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Radially Separated, Static, Finite-Energy Yang-Mills-Higgs Fields

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Abstract

For the above system, the radial and angular equations are exhibited for any Yang-Mills group G, and the angular equations are solved for G=SU(2). For all the solutions in this case, the magnetic charge is $4\pi/e$.

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Let G be a compact Lie group and

$$\mathcal{L} = -\frac{1}{4}(\underset{\sim}{F}_{\mu\nu}, \underset{\sim}{F}_{\mu\nu}) - \frac{1}{2}(\underset{\sim}{D}_\mu \Phi, \underset{\sim}{D}_\mu \Phi) - V(\Phi), \tag{1}$$

a Yang-Mills-Higgs Lagrangian density for G, where (A,B) denotes inner product in G-spaces, Φ belongs to any real representation T(g) of G, and

$$\underset{\sim}{D}_\mu \Phi \equiv \underset{\sim}{D}_\mu \Phi^A = \nabla_\mu \Phi^A + e\, A^\alpha_\mu t^\alpha_{AB} \Phi^B, \quad \underset{\sim}{D}_\mu \underset{\sim}{A}_\nu \equiv \underset{\sim}{D}_\mu A^\alpha_\nu = \nabla_\mu A^\alpha_\nu + e(\tau^\beta)^{\alpha\gamma} A^\beta_\mu A^\gamma_\nu \tag{2}$$

$$\underset{\sim}{F}_{\mu\nu} = \frac{1}{e}[\underset{\sim}{D}_\mu, \underset{\sim}{D}_\nu] = \nabla_\mu \underset{\sim}{A}_\nu - \nabla_\nu \underset{\sim}{A}_\mu + e[\underset{\sim}{A}_\mu, \underset{\sim}{A}_\nu], \qquad \underset{\sim}{A}_\mu = A^\alpha_\mu \tau^\alpha \tag{3}$$

where τ^α and t^α are the (real anti-symmetric) generators of the adjoint $\tau(g)$ and T(g) representations of G respectively. Classically there is no restriction on $V(\Phi)$ except that it be G-invariant and bounded below, but for QFT renormalizability requires that it be a polynomial of fourth degree. Furthermore for the occurrence of Higgs mechanism, the second degree term in $V(\Phi)$ must have a negative coefficient.[1]

We are interested in finite-energy (FE) solutions[2,3] in the case when all the time-derivatives are zero. For the moment we shall also assume that $A_0=0$, since, as we shall see, any solution with $A_0=0$ can be generalized to a (dyon[4]) solution with $A_0 \neq 0$. In this case the Hamiltonian density corresponding to $\mathcal{L}$ is just the negative of $\mathcal{L}$ and the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \int \mathcal{H}(x)\, d^3x = \int \left(\frac{1}{2}(\vec{\underset{\sim}{F}}, \vec{\underset{\sim}{F}}) + \frac{1}{2}(\vec{\underset{\sim}{D}}\Phi, \vec{\underset{\sim}{D}}\Phi) + V(\Phi)\right) d^3x, \tag{4}$$

where arrow denotes a three-vector. Our primary aim is to find FE solutions of (4) for factored fields. However, we find it instructive to first find the conditions that are imposed by the spontaneous breakdown and the finite energy without factorization. First, the spontaneous breakdown requires that $(\Phi(x),\Phi(x))$ have a finite limit when $r\to\infty$, that is

$$\lim_{r\to\infty} \Phi(x)\to c\ \phi(\omega) \text{ where } (\phi(\omega),\phi(\omega))=1 \text{ and finite } c\neq 0, \tag{5}$$

where ω denotes the polar angles (θ,φ). From (2) we then see that the gauge-field mass-matrix is just $e^2M_{\alpha\beta}$ where

$$M_{\alpha\beta} = c^2(t^\alpha\phi,t^\beta\phi). \tag{6}$$

Next, since finite energy requires that the integral in (4) converges, i.e., $r^3\mathcal{H}(x)\to 0$ as $r\to\infty$, and each term in $\mathcal{H}(x)$ is separately positive, we see that (subject to reasonable assumptions of smoothness) finite energy requires as $r\to\infty$,

$$\text{a).}\quad e\vec{\underset{\sim}{A}}(x)\to\vec{\underset{\sim}{a}}(\omega)r^{-1}, \qquad \text{b).}\quad \vec{\underset{\sim}{d}}\phi(\omega)\equiv[\vec{\partial}+\vec{a}^{\alpha}(\omega)t^{\alpha}]\phi(\omega)=0, \tag{7}$$

where

$$\vec{\partial}=r\vec{\nabla}.$$

Conversely, if (5) and (7) are satisfied $r^3\mathcal{H}(x)\to 0$, as $r\to\infty$. The physical meaning of (7b) is that, at least locally, the ω-dependence of $\phi(\omega)$ can be gauged away, a result that can be written in a perhaps more familiar form as $\phi(\omega)=T(g(\omega))\phi(0)$, $g(\omega)\varepsilon\ G$. Further Eq.(7b) implies

$$\vec{\underset{\sim}{J}}\phi = 0, \tag{8}$$

where

$$\vec{\underset{\sim}{J}} = -(\hat{r}\times\vec{\underset{\sim}{d}})=\vec{\underset{\sim}{L}}+\vec{\underset{\sim}{b}} \tag{8'}$$

is the covariant angular momentum operator[5].

It turns out to be useful to consider the integrability conditions for (7b) which are easily computed to be

$$f^{\alpha}_{ij}\,t^{\alpha}\phi = 0, \tag{9}$$

where

$$\underset{\sim}{f}_{ij} \equiv f^{\alpha}_{ij}\tau^{\alpha} = [\underset{\sim}{J}_i, \underset{\sim}{J}_j] - \varepsilon_{ijk}\underset{\sim}{J}_k = \partial_i \underset{\sim}{a}_j - \partial_j \underset{\sim}{a}_i + [\underset{\sim}{a}_i, \underset{\sim}{a}_j] + \varepsilon_{ijk}\underset{\sim}{b}_k - \varepsilon_{ijk}[\underset{\sim}{J}_k, \hat{r}\cdot a]$$

$$= \lim_{r\to\infty} r^2 \underset{\sim}{F}_{ij} - \varepsilon_{ijk}[\underset{\sim}{J}_k, \hat{r}\cdot\underset{\sim}{a}]. \qquad (9')$$

The field $\underset{\sim}{f}_{ij}$ is the analogue on the unit sphere of the field $\underset{\sim}{F}_{\mu\nu}$ in Minkowski space, and just as $\underset{\sim}{F}_{\mu\nu}=0$ implies that $\underset{\sim}{A}_{\mu}$ can be gauged to zero, $\underset{\sim}{f}_{ij}=0$ implies that $\underset{\sim}{a}_i(\omega)$ can be gauged to zero. Similarly just as $e\underset{\sim}{F}_{\mu\nu}$ measures the failure of the Poincaré commutator $[\underset{\sim}{D}_{\mu}, \underset{\sim}{D}_{\nu}]$ to close, $\underset{\sim}{f}_{ij}$ measures the failure of the covariant angular momentum operators to close. The importance of f^{α}_{ij} is that the nontrivial solutions of (7b) exist only if (9) is satisfied with nonzero f^{α}. We also note that

$$\hat{r}_i f^{\alpha}_{ij} = 0, \text{ hence } \quad f^{\alpha}_{ij} = \varepsilon_{ijk}\, \hat{r}_k f^{\alpha}, \qquad (10)$$

where

$$\underset{\sim}{f} = -(\vec{\underset{\sim}{L}}\cdot\vec{\underset{\sim}{a}} + \underset{\sim}{v}) \text{ with } \underset{\sim}{v} = -\varepsilon_{ijk}\, \hat{r}_i \underset{\sim}{a}_j \underset{\sim}{a}_k \quad . \qquad (10')$$

Thus $f(\omega)$ is a scalar field valued in the Lie Algebra $\mathcal{G}$ and from (9)

$$f^{\alpha}(\omega)t^{\alpha}\phi(\omega) = 0, \qquad (11)$$

or equivalently

$$M_{\alpha\beta}f^{\beta} = 0. \qquad (11')$$

The above equation shows that $\underset{\sim}{f}$ qualifies as a candidate for defining the electromagnetic direction in $\mathcal{G}$. Further it is the sole candidate, if the mass matrix $M_{\alpha\beta}$ has only one zero eigenvalue.

Finally by forming the inner product of (7b) with $t_{\beta}\phi(\omega)$ we obtain

$$M_{\alpha\beta}a^{\beta}_i = c^2(\phi, t^{\alpha}\partial_i\phi), \qquad (12)$$

which expresses the gauge field $a_i(\omega)$ in terms of the mass matrix and the Higgs currents. It is also easily verified from (7b) that the mass matrix is rotationally invariant in the sense that

$$\underset{\sim}{J}_i M_{\alpha\beta} = 0 . \tag{13}$$

To proceed further we must add to the FE condition (7b) the minimality conditions for the Hamiltonian (4). In the static case these are just the equations of motion, which may be written as

$$D^2\Phi = \frac{\partial V}{\partial \Phi} \quad \text{and} \quad \underset{\sim}{D}_i F^{\alpha}_{ij} = -e(\Phi, t^{\alpha}\underset{\sim}{D}_j \Phi) . \tag{14}$$

The ideal program would be to solve the equations in complete generality, but since this is too difficult, we return at this point to our original program, which is to seek solutions of the form

$$\Phi^A(x) = c\phi^A(\omega)\left(\frac{S(r)}{r}\right), \qquad e\, A^{\alpha}_i(x) = a^{\alpha}_i(\omega)\left(\frac{R(r)}{r}\right) \tag{15}$$

in the Landau gauge, which in the static case reduces to

$$\vec{\nabla}\cdot\vec{A} = 0. \tag{16}$$

Eq.(15) is a natural generalization of that of 't Hooft and Polyakov,[2] but here no apriori assumption is made about the form of the angular functions $\phi(\omega)$ and $a(\omega)$. In particular, they are not assumed to be spherically symmetric[6].

We now insert the ansatz (15) into the equations of motion and the gauge condition (16), and after a lengthy calculation, which will be given in detail elsewhere, we obtain the following results: first, the gauge condition yields the two separate conditions

$$\vec{\partial}\cdot\vec{\underset{\sim}{a}} = 0 \quad \text{and} \quad \vec{r}\cdot\vec{\underset{\sim}{a}} = 0. \tag{17}$$

From the equations of motion we obtain, in addition to the FE condition (7b), the six angular equations,

$$\begin{aligned} &a).\ \partial^2\vec{\underset{\sim}{a}} = -L(L+1)\vec{\underset{\sim}{a}}, && c).\ (\underset{\sim}{a}_i, \underset{\sim}{a}_i) = \nu^2, && e).\ a^{\alpha}_i = \frac{1}{n}(\phi, t^{\alpha}\partial_i\phi), \\ &b).\ \partial^2\phi = -\ell(\ell+1)\phi, && d).\ [\underset{\sim}{a}_i[\underset{\sim}{a}_i, \underset{\sim}{a}_j]] = -\frac{\nu^2}{N}\underset{\sim}{a}_j, && f).\ \underset{\sim}{d}_i \underset{\sim}{f}_{ij} = 0, \end{aligned} \tag{18}$$

where L, ℓ, ν, N and n are constants with

$$\ell(\ell+1) = n\nu^2 . \tag{18'}$$

The two radial equations are

$$r^2 S'' = S[\ell(\ell+1)+\mu^2(S-r)(\sigma S+r)], \tag{19a}$$

$$r^2 R'' = (1-R)[L(L+1)R(1-\varepsilon R)-nc^2 S^2], \tag{19b}$$

$$\varepsilon = \ell(\ell+1)/L(L+1)Nn \tag{20}$$

and where the dimensionless positive constant σ depends on the precise form of the potential $V(\Phi)$. If there are no third degree terms present in $V(\Phi)$, then $\sigma=1$. Thus when

$$V(\Phi)=\frac{\lambda}{4}(\Phi,\Phi)^2-\frac{\mu^2}{2}(\Phi,\Phi),\ \sigma=1 \text{ and } c^2=\mu^2/\lambda.$$

Conversely, if (7b), (17), (18) and (19) are satisfied, the minimalization conditions are automatically satisfied (at least in the sector (15)).

Before attempting to solve (18), which we shall actually do in this letter only for G=SU(2), we analyze them briefly for general G. First (a) and (b) show that $\underset{\sim}{a}(\omega)$ and $\phi(\omega)$ are definite spherical harmonics, of order L and ℓ respectively (so that L and ℓ are non-negative integers). In particular, this means that we can obtain the explicit form of $\phi(\omega)$ from the following lemma which can be proved using direct Clebsch-Gordan analysis:

Lemma: if $\phi(\omega)$ is subject to the normalization condition in (5) and satisfies (18b), then there exists a $(2\ell+1)$ dimensional representation of the rotation group R such that

$$D^{(\ell)}(R)\phi(\omega) = \phi(R^{-1}\omega).$$

The above statement implies that

$$\phi^A(\omega) = \sum_m \eta^A_m Y^m_\ell(\omega), \quad A=1,2,\ldots K,\ K=\dim T(g),$$

where η^A_m are constants which satisfy the reality conditions

$$\bar{\eta}^A_m = (-1)^m \eta^A_{-m} \text{ and } \sum_A \eta^A_m \eta^A_{m'} = \frac{4\pi}{2\ell+1}(-1)^m \delta_{m,-m'}. \tag{21}$$

It then follows[7] in particular that $2\ell+1\leq K$.

Next we note from (17) that $\underline{v}(\omega)$ defined in (10') satisfies $[\underline{a}_i(\omega),\underline{a}_j(\omega)]=-\varepsilon_{ijk}\, r_k\underline{v}(\omega)$. Then (18d) shows that at each ω, the $\underline{a}_i(\omega)$'s and $\underline{v}(\omega)$ span an $SU(2)_\omega$ Lie subalgebra of $\mathcal{G}$. For different ω, these $SU(2)_\omega$ subalgebras are conjugated by G. The adjoint representation of $\mathcal{G}$ reduces into a direct sum of irreducible representations of these $SU(2)_\omega$ with multiplicities c_j $(c_1\geq 1)$ and N given by

$$N = \sum_{j=0}^{\infty} \tfrac{1}{3}c_j\ j(j+1)(2j+2) = \sum_{j=0}^{\infty} \tfrac{1}{2}c_j \binom{2j+2}{3} \quad \text{is an integer} \geq 2.$$

A simple interpretation of (18e) is obtained if we note from (12) that it can be written in the alternative form

$$M_{\alpha\beta}(\omega)a^\beta_i(\omega) = c^2 n\ a^\alpha_i(\omega). \tag{22}$$

Thus the nonvanishing gauge fields are actually eigenfields of the mass-squared operator with the same mass-squared $m^2=c^2n$. The constant Nn is related to the Casimir operator for the ϕ-representation T(g) as will be seen below. Finally using (10) we see that (18f) which is just the matter-free Yang-Mills equation on the unit sphere, can be written in the much simpler form

$$\underset{\sim}{d}_i f^\alpha = 0\ . \tag{23}$$

From (23) it follows in particular that

$$f^\alpha(\omega) = (\tau(g(\omega)))^{\alpha\beta} f^\beta(0) \quad \text{and} \quad (\underset{\sim}{f}(\omega),\ \underset{\sim}{f}(\omega)) = \kappa^2\ , \tag{24}$$

where κ is a constant. Comparing (23) with (7b) we see that $\underset{\sim}{f}(\omega)$ satisfies the same equation as the Higgs field. But, in contrast to the Higgs field, $\underset{\sim}{f}(\omega)$ always lies in the adjoint representation.

We now turn to the specific case of G=SU(2) for which we can solve the equations (18). For SU(2)=G, the only non-trivial little group is U(1) and the representations T(g) which allow this little group are just the real (integral spin) representations. Since, from (22), the fields $a_i(\omega)$ have definite non-zero mass, they are orthogonal to $U(1)_\omega$. Hence $[\underline{a}_i(\omega),\underline{a}_j(\omega)]$, which is orthogonal to the $\underline{a}_i(\omega)$, must lie in $U(1)_\omega$. Hence, for SU(2), $[\underline{a}_i(\omega),\underline{a}_j(\omega)]$ must have zero mass, and so we have

$$M_{\alpha\beta}\, v^\beta = 0 \ . \qquad (25)$$

But then from (11),(11'),(13),(22) and (25) we have

$$c^2 n\, J_k a_k^\alpha = J_k M_{\alpha\beta} a_k^\beta = M_{\alpha\beta} J_k a_k^\beta = -M_{\alpha\beta}(f^\beta + v^\beta) = 0 \qquad (26)$$

and hence from (11) again

$$f^\alpha = v^\alpha = -\frac{1}{2} L_k a_k^\alpha \ . \qquad (27)$$

But then f^α is a spherical harmonic of the same definite order L as $a_j^\alpha(\omega)$. Since from (24), $\underset{\sim}{f}(\omega)$ also has a constant norm in the real inner product space of the adjoint representation, the lemma applied above to $\phi(\omega)$, is applicable also to $\underset{\sim}{f}(\omega)$ and we obtain in analogy to (21)

$$2L+1 \le \dim \mathcal{G} \qquad (28)$$

But for SU(2), the adjoint representation is 3-dimensional. Hence $2L+1 \le 3$, and if we discard the trivial case L=0, we have L=1 and[8]

$$f^\alpha = -\kappa \hat{r}^\alpha . \qquad (29)$$

Further, since L=1 for $a(\omega)$ also, we have, using (17) and (27)

$$a_i^\alpha = \kappa \varepsilon_{\alpha i j} \hat{r}_j \ . \qquad (30)$$

If we now reinsert (21),(29) and (30) into the system (7b) and

(18) we see that we have a consistent solution provided only that the constants satisfy the conditions

$$\nu^2=N=2, \quad \kappa=1, \quad 2n=\ell(\ell+1) \tag{31}$$

and the $\phi(\omega)$-field satisfies

$$(L_i+t_i)\phi(\omega)=0. \tag{32}$$

Equations (21),(29),(30),(31) and (32) constitute, therefore, the complete set of solutions of the angular equations (7b) and (18) for G=SU(2). These solutions differ from that of 't Hooft and Polyakov[2] only in that the field $\phi(\omega)$ can have arbitrary integral isospin t, and then only provided $\ell=t$ and the spin and isopin compensate as in (32). In particular, the magnetic charge is $4\pi/e$ as in Ref.2, since it is determined by $\varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma}\,\partial_i u^\alpha\partial_j u^\beta\partial_k u^\gamma$ where u^α is the unit vector in the EM direction and, as discussed above, $u^\alpha = f^\alpha = -\kappa\hat{r}_\alpha$.

Using (31) and replacing $\ell(\ell+1)$ by its gauge-invariant counterpart $t(t+1)$, the radial equations (19) can be reduced to

$$r^2H'' = H[t(t+1)K^2+\frac{\mu^2}{c^2}(H-cr)(\sigma H+cr)], \qquad H=cS\,,$$

$$r^2K'' = K[(K^2-1)+t\frac{(t+1)}{2}\,H^2]\,, \qquad K=(1-R)\,, \tag{33}$$

where the boundary conditions are $K(0)=1$, $H(0)=0$ and $K\to 0$, $H\to cr$ as $r\to\infty$. These equations are derivable from the Hamiltonian (4) which now takes the form

$$= \frac{4\pi}{e^2}\int_o^\infty \frac{dr}{r^2}[(rK')^2+\frac{1}{2}(rH'-H)^2+\frac{1}{2}(K^2-1)^2+\frac{t(t+1)}{2}\,H^2K^2$$
$$+\ \frac{1}{4}\mu^2r^2H^2\ U(\frac{H}{cr})]\,, \tag{34}$$

where

$$U(x) = (\sigma x^2+\frac{4}{3}(1-\sigma)x-2) \tag{34'}$$

Eqs.(33) reduce to the usual ones[9] for $t=1$, $\sigma=1$ and although it is probable that they have regular finite energy solutions for other values of t and σ, we do not know for sure at present whether they do or not. If they have, these solutions give a mass-formula $m(t)=H$ on account of the t-dependence in (34).

In conclusion we note that the above formalism can be extended to include dyons by making the ansatz

$$A_o^\alpha = \hat{r}^\alpha \left(\frac{J(r)}{r}\right). \tag{35}$$

Since A_o^α plays the role of a second Higgs field, which lies in the adjoint representation and which has no potential and no interaction with $\phi(\omega)$ (since $-\kappa\hat{r}_\alpha = f^\alpha$ is in the little group of $\phi(\omega)$), one sees by analogy with ϕ that the only effect of A_o^α on the field equations is to add to the Lagrangian and energy densities, a term of the form

$$\frac{1}{e^2 r^2}\left[\frac{1}{2}(r\,J'-J)^2 + \frac{1}{2}\,J^2 K^2\right]. \tag{36}$$

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References

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5. We have defined $\vec{\underset{\sim}{L}}=-\hat{r}\times\vec{\partial}$ with the commutation relations $[L_i,L_j]=\varepsilon_{ijk}L_k$. Then $[\partial_i,\partial_j]=-\varepsilon_{ijk}L_k$; $\vec{\underset{\sim}{b}}=-\hat{r}\times\vec{a}^{\alpha}t^{\alpha}$.

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7. See E.Cremmer, F.Schaposnik, and J.Scherk, Ecole Normale Superieure preprint (1976), who have independently proved this result.

8. Strictly speaking f is related to $-\kappa\hat{r}$ by an arbitrary orthogonal matrix which we take to be 1 by convention.

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