \cdot & -1 & \cdot \end{pmatrix} \qquad Q^4_{-1} = i\sqrt{\frac{5}{112}} \begin{pmatrix} \cdot & -1 & \cdot & \cdot & \cdot \\ 1 & \cdot & \sqrt{6} & \cdot & \cdot \\ \cdot & -\sqrt{6} & \cdot & -\sqrt{6} & \cdot \\ \cdot & \cdot & \sqrt{6} & \cdot & 1 \\ \cdot & \cdot & \cdot & -1 & \cdot \end{pmatrix}$$

$$Q^4_2 = \sqrt{\frac{5}{112}} \begin{pmatrix} \cdot & \cdot & \sqrt{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\sqrt{8} & \cdot \\ \sqrt{3} & \cdot & \cdot & \cdot & \sqrt{3} \\ \cdot & -\sqrt{8} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{3} & \cdot & \cdot \end{pmatrix} \qquad Q^4_{-2} = i\sqrt{\frac{5}{112}} \begin{pmatrix} \cdot & \cdot & -\sqrt{3} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{8} & \cdot \\ \sqrt{3} & \cdot & \cdot & \cdot & -\sqrt{3} \\ \cdot & -\sqrt{8} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{3} & \cdot & \cdot \end{pmatrix}$$

$$Q^4_3 = \sqrt{\frac{5}{16}} \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \end{pmatrix} \qquad Q^4_{-3} = i\sqrt{\frac{5}{16}} \begin{pmatrix} \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$Q^4_4 = \sqrt{\frac{5}{8}} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad Q^4_{-4} = i\sqrt{\frac{5}{8}} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Part I. APPENDIX 6

POSITIVITY OF THE DENSITY MATRICES

AND CONVEXITY OF THE POLARIZATION DOMAIN

In this appendix we consider the Hermitian operators on the Hilbert space $\mathcal{H}_n$ of dimension n. They form a n^2 dimensional vector space $\mathcal{E}_{n^2}$ on which we can put the Euclidian scalar product (as we did in equation I(4)),

$$(\rho_1, \rho_2) = \operatorname{tr} \rho_1 \rho_2 . \tag{1}$$

So $\mathcal{E}_{n^2}$ is an Euclidian space; the distance between the two points ρ_1 and ρ_2 is $|\rho_1 - \rho_2| = (\rho_1 - \rho_2, \rho_1 - \rho_2)^{1/2}$.

A Hermitian matrix can be written

$$R = \Sigma_i \lambda_i P_i \tag{2}$$

where λ_i are its eigenvalues (real numbers) and P_i are Hermitian projectors, i.e.,

$$P_i^* = P_i = P_i^2 \tag{3}$$

the multiplicity of the λ_i eigenvalue is given by

$$\operatorname{tr} P_i = \operatorname{rank} P_i \tag{4}$$

Definition. A Hermitian matrix is positive (or semi-positive) if all its eigenvalues λ_i are > 0 (or $\geqslant 0$).

Outside this appendix we generally use the word positive for semi-positive. In this more technical appendix we shall distinguish the two notions. We shall note $R > 0$, $R \geqslant 0$ for R positive, R semi-positive.

Theorem 1. $R > 0$ (or $R \geqslant 0$) $\Leftrightarrow$ any $|x\rangle \in \mathcal{H}_n$, $\langle x | Rx \rangle > 0$ (or $\geqslant 0$).

Let $R = \Sigma_i \lambda_i P_i$

then $\langle x | Rx \rangle = \Sigma_i \lambda_i \langle x | P_i x \rangle = \Sigma_i \lambda_i \langle x | P_i P_i x \rangle = \Sigma_i \lambda_i \langle P_i x | P_i x \rangle$

since $\langle P_i x | P_i x \rangle = \| P_i x \|^2 \geqslant 0$; $\lambda_i \geqslant 0 \Rightarrow \langle x | Rx \rangle \geqslant 0$

(Note $|x\rangle \neq 0 \Rightarrow$ some $P_i |x\rangle \neq 0$ since $\Sigma_i P_i = 1$, so $\lambda_i > 0 \Rightarrow \langle x | Rx \rangle > 0$).

Conversely if for all $|x\rangle \in \mathcal{H}_n$, $\langle x|Rx\rangle > 0$ (or $\geqslant 0$) this is true for the eigenvectors of R; e.g. if $P_i|x_i\rangle = |i\rangle$, then $\langle i|Ri\rangle = \lambda_i > 0$ (or $\geqslant 0$), hence all eigenvalues are > 0 (or $\geqslant 0$).

Theorem 2. If $\alpha_i > 0$, $R_i > 0$, then $\sum \alpha_i R_i > 0$. Indeed for every $x\rangle \in \mathcal{H}_n$, $\langle x|R_i x\rangle > 0$ so $\sum_i \alpha_i \langle x|R_i x\rangle = \langle x|\sum_i \alpha_i R_i x\rangle > 0$.

Definition. A cone $\mathcal{C}$ in a vector space is a domain such that if $a \in \mathcal{C}$, $b \in \mathcal{C} \Rightarrow a+b \in \mathcal{C}$.

As a corollary of theorem 2 we see that the positive matrices on $\mathcal{H}_n$ form a cone $\mathcal{C}_n$ in $\mathcal{E}_{n^2}$, which is an open set of $\mathcal{E}_{n^2}$. The boundary of the cone, $\partial\mathcal{C}_n$, is the set of semi-positive matrices which are not positive; those matrices then have some eigenvalues zero. The rank of a matrix ρ is the dimension of its image space $\rho\mathcal{H}_n$. Positive matrices have rank n; matrices of $\partial\mathcal{C}_n$ have rank $< n$. Let us call $\partial_k \mathcal{C}_n$ $(k<n)$ the set of semi-positive matrices of rank k.

$$\partial \mathcal{C}_n = \bigcup_{0<k<n} \partial_k \mathcal{C}_n \qquad (5)$$

As we saw in I, in some experiments an upper limit r of the rank of the polarization matrix is known. So it belongs to $\bigcup_{0<k\leqslant r} \partial_k \mathcal{C}_n$. We denote $\bar{\mathcal{C}}_n = \mathcal{C}_n \cup \partial\mathcal{C}_n$ the closure of $\mathcal{C}_n$. It is the set of $\rho \geqslant 0$. Note that if $\rho_1 \geqslant 0$ and $\rho_2 \geqslant 0$, $\rho_1\rho_2$ is not necessary $\geqslant 0$. However :

Theorem 3. If $\rho_1 \geqslant 0$ and $\rho_2 \geqslant 0$, then $\operatorname{tr}\rho_1\rho_2 \geqslant 0$. Indeed ρ_1 can be written $\rho_1 = \sum_i \lambda_i P_i = \sum_i \lambda_i |i\rangle\langle i|$ where the $|i\rangle \in \mathcal{H}_n$ form an orthonormal basis of eigenvectors of ρ_1 and $\lambda_i \geqslant 0$. Then

$$\operatorname{tr}\rho_1\rho_2 = \sum_i \lambda_i \langle i|\rho_2|i\rangle$$

which is $\geqslant 0$ by theorem 1. More specifically, if $\rho_1 > 0$, $\rho_2 \geqslant 0$, theorem 1 shows that $(\rho_1, \rho_2) > 0$. Indeed

$$\operatorname{tr}\rho_1\rho_2 = \sum_i \lambda_i \rho_2 P_i \qquad \text{with } \sum_i P_i = I,\ \lambda_i > 0.$$

since $\sum_i \operatorname{tr}\rho_2 P_i = \operatorname{tr}\rho_2 = 1$ at least one of the terms $\operatorname{tr}\rho_2 P_i = \langle i|\rho_2|i\rangle$ is > 0 so $\operatorname{tr}\rho_1\rho_2 > 0$.

Hence as a corollary, if two polarization states are orthogonal, they are both in $\partial \mathcal{C}_n$.

Consider now equations (6) to (8) of part I, where $\mathcal{E}_N$ is the n^2-1 dimensional subspace of $\mathcal{E}_{n^2}$ defined by the condition $\operatorname{tr} \rho = 1$. It is an Euclidian subspace of $\mathcal{E}_{n^2}$ and the polarization domain $\mathcal{D}_n$, i.e., the set of density matrices $\rho \geqslant 0$, $\operatorname{tr} \rho = 1$ is

$$\mathcal{D}_n = \bar{\mathcal{C}}_n \cap \mathcal{E}_N . \qquad (6)$$

The density matrix of the unpolarized state is

$$\rho_o = \frac{1}{n} I \qquad (7)$$

Since the vector ρ_o in $\mathcal{E}_{n^2}$ is orthogonal to the subspace $\mathcal{E}_N$ of the Euclidian space $\mathcal{E}_{n^2}$, with

$$\rho = \sqrt{\frac{n-1}{n}}\, \rho' + \rho_o \qquad (8)$$

we deduce

$$(\rho, \rho) = \frac{n-1}{n} \operatorname{tr} \rho'^2 + \frac{1}{n}$$

so

$$1 \geqslant (\rho, \rho) \geqslant \frac{1}{n} = (\rho_o, \rho_o) \qquad (9) .$$

Relation (9) satisfied by spin density matrices has been used in the physics literature. It is much weaker than $\operatorname{tr} \rho = 1$, $\rho \geqslant 0$. We saw in Fig. 1 page I. 2 - 3, that $\mathcal{D}_n$ is inside the sphere $\mathcal{S}_{N-1}$, intersection of $\mathcal{E}_N$ ($\operatorname{tr} \rho = 1$) and of the unit sphere $\mathcal{S}_N$ ($(\rho, \rho) = 1$) of $\mathcal{E}_{n^2} = \mathcal{E}_{N+1}$. Furthermore, the domain

$$\mathcal{D}_n \cap \mathcal{S}_N = \partial_1 \mathcal{D}_n \qquad (10)$$

is the set of density matrices of pure states, i.e., rank one projector $P_i = P_i^2$ $\operatorname{tr} P_i = \operatorname{tr} P_i^2 = 1$.

In part I and in all applications, we will multiply the length in the N dimensional Euclidian space $\mathcal{E}_N$ by the factor $\sqrt{\frac{n}{n-1}}$ so that the sphere $\mathcal{S}_{N-1}$ of center ρ_o has radius one in the new scale.

Definition of convexity. A domain D of a real vector space $\mathcal{E}$ is convex if all vectors $\alpha a + \beta b$ with $0 \leqslant \alpha$, $0 \leqslant \beta$, $\alpha + \beta = 1$ are elements of D when $a \in D$, $b \in D$. We can also say that all points between a and b on the straight line joining a and b belong to the domain when a and b do.

Example of a convex domain : the linear manifolds of $\mathcal{E}_n$ that we also call k-planes when their dimension is $k \leq n$.

It is easy to check that :

a) The intersection of convex domains is a convex domain.

b) The linear transformed of a convex domain is a convex domain.

c) A convex domain is connex.

Theorem 2 shows that the cone $\mathcal{C}$ of positive (also the cone $\bar{\mathcal{C}}$ of semi-positive) matrices in $\mathcal{H}_n$ is convex.

From property a) and equation (6) the polarization domain $\mathcal{D}_n$ is convex. Since $\mathcal{D}_n$ is in the Euclidian spaces $\mathcal{E}_N \subset \mathcal{E}_{n^2}$, we want to make some geometrical remarks on convex domains of Euclidian space $\mathcal{E}$ and their orthogonal projection on a subspace $\mathcal{P}$. Let P be this orthogonal projection; it is the identity on $\mathcal{P}$ and $P^2 = P$. Let $\mathcal{D}$ be a domain of $\mathcal{E}$, the domain $\Gamma = P\mathcal{D} \subset \mathcal{P}$ is called the projection of $\mathcal{D}$ on $\mathcal{P}$. From remark b), $\mathcal{D}$ convex $\Rightarrow$ $P\mathcal{D}$ convex, of course $C = \mathcal{D} \cap \mathcal{P} \subset \Gamma$. We can always consider an Euclidian space $\mathcal{E}$ as a vector space, after we have chosen a point 0 of $\mathcal{E}$ to be the origin of the vector space, so if $a, b \in \mathcal{E}$, $a+b$ is defined. An involution K of $\mathcal{E}$ is an Euclidian transformation (i.e., it preserves the distance) whose square is the identity I in $\mathcal{E}$ i.e. $K^2 = I$. It can be shown that the fixed points of K form an Euclidian subspace $\mathcal{P} \subset \mathcal{E}$ and K is the symmetry through $\mathcal{P}$ i.e., if P is the projection on $\mathcal{P}$ and $x \in \mathcal{E}$ one sees (see Fig. 1)

$$Kx = x-2(x-Px) = -x + 2Px \qquad (11)$$

$$\text{i.e.} \qquad K = -I + 2P . \qquad (11')$$

(Note that $\mathcal{P}$ may be reduced to a point !). From now on we denote by $K_{\mathcal{P}}$ the symmetry through $\mathcal{P}$.

<u>Definition.</u> If $K_{\mathcal{P}}\mathcal{D} = \mathcal{D}$, then $\mathcal{P}$ is called a symmetry p-plane for $\mathcal{D}$, where $p = \dim \mathcal{P}$.

<u>Theorem 4</u> . If $\mathcal{P}$ is a symmetry plane of the convex domain $\mathcal{D}$, then

$$C = \Gamma \quad \text{where} \quad C = \mathcal{D} \cap \mathcal{P} \quad \text{and} \quad \Gamma = P_{\mathcal{P}}\mathcal{D} .$$

Let $a \in \Gamma$, there exists $b \in \mathcal{D}$ such that $Pb = a$. Since $\mathcal{P}$ is a symmetry plane $K_{\mathcal{P}} b \in \mathcal{D}$ and from the convexity $\frac{1}{2}(b + K_{\mathcal{P}} b) \in \mathcal{D}$ and from (11)

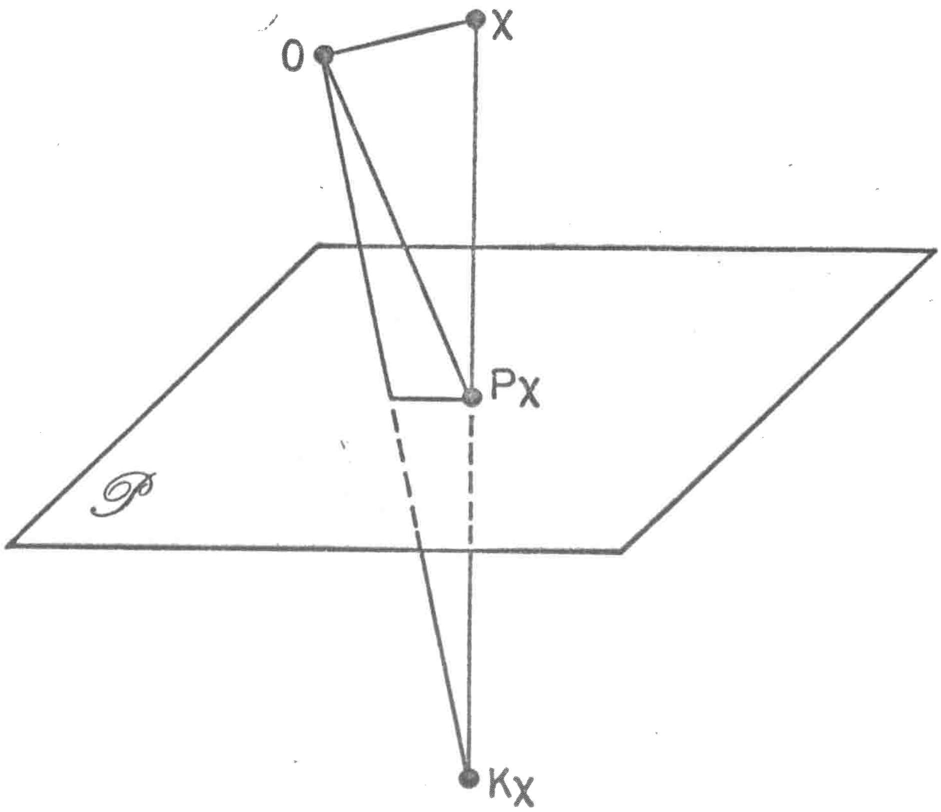

FIG.1

Fig. 1. - Point x of a Euclidian vector space, with its projection P x on the subspace $\mathcal{P}$, and its symmetric K x through $\mathcal{P}$. Remark that K x = 2 P x - x .

$\frac{1}{2}(b + Kb) = Pb = a$ so $a \in C$, hence $\Gamma \subset C$ and since $C \subset \Gamma$, therefore $C = \Gamma$. Of course, this was geometrically obvious.

The diagonal matrices of $\mathcal{D}_n$.

Let us remark that there are p-planes $\wp$ such that $\mathcal{D}_n \wp = P_\wp \mathcal{D}$ which are not symmetry planes of $\mathcal{D}$. Such an example is obtained by the domain Δ_n of the semi-positive diagonal matrix of trace 1. So $\Delta_n \subset \mathcal{D}_n$ the domain of density matrices. Let $\wp$ be the n-plane of $\mathcal{E}_{n^2}$ containing all diagonal Hermitian matrices on $\mathcal{H}_n$. By definition $\Delta_n = \wp_\cap \mathcal{D}_n$. We now remark that if $\rho \geqslant 0$, each diagonal matrix element ρ_{ii} is positive. Indeed let $|i\rangle$ the vector of coordinates $\xi^\alpha = \delta^\alpha_i$ ($\alpha = 1$ to n) then $\langle i|\rho|i\rangle =$ $= \rho_{ii} \geqslant 0$ by theorem 2. Hence $P_{\wp\cap} \mathcal{D}_n = \Delta_n$.

If $n = 1$ or $n = 2$, the n-plane of diagonal matrices is a symmetry plane of $\mathcal{D}_n$. This is not true for $n \geqslant 0$ as it is shown by the two Hermitian matrices λ, λ' with $\mathrm{tr}\,\lambda = \mathrm{tr}\,\lambda' = 1$, $\lambda' = K_\wp \lambda$ (i.e., the non diagonal elements are changed of sign).

$$\lambda = \frac{1}{5}\left(\begin{array}{ccc|c} 1 & 1 & 1 & \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & \\ \hline & 0 & & 0 \end{array}\right) \qquad \lambda' = K_\wp \lambda = \left(\begin{array}{ccc|c} 1 & -1 & -1 & \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & \\ \hline & 0 & & 0 \end{array}\right)$$

the eigenvalues of λ are $\frac{1}{5}$, $\frac{1}{5}(2+\sqrt{2})$, $\frac{1}{5}(2-\sqrt{2})$, and n-3 times zeros, those of λ' are $\frac{3}{5}$, $\frac{1}{5}(1+\sqrt{2})$, $\frac{1}{5}(1-\sqrt{2})$, and n-3 times zeros and $1-\sqrt{2} < 0 \Rightarrow \lambda'$ is not semi-positive.

Note that the convex domain Δ_n is the regular polyhedra in $\mathbb{R}^n \subset \mathcal{E}_{n^2}$ $0 \leq \lambda_i$, $\Sigma_{i=1}^n \lambda_i = 1$. It is in $\mathcal{E}_N$ ($\mathrm{tr}\,\rho = 1$). It is the regular n-hedron (for $n = 3$: equilateral triangle, $n = 4$: tetrahedron, etc...) whose center is $\rho_0 = \frac{1}{n} \mathbb{1}$. Its n vertices are the n diagonal rank one projectors P_i (one $\lambda_i = 1$, all others are zero). The straight line $P_i \rho_0$ cut the n-hedron in Q_i at the center of the face opposite to the vertex P_i and it is perpendicular to it. (Note that $(Q_i)_{\alpha\beta} = \frac{1}{n-1}(\delta_{\alpha\beta} - \delta_{i\alpha}\delta_{i\beta})$)

the square of the distance $\rho_o P_i$ in $\mathcal{E}_N$ is $tr(\frac{1}{n} - P_i)^2 = \frac{n-1}{n}$, in $\mathcal{E}'_N$ is 1

the square of the distance $\rho_o Q_i$ in $\mathcal{E}_N$ is $tr(\rho_o - Q_i)^2 = \frac{1}{n}$, in $\mathcal{E}'_N$ is $\frac{1}{n-1}$

finally, the scalar product in $\mathcal{E}'_N$ of two distinct pure states is

$(i \neq j)$, $\frac{n}{n-1} tr(P_i - \rho_o)(P_j - \rho_o) = -\frac{1}{n-1}$ as equation I (8) shows since

$$P_i P_j = 0 \Rightarrow (P_i, P_j) = 0 .$$

Involutions in $\mathcal{E}$.

We will now study some physically interesting examples of involution in $\mathcal{E}_{n^2}$.

Let $U(n)$ be the unitary group of $n \times n$ matrices on $\mathcal{H}_n$. It acts on $\mathcal{E}_{n^2}$ as

$$\rho \rightsquigarrow u \rho u^* = u \rho u^{-1} \tag{12}$$

Since such a unitary transformation respects the scalar product on $\mathcal{E}_{n^2}$

$$(\rho_1, \rho_2) = tr\, \rho_1 \rho_2 = tr\, u \rho_1 u^{-1} u \rho_2 u^{-1} = (u \rho_1 u^{-1}, u \rho_2 u^{-1}) \tag{13}$$

it is an isometry of $\mathcal{E}_{n^2}$. Furthermore it preserves the eigenvalues of ρ so $U(n)$ transforms $\mathcal{D}_n$ in itself. If $u^2 = 1$, the transformation (12) is an involution on $\mathcal{E}_{n^2}$. Let us give examples of such involution :

The B-symmetry.

For the B-symmetry, that we shall denote K_B

$$u = D^{(j,n)}(S_n) \quad \text{or} \quad u = \bigotimes_k D^{(j_k, n_k)}(S_n) \tag{14}$$

where S_n is the symmetry in space-time through the reaction plane (three-plane of $\underline{p}_A, \underline{p}_B, \underline{p}_C$, the energy momentum of the beam particle A, of the target particle B and of the observed particle C), $\underline{n}$ is the unit (space-like) vector normal to this three-plane. Since $S_n^2 = 1$, $u^2 = (-1)^{2j}$ (see equation A1 (25) or A1 (26)) then $\rho \rightsquigarrow u^2 \rho u^{-2} = \rho$, the B-symmetry does induce an involution on $\mathcal{E}_{n^2}$. The fixed points of this involution form the p-plane $\mathcal{B}$ of B-symmetric matrices, with

$$\left.\begin{aligned} p &= \tfrac{1}{2} n^2 && \text{if } n \text{ is even} \\ p &= \tfrac{1}{2}(n^2+1) && \text{if } n \text{ is odd} \end{aligned}\right\} \tag{15}$$

So

$$C_{\mathcal{B}} = \mathcal{B} \cap \mathcal{D}_n = \Gamma_{\mathcal{B}} \quad (16)$$

is the set of B-symmetric density matrices.

Indeed, (as we shall see in part IV) in an experiment with polarized target if we observe only the B-symmetric part of the polarization matrix, it is a positive matrix which is that which would have been observed if the target had not been polarized.

<u>The set of alignment matrices.</u>

As we have seen in I.4, if $\mathcal{H}_n(\underline{p})$ is the 2j+1 dimensional Hilbert space of polarization states of a particle of spin j and energy momentum $\underline{p}$, the little group $\mathcal{L}_p$ of the "rotations" and space-like symmetries of the Lorentz group $\mathcal{L}$ which leave $\underline{p}$ invariant, acts on $\mathcal{H}_n(\underline{p})$ through the 2j+1 dimensional irrep of the orthogonal O(3) group (three-dimensional rotation and symmetry group) $D^{(j,\eta)}$. Therefore this group acts on $\mathcal{E}_{n^2}$ by the representation $D^{(j,\eta)} \otimes \overline{D}^{(j,\eta)} \sim D^{(j,\eta)} \otimes D^{(j,\eta)} \sim \oplus_{L=0}^{2j} D^{(L,+1)}$. This induces the decomposition of $\mathcal{E}_{n^2}$ into the direct sum of space

$$\mathcal{E}_{n^2} = \oplus_{L=0}^{2j} \mathcal{E}^{(L)} \quad (17)$$

$$\mathcal{E}_N = \oplus_{L=1}^{2j} \mathcal{E}^{(L)} \quad (17')$$

and the corresponding decomposition of $\rho \in \mathcal{E}_{n^2}$

$$\rho = \rho_0 + \Sigma_{L=1}^{2j} \rho^{(L)} \quad (18).$$

A density matrix ρ is an alignment matrix if all its components $\rho^{(L)} = 0$ for L odd ; we also say ρ has only even-multipoles. The generalization to $\mathcal{H}_n = \otimes_{k=1}^{r} \mathcal{H}_{2j_k+1}(p_k)$, the Hilbert space of polarization states of r particles with spin j_k, energy momentum p_k, is straightforward. (See I(11') for two particles). The irrep of O(3) on $\mathcal{H}_n$ is $\otimes_k D^{(j_k,\eta_k)}$ and ρ can be expanded in r-uple multipoles

$$\rho = \Sigma_i \Sigma_{L_{k_i}=0}^{2j_k} \rho^{(L_{k_1}, L_{k_2}, \ldots, L_{k_r})} \quad (19)$$

ρ is an alignment matrix if $\rho^{(L_{k_1}, \ldots, L_{k_r})} = 0$ when $\Sigma_i L_{k_i}$ is odd. Note that the polarization matrix of the i-particle is

$$\rho_i = \Sigma_{L_k=0}^{2j_i} \rho^{(0,0,\ldots,L_k,\ldots,0)} \tag{20}$$

so that the alignment matrix of ρ_i is just that obtained from the single-particle polarization matrix.

We can define the involution $K_{\mathcal{a}}$ on $\mathcal{E}_{n^2}$

$$K_{\mathcal{a}}\rho = \Sigma_{L=0}^{2j} (-1)^L \rho^{(L)} \tag{21}$$

when ρ is given by (18) and

$$K_{\mathcal{a}}\rho = \Sigma \; \Sigma_{L_{k_i}=0}^{2j_i} (-1)^{\Sigma_i L_{k_i}} \rho^{(L_{k_1}, \ldots, L_{k_n})} \tag{22}$$

when ρ is given by (19).

Note that the fixed points of $K_{\mathcal{a}}$ form the p-plane $\mathcal{a}$ of aligned matrices, and $K_{\mathcal{a}}$ does not change the length of the matrices $(K_{\mathcal{a}}\rho, K_{\mathcal{a}}\rho) = (\rho, \rho)$.

We want to show that $K_{\mathcal{a}}$ leaves $\mathcal{D}_n$ invariant, so $\mathcal{a}$ is a symmetry plane of $\mathcal{D}_n$.

Consider the involution $K_{\mathcal{T}}$ which transposes the matrices of $\mathcal{E}_{n^2}$:

$$K_{\mathcal{T}}\rho = \rho^T \tag{23}$$

Its fixed points form the subspace of the symmetrical matrices of $\mathcal{E}$; $K_{\mathcal{T}}$ does not change the eigenvalues of the matrices so it transforms $\mathcal{D}_n$ in itself.

Let us call $K_{\mathcal{J}}$ the involution induced (as in 12) by

$$u = D^{(j,\eta)}(S_{n^{(2)}}) = \eta \Gamma^j, \qquad \text{A1.(53)}$$

where

$$(\Gamma^j)^{\lambda}_{\lambda'} = (-1)^{j-\lambda} \delta^{-\lambda}_{\lambda'}, \qquad \text{A1.(50)}$$

indeed $u^2 = (\Gamma^j)^2 = (-1)^{2j} \mathbb{1}$.

We remark that

$$K_{\mathcal{T}} K_{\mathcal{J}} = K_{\mathcal{J}} K_{\mathcal{T}}, \tag{24}$$

indeed

$$(\Gamma^j \rho (\Gamma^j)^{-1})^T = \Gamma^j \rho^T (\Gamma^j)^{-1}$$

because

$$(\Gamma^j)^T = (-1)^{2j} \Gamma^j \qquad A : (51)$$

Furthermore the product of two commuting involutions is an involution. This is the case of $K_{\mathcal{T}} K_{\mathcal{J}} = K_{\mathcal{J}} K_{\mathcal{T}}$. We have shown in A2. that

$$K_{\mathcal{A}} = K_{\mathcal{T}} K_{\mathcal{J}} \qquad (25)$$

Indeed equation A2.(57) is

$$K_{\mathcal{J}} K_{\mathcal{T}} T^L_M = \Gamma^j (T^L_M)^T (\Gamma^j)^{-1} = (-1)^L T^L_M .$$

Since both $K_{\mathcal{T}}$ and $K_{\mathcal{J}}$ transform $\mathcal{D}_n$ into itself, this is also the case of $K_{\mathcal{A}}$.

To summarize, the subspace of alignment matrices is a symmetry plane of $\mathcal{D}_n$ and $C_{\mathcal{A}} = \Gamma_{\mathcal{A}}$.

Theorem 5. Let $\mathcal{V}$ and $\mathcal{W}$, be two symmetry planes (which might be of different dimension) of the convex domain $\mathcal{D}_n$. If

$$P_{\mathcal{V}} P_{\mathcal{W}} = P_{\mathcal{W}} P_{\mathcal{V}} \qquad (26)$$

then $\mathcal{W}$ is a symmetry plane of $C_{\mathcal{V}} = \mathcal{V} \cap \mathcal{D} = P_{\mathcal{V}} \mathcal{D} = \Gamma_{\mathcal{V}}$. (By syntactic symmetry $\mathcal{V}$ is a symmetry plane of $\mathcal{W} \cap \mathcal{D}$).

From (11'), $K_{\mathcal{T}} = -I + 2P_{\mathcal{T}}$ we obtain that $P_{\mathcal{V}}, P_{\mathcal{W}}, K_{\mathcal{V}}, K_{\mathcal{W}}$ form a set of commuting operators. Let $a \in \mathcal{V} = P_{\mathcal{V}} \mathcal{E}$. So there exists x such that $a = P_{\mathcal{V}} x$

$$K_{\mathcal{W}} a = K_{\mathcal{W}} P_{\mathcal{V}} x = P_{\mathcal{V}} K_{\mathcal{W}} x \in P_{\mathcal{V}} \mathcal{E} = \mathcal{V} .$$

Hence $\mathcal{W}$ is a symmetry plane of $\mathcal{V}$; it is also a symmetry plane of $\mathcal{D}_n$, hence it is a symmetry plane of their intersection : $\mathcal{V} \cap \mathcal{D}$.

Corollary. Of course, instead of (26) we could have used

$$K_{\mathcal{V}} K_{\mathcal{W}} = K_{\mathcal{W}} K_{\mathcal{V}} \qquad (26')$$

in the theorem, since $P_{\rho} = \frac{1}{2} (I + K_{\rho})$.

We note that $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$ commute. Indeed for one particle states, if we shorten $D^{(j,\eta)}(S_n)$ into D^j we have

$$K_{\mathcal{A}} K_{\mathcal{B}} \rho = (\Gamma^j D^j \rho (D^j)^{-1} (\Gamma^j)^{-1})^T = \bar{\Gamma}^j \bar{D}^j \bar{\rho} (\bar{D}^j)^{-1} (\bar{\Gamma}^j)^{-1} = \Gamma^j D^j \rho^T D^{j-1} \Gamma^{j-1}$$

since ρ and $K_{\mathcal{A}} K_{\mathcal{B}} \rho$ are Hermitian matrices and $\bar{\Gamma}^j = \Gamma^j$. (A 1.50) ; using equations (A 1.49) : $\bar{D}^j = \Gamma D \Gamma^{-1}$, and (A 1.51) : $\Gamma^{j-1} = (-1)^{2j} \Gamma^j = (\Gamma^j)^T$

we can transform this equation into :

$$K_{\mathcal{A}} K_{\mathcal{B}} \rho = D^j \Gamma^{j-1} \rho^T \Gamma^j D^{j-1} = K_{\mathcal{B}} (\Gamma^{j-1} \rho^T \Gamma^j) = K_{\mathcal{B}} (\Gamma^j \rho^T \Gamma^{j-1}) = K_{\mathcal{B}} K_{\mathcal{A}} \rho.$$

For r particle states

$$D^j = \bigotimes_i D^{(j_i , \eta_i)} (S_n)$$

and Γ^j is to be replaced by

$$\otimes_i \Gamma^{j_i}$$

and the proof still holds.

So theorem 5 tells us that $\mathcal{B}_\cap \mathcal{D}_n$ the domain of B-symmetric density matrices, has $\mathcal{A}$, the domain of aligned matrices as symmetry plane and also $\mathcal{B}$ is a symmetry plane of $\mathcal{A}_\cap \mathcal{D}_n$.

From now on we will prefer the expression even polarization to alignment.

One particle-state, B-symmetry, even polarization and pure states.

As we have seen in A3, for one particle, B-symmetry imposes to ρ the conditions :

in transversity quantization $\rho^m_{\ m'} = (-)^{m-m'} \rho^m_{\ m'}$, (27)

in helicity quantization $\rho = \Gamma \rho \Gamma^{-1}$. (28)

For a pure state $\rho = x \rangle \langle x$. Let ξ be the components of x .

In transversity, B-symmetry is equivalent to

either $\xi^{j-1} = \xi^{j-3} = \xi^{j-5} = \dots = 0$, (29)

or $\xi^j = \xi^{j-2} = \xi^{j-4} = \dots = 0$. (29')

In helicity, B-symmetry is equivalent to $\Gamma |x\rangle = \lambda |x\rangle$ with $\lambda^2 = (-)^{2j}$ since $\Gamma^2 = (-)^{2j} I$ so, explicitly

$$\lambda \xi^m = (-)^{j-m} \xi^{-m} . \quad (30)$$

As we have just seen, to say that the polarization matrix ρ has only even polarization is equivalent to the condition

$$\rho^T = \Gamma \rho \Gamma^{-1} = \bar{\rho} . \tag{31}$$

It implies for pure states

$$\Gamma | x \rangle = \lambda | \bar{x} \rangle .$$

If we multiply by Γ both members we obtain

$$(-)^{2j} | x \rangle = \lambda \Gamma | \bar{x} \rangle ,$$

which is equivalent to

$$(-)^{2j} \bar{\lambda}^{-1} | x \rangle = \Gamma | x \rangle ,$$

so $\lambda \bar{\lambda} = (-)^{2j}$. Hence,

Theorem 6. For half integer spin, there are no aligned pure states, i.e., pure states with even polarization only.

We had noticed in A 3 that the one-particle polarization matrix can be considered as a checker board with :

- black squares if (m-m') is even (this includes the diagonal)
- white squares if (m-m') is odd.

The Bohr symmetry implies conditions on the density matrix elements :

In transversity quantization :

B-condition ⟺ white squares have zeros

In helicity quantization :

B-condition ⟺ { i) the black square matrix is symmetrical through the center
ii) the white square matrix is antisymmetrical through the center

Even polarization ⟺ { i) the black square matrix is symmetrical through the second diagonal
ii) the white square matrix is antisymmetrical through the second diagonal.

Even polarization + B-condition ⟺ The polarization matrix is real and symmetrical through the center and the two diagonals

ERRATUM

The definition, page A6 - 2, line 6, should be corrected :

<u>Definition</u>. A cone $\mathcal{C}$ in a vector space with vertex at the origin is a set of points such that $a \in \mathcal{C}$ and $\lambda \geqslant 0$ implies $\lambda a \in \mathcal{C}$.

B I B L I O G R A P H Y

Ademollo - Gatto - Preparata - 65
M. Ademollo, R. Gatto and G. Preparata
Phys. Rev. 140, B192 (1965)

Basel - 61
Proceedings of the International Symposium on Polarization Phenomena of Nucleons, Basel, Switzerland, 1960
Helv. Phys. Acta, Suppl. 6, 436 (1961)

Byers - Fenster - 63
N. Byers and S. Fenster
Phys. Rev. Let. 11, 52 (1963)

Cohen Tannoudji - Morel - Navelet - 68
G. Cohen-Tannoudji, A. Morel, and H. Navelet
Ann. Phys. 46, 2, 239 (1968)

Condon - Shortley - 35
E.U. Condon and G.H. Shortley
The theory of Atomic Spectra
University Press, Cambridge (1935)

Dalitz - 66
R.H. Dalitz
Nucl. Phys. 87, 89 (1966)

De Rafael - 66
E. de Rafael
Théorie relativiste des corrélations angulaires et application à l'analyse phénoménologique des particules et résonances
Ann. Inst. Henri Poincaré, V2, 83 - 204

Gottfried - Jackson - 64
K. Gottfried and J.D. Jackson
Nu. Cim. 34, 735 (1964)

Jackson - 65
J.D. Jackson
in High Energy Physics, Les Houches 1965
Gordon and Beach, Science Publishers, Inc., New York (1966)

Jacob - Wick - 59
M. Jacob and G.C. Wick
Ann. of Phys. 7, 404 (1959)

Henry - de Rafael - 65
C. Henry and E. de Rafael
Ann. Inst. Henri Poincaré **2** , 87 (1965)

Kotanski - Zalewski - 68
A. Kotanski and K. Zalewski
Nucl. Phys. **B4** , 559 (1968)

Michel - 59
L. Michel
Suppl. Nu. Cim. **14** , 95 (1959)

Racah - 42
G. Racah
Phys. Rev. **62** , 437 (1942)

Rose - 57
M.E. Rose
Elementary theory of angular momentum
Wiley, New York, 1957

Van Hove - 69
L. Van Hove
Nucl. Phys. **B 9** , 331 (1969)

Wigner - 39
E.P. Wigner
Ann. Math. **40** , 149 (1939)

Wigner - 59
E.P. Wigner
Group theory and its application to the quantum mechanics of atomic spectra
Academic Press, New York (1959)

The bibliography is neither complete nor systematic. We would be thankful to the remarks of our readers who will help us to improve the bibliography of the final version.