SYMMETRY BREAKING BY HIGGS FIELDS

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These lectures study the following problem : given a G-symmetric Lagrangian containing gauge fields and Higgs fields, what is the nature of the subgroup H of G obtained by spontaneous symmetry breaking through the Higgs mechanism ?

We first recall briefly what is this mechanism ; we also show that the problem we study is very similar to the symmetry breaking occuring in Landau theory of second order phase transition. These lectures explain basic facts about orthogonal actions of compact Lie groups and the structure of the ring of invariant polynomials. They finally give known results on this problem and the main open conjecture. In an appendix, we establish some relations between a compact group action and the corresponding action of the finite generalized Weyl group.

1. Spontaneous symmetry breaking in gauge field theory

A gauge field with symmetry G is a vector field $A_\mu(x)$ defined on Minkowski space time and each vector component is valued in G , the Lie algebra of G . The corresponding gauge invariant field is

$$F_{\mu\nu}(x) = \frac{\partial}{\partial x^\mu} A_\nu(x) - \frac{\partial}{\partial x^\nu} A_\mu(x) + e \, [A_\mu(x),A_\nu(x)] \quad . \qquad (1)$$

More generally any field (e.g. a Dirac field $\psi_\alpha(x)$, a scalar field $\phi(x)$) is valued in the vector space E carrying the linear representation U(g) of the gauge group G and therefore the linear representation $a \longmapsto L(a)$ of the Lie algebra :

$$L(a) = \frac{d}{d\theta} U(e^{a\theta})\Big|_{\theta=0} \quad , \qquad [L(a),L(b)] = L([a,b]) \qquad (2)$$

(where [a,b] is the Lie product of a,b). The covariant derivative of the field is e.g. for ϕ .

$$D_\mu \phi(x) = \frac{\partial}{\partial x^\mu} \phi(x) + e \, L(A_\mu(x)) \, \phi(x) \quad . \qquad (3)$$

If in the usual G-invariant Lagrangian density, e.g. $L_0(A_\mu, \psi_a)$ with zero mass field A_μ , derivatives are replaced by covariant derivatives, this Lagrangian density is invariant by G(x) (the group transformation g ∈ G is an arbitrary function of x).

We assume that G is a compact group and that the representations on the field value spaces E are orthogonal $(U^T(g) \, U(g) = I$ so $L(a)^T = -L(a))$. We denote by

$(\ , \ )$ a G-invariant orthogonal scalar product in E .

Spontaneous symmetry breaking by the Higgs mechanism is obtained with the introduction of the Higgs scalar field $\phi(x)$. The full Lagrangian density is $L = L_0 + L_H$ with

$$L_H = -\frac{1}{2}(D_\mu \phi(x), D^\mu \phi(x)) - P(\phi(x)) \tag{4}$$

where

$$P(\phi(x)) \text{ is a G-invariant, bounded below, polynomial in } \phi . \tag{5a}$$

Renormalizability of the theory requires moreover :

$$P(\phi) \text{ is a fourth degree polynomial.} \tag{5b}$$

We moreover assume that E does not contain non-trivial G-invariant vectors. This implies that

$$P(\phi) \text{ has no linear term in } \phi . \tag{5c}$$

We then write

$$P(p) = -\frac{\mu^2}{2}(\phi, \phi) + p_3(\phi) + p_4(\phi). \tag{5d}$$

We will later explicit the 3 rd and 4 th degree terms $p_3(\phi)$ and $p_4(\phi)$. Since the quadratic form $-\frac{\mu^2}{2}(\phi, \phi)$ is definite negative, $P(\phi)$ has a maximum at the origin $\phi = 0 \in E$. Let ϕ_0 a value of $\phi(x)$ at which $P(\phi)$ reaches its lower bound. From 4c, the isotropy group of ϕ_0 (i.e. the set of $g \in G$, $U(g)\phi_0 = \phi_0$) is a strict subgroup $H \subset G$. In the Higgs mechanism, we are interested by quantum field theory around the classical solution $\phi(x) = \phi_0$ (This constant is the vacuum expectation value of $\phi(x)$). The symmetry is broken from the compact Lie group G (of dimension $d(G)$) to the subgroup $H \subset G$ (of dimension $d(H)$).

Remark that the first term in (4) contains quadratic term in $A_\mu(x)$ (they come from the second term in (3)). In the direction $a \in H$, the Lie algebra of H ,

$$a \in H \quad , \quad L(a) \ \phi(x) = 0 \quad . \tag{6}$$

So there are $d(H)$ linear independent gauge fields A_μ on (valued in H) which keep this zero mass. Indeed they correspond to the preserved gauge symmetry. The directions of the orthogonal subspace $H^\perp \subset G$ for the Cartan-Kelling metric corresponds to massive gauge field. More precisely, the

$$d(G)-d(H) \text{ dimensional subspace,} \qquad \{L(a) \ \phi_0 \ , a \in H^\perp\} \tag{7}$$

is parallel to the tangent plane, at ϕ_0 of the orbit $G \cdot \phi_0 = \{U(g) \ \phi_0\}$ of ϕ_0 . The Higgs field in this direction have zero mass and could correspond to Goldstone bosom. However, by a gauge transformation they can be reinterpreted as part of the

massive gauge field

$$a \in H^1 \quad , \quad A_\mu(x) \longmapsto A_\mu(x) + \frac{\partial}{\partial x^\mu} \phi(x) \quad . \tag{8}$$

Finally by expansion of $P(\phi)$ around ϕ_0 we see that the mass matrix of the Higgs field is in (4)

$$M_H = \frac{d^2 P(\phi)}{d\phi^2}\bigg|_{\phi=\phi_0} \quad ; \tag{9}$$

it is the Hessian of the Higgs polynomial. Its kernel contains the tangent plane (7) to the orbit. The orthogonal subspace in E correspond to

$$d = \dim E - (d(G) - d(H)) \tag{10}$$

massive Higgs bosons. When the kernel of (9) has dimension $\delta > d(G) - d(H)$, there are $\delta - (d(G)-d(H))$ massless Higgs bosons which are "pseudo-Goldstone" (see e.g. S. Weinberg (1971)). They correspond to a nearly preserved symmetry $\widetilde{H} > H$ (see below where $\widetilde{H}$ will be emphasized again).

To summarize : in the Higgs mechanism the gauge symmetry G is spontaneously broken on the subgroup H, isotropy group of a minimum of the Higgs polynomial $P(\phi)$, defined by the equations (5), on the space E carrying an orthogonal representation of G.

The aim of these lectures is to give all relevant mathematical information known on the nature of the subgroups of G which are isotropy group of Higgs polynomial minima. I have already written a paper (L. Michel (1979)) on this subject. I refer the reader to it for some details, examples or proofs.

Let us recall now some well known facts on :

2. Orthogonal action of compact groups

We first recall that the isotropy groups of a G-orbit are conjugated. Moreover a natural definition of equivalence on the G-orbits leads to the theorem : Two G-orbits are equivalent (we also say "are of the same type") when their isotropy groups are conjugated. We shall denote by $[H]$ the class of G-subgroups conjugated to H, i.e.

$$H < G \quad , \quad [H] = \{gHg^{-1} , \; \forall g \in G\} \quad . \tag{11}$$

Hence there is a bijective correspondance between the orbit types of the G-action and the conjugation classes of the subgroups of G. We shall denote by $[G:H]$ an orbit of type $[H]$. In any G-action it is convenient to call stratum the union of all orbits of the same type. Hence all points of a stratum have conjugated isotropy groups.

Here we are only interested by compact Lie groups G and their linear actions on real vector spaces E . (These linear actions are therefore orthogonal. By a famous theorem of Mostow (1957) any G smooth action can be embedded in an G orthogonal action). The isotropy groups are closed subgroups of G and the orbits are closed compact manifolds.

There is a natural order on the set of conjugation classes of the closed sub-groups of compact groups $^{*)}$: [H] < [H']   if there is a subgroup H ∈ [H] which is contained in a subgroup H' ∈ [H'] . This implies an order on the set K of conjugation classes of isotropy groups — or equivalently of the strata — which appear on a group action. It is a non trivial theorem that for compact G smooth action there is one minimal isotropy group and the corresponding stratum is open dense (for reviews see (R.S. Palais (1960), D. Mongommery (1964), L. Michel (1972)). It is easy to show that the union of strata with [H] $\geq$ [H$_0$] is a closed set.

We will often consider the case of an orthogonal G representation on E irreducible on the real (it may be reducible on the complex into the direct sum of an irreducible complex representation and its complex conjugate). Let K be intersection of the isotropy groups of an arbitrary orbit. The elements of K acts trivially on the orbit and therefore on its linear span : the whole space E ; so K is in the kernel of the representation. Let k in this kernel : then it has to be in any K . Hence the kernel K of the representation is the intersection of the isotropy groups of any orbit of an irreducible representation. We remark that all these results apply to finite groups which are compact Lie groups of dimension zero. In that particular case, one shows easily that the minimal isotropy group of an orthogonal action is K , the kernel of the representation. In the appendix we study the following problem : when the decomposition into orbit and strate of an orthogonal representation of a compact Lie group can be reduced to that for a finite group.

3. Landau theory of second order phase transitions

Although the audience is mainly interested by gauge field theory, it is worth-while to point out that everything we will learn on the Higgs mechanism of symmetry breaking is also applicable to the Landau theory of second order phase transitions (see e.g. L.D. Landau et. al. (1958)). Indeed this theory is really the forerunner of the Higgs mechanism and is over forty years old. It was made by Landau to study symmetry change in crystals, but it can be extended to all second order phase transitions and even to many other type of bifurcations with symmetry change (e.g. the Jacobi ellipsoid for rotating celestial bodies, as shown by G. Bertin et. al. (1976). Essentially one studies a G-invariant function V (usually Gibbs free energy) which depends also of external parameters (as temperature T) and the equilibrium state is

$^{*)}$ It is easy to see that this is not possible for some non compact groups, e.g. the affine group in n $\geq$ 2 dimensions.

given by the minima of this function. With the usual Landau analysis (explained e.g. in L.D. Landau et. al. (1958)), this is reduced to the study of the minima of a G-invariant polynomial built over an irreducible G-linear representation on a real vector space E .

In a second order phase transition from crystal to crystal, the spontaneous symmetry breaking is from the crystallographic group G to a subgroup $H \subset G$. The subgroup H contains a three dimensional lattice of translations K which is invariant subgroup of G . So K will be in the intersection of all subgroups of the conjugation class $[H]$ and as we have remarked in the previous subsection K is in the kernel of the irreducible orthogonal representation of G in E . This implies that the image $G' = G/K$ of this representation is finite.

In recent years, many second order transitions from crystals to incommensurate structures have been discovered and studied. Then H no longer contains a 3-dimensional lattice of translation, and the image G' is not closed in the orthogonal group of the finite dimensional space E (all irreducible representations of a crystallographic group are finite dimensional). In that case we are led to study its closure $\overline{G'}$ which is again a compact Lie group.

To summarize, the mathematical results we shall present applies to all second order transitions when the action of the symmetry group G on the finite dimensional real vector space E is orthogonal. The image G' of the action is a subgroup of $O(E)$, the orthogonal group on E . If this image is not closed, we consider its closure that we will also denote by G' .

4. Geometry of orthogonal group actions

We consider a faithful orthogonal representation $g \longmapsto \Delta(g)$ of the compact group G' on the finite dimensional real vector space E . One shows that the number of strata is finite : G.D. Mostow (1957). Given the orbit $G(m)$ of m , there exists a tubular neighborhood $V_{G(m)}$ such that for any point m' of it, there is a unique nearest point to m' on $G(m)$. We call it $r(m)$; the surjective map $V_{G(m)} \xrightarrow{r} G(m)$ is an equivariant retraction :

$$\forall m' \in V_{G(m)} , \quad \forall g \in G , \quad r(\Delta(g) \cdot m') = \Delta(g) \cdot r(m') . \tag{12}$$

Note that $g \in G_{m'}$ (the isotropy group of m') implies $g \in G_m$ so

$$\forall m' \in V_{G(m)} , \quad [G_{m'}] \le [G_m] . \tag{13}$$

So when $[G_m]$ is minimal in K , the strata contains an open set. We have quoted the stronger result in § 2 : there is a unique minimal element in K and the corresponding stratum is open dense. We call it the generic stratum. The set $r^{-1}(m)$ of

points of $V_{G(m)}$ whose image is m, is called the local slice in m. When m is in the generic stratum, the slice $r^{-1}(m)$ cuts every orbit in $V_{G(m)}$ in one point only.

We denote by $T_{G(m)}$ and $N(m)$ the tangent plane and the normal plane to the orbit at m. The local slice $r^{-1}(m)$ is in $N(m)$. The set of G-orbits is denoted E/G and is called the orbit space. It carries a natural metric :

$$\hat{d}(G(x),G(y)) = \inf_{x'\in G(x),y'\in G(y)} (x'-y',x'-y') \quad . \tag{14}$$

Since orbits are compact this minimum does exist and one verifies that $\hat{d}$ is a distance. If $d(x,y)$ is such a minimum the segment is orthogonal to both orbits $G(x)$, $G(y)$; i.e.

$$\overline{xy} \subset N(x) \cap N(y) \quad . \tag{15}$$

We call also the normal plane $N(x)$ a global slice ; we have implicitly proven :

Lemma (1) : Any global slice $N(x)$ cuts every orbit of G in E. One can then define (e.g. L. Michel (1971)) a G-invariant smooth function f vanishing outside a compact in $V_{G(m)}$, with $-1 \leq f(m') \leq 0$ on this compact and such that $f(m') = -1 \Leftrightarrow m' \in G(m)$. One says that the smooth functions separate the orbits. More generally, since polynomials are dense in smooth functions one can show (cf. G. Schwarz (1975)) that every G-invariant smooth function is a smooth function of G-invariant polynomials. So finally, invariant polynomials separate the orbits.

We give again a precise formulation of the mathematical problem which arises from the physical phenomena of symmetry breaking that we have described. We consider G' -invariant polynomials only ; from what we just stated it would be easy to extend our study to G'-invariant smooth function. Given the orthogonal representation $g \longmapsto \Delta(g)$ of G, with compact image G', on E, the symmetry is broken down to H where $\Delta(H) = H'$ is the isotropy group of

$$P \quad , \quad \text{a G'-invariant polynomial, bounded below, maximum at 0.} \tag{16a}$$

This requires that the terms of higher degree $P_n(x)$ are of even degree n and they are non-negative. For technical commodity we require :

$$p_n(x) > 0 \qquad \text{for } x \neq 0 \tag{16b}$$

and also that

$$E \quad \text{has no nonzero G'-invariant vector.} \tag{16c}$$

From our geometrical study one can easily build such a polynomial with its lower bound on an arbitrary G orbit ($\neq 0$) of E. (This is done explicitly in L. Michel (1979)). So in order to be able to say more on the minima of P we will require

later on either or both conditions :

$$n = \text{degree of } P = 4 \quad . \tag{17a}$$

The orthogonal representation of G' on E is irreducible
on the real. $\tag{17b}$

Restriction (17a) is usually made in physics, as well as for the Higgs mechanism as
for Landau theory. This leads to an important remark : although it is not difficult
to write a polynomial invariant by G' and not invariant by a larger closed subgroup
of the orthogonal group $O(E)$, this may become impossible with the restriction to
fourth degree polynomials. Obviously every mathematical property depends on the exact
invariance group $\widetilde{G}$ of the polynomial P ; this is the isotropy group in the action
of $O(E)$ on the set of polynomials on E . A minimum of the $\widetilde{G}$-invariant polynomial
P will have $\widetilde{H} < \widetilde{G}$ as isotropy group for the action of $\widetilde{G}$, $H' = \widetilde{H} \cap G'$ as isotro-
py group for the action of the image $G' = G/K$ and the physical invariance group of
the broken symmetry will be $H < G$ such that $H' = H/K$. For the dimension of E

$$\dim(E) \leq 4 \tag{18}$$

the list of irreducible groups $\widetilde{G}$ has been established in L. Michel et.al.(1981). Their
number is given in table 1 :

Table 1 :

Number of conjugated classes in $O(E)$ of irreducible isotropy
groups $\widetilde{G}$ of fourth degree polynomials ($O(E)$ is omitted !)

dim E	1	2	3	4	
without 3rd degree terms	0	1	1	13	(19)
with 3rd degree terms	1	1	1	17	

For a $\widetilde{G}$-invariant polynomial P satisfying the conditions (16) we can prove

Theorem 1 : Every $\widetilde{H}$ of a maximal element of K is the isotropy group of an extre-
mum of P . (We exclude the origin of E for the definition of K).

This theorem is an easy consequence of a theorem of L. Michel (1971). We sketch
here a proof. (A variation of it is given in L. Michel (1979), p. 166) Consider the
real algebraic manifold M of equation

$$M : (\phi, \frac{dP}{d\phi}) = 0 \tag{20}$$

on which the radial gradient of P vanishes. From (16a,b) it is bounded and there-
fore compact. The origin is an isolated point that we neglect. M is $\widetilde{G}$-invariant and

the $\widetilde{G}$-strata on M are the intersection by M of the strata in E. Those which correspond to maximal elements of K are closed and therefore compact. The restriction of P on each connected component of these compact strata is either constant or it has at least a maximum and a minimum. One also proves easily (see e.g. L. Michel (1971)) that the gradient of a $\widetilde{G}$ smooth function is at each point tangent to the stratum ; this implies that the extrema of the restriction of P to the compact strata of M are extrema of the whole polynomial P. To say more on extrema of P we study now the nature of :

5. The ring of G-invariant polynomials

We recall that G is compact and acts by an orthogonal representation Δ on the real vector space E of dimension m. Let $T(E)$ be the ring of polynomial on E and $T_n(E)$ be the vector space of homogeneous degree n polynomials. Its dimension is

$$\dim T_n(E) = \binom{n+m-1}{n} \qquad . \tag{21}$$

We denote by $T(E)^G$ the set of G-invariant polynomials. It is also a ring. We define

$$T_n(E)^G = T_n(E) \cap T(E)^G \tag{22}$$

If

$$c_n = \dim T_n(E)^G \tag{23}$$

it is given by the generating function (called Molien function or Poincaré function in the literature).

$$\sum_{n=0}^{\infty} c_n t^n = M(t) = \int_G \frac{d\mu(g)}{\det(I-t\Delta(g))} \qquad , \text{ where } \int_G d\mu(g) = 1 \tag{24}$$

For a finite group G of order $|G|$, $M(t) = \frac{1}{|G|} \sum \frac{1}{\det(I-t\Delta(g))}$. $\tag{24'}$

See T.A. Springer (1977), R.P. Stanley (1979), L. Michel (1977), (and also L. Michel (1979)) for different reviews on the subject of this section. $M(t)$ is a rational fraction which can be put on the form

$$\left. \begin{array}{l} M(t) = \dfrac{N(t)}{D(t)} \qquad , \qquad N(t) = \displaystyle\sum_{\alpha=0}^{\nu} t^{\delta_\alpha} \quad , \quad \delta_0 = 0 \\[4mm] \qquad\qquad\qquad D(t) = \displaystyle\prod_{i=1}^{m'} (1-t^{d_i}) \quad m' \le m \end{array} \right\} \quad . \tag{25}$$

(In this form $N(t)$ and $D(t)$ may have a common factor), which correspond to the following structure of $T(E)^G$: Every G-invariant polynomial of $T(E)^G$ is of the form

form

$$p(x) = \sum_{\alpha=0}^{\nu} q_\alpha(\theta_1(x),\ldots,\theta_{m'}(x)) \, \varphi_\alpha(x) \quad , \quad \varphi_0(x) = 1 \tag{26}$$

where the q_α are arbitrary polynomials of m' variables, the $\theta_i(x)$, $1 \le i \le m'$ are algebraically independent polynomials of degree d_i. The $\varphi_\alpha(x)$ are G-invariant polynomials of degree δ_α. Then φ_α^2, φ_α^3 .... are other φ_α's of degree $2\delta_\alpha$, $3\delta_\alpha$ ... up to $\nu_\alpha > 0$ where

$$\varphi_\alpha^{\nu_\alpha}(x) = \phi_\alpha(\theta_1(x),\ldots,\theta_{m'}(x)) \quad . \tag{27}$$

In other words, $T(E)^G$ is finitely generated and it is a free module of dimension $\nu+1$ on a ring of m' variable polynomials. When G is finite $m' = m$. We call reflection an orthogonal operator on E with eigen values -1, multiplicity 1, and 1 multiplicity $m-1$. For groups G generated by reflections (Coxeter groups) $\nu = 0$ so

$$p(x) = p(\theta_1(x),\ldots,\theta_m(x)) \quad . \tag{28}$$

More generally for a finite group, the number of reflections is

$$r(G) = \sum_{i=1}^{m} d_i - m - \frac{2N'_t(1)}{N(1)}$$

(where $N'_t = dN/dt$).

From (26) and (27) we can compute the gradient of a G-invariant polynomial as

$$\frac{dp(x)}{dx} = \sum_{i=1}^{m'} F_i(x) \frac{d\theta_i}{dx} \quad \text{with} \quad F_i(x) = \sum_{\alpha=0}^{\nu} \left(\frac{\partial q_\alpha}{\partial\theta_i} + \frac{1}{\nu_\alpha} q_\alpha \phi^{-1} \frac{\partial\phi_\alpha}{\partial\theta_i}\right)\varphi_\alpha . \tag{29}$$

The algebraic independence of the θ_i implies the linear independence of their gradient on the generic (open dense stratum) so

Lemma 2': If one of the $F_i(x)$ in (29) is a non-vanishing constant the G invariant polynomial $p(x)$ has no extrema in the open dense stratum.

(This lemma has also been obtained independently by M. Jarič, conference on group theory, Mexico (1980)).

If we want to exclude more isotropy groups we must add to the conditions (16) at least one of the condition (17). Physically, the condition (17a) limiting the degree to 4 seems the most important. With these conditions (16) and (17a) we can obtain the minimum of the Higgs polynomial in the generic stratum when the representation of G is reducible. We can show it in a simple example.

Let

$$E = \bigotimes_{\alpha=1}^{n} E_\alpha \quad , \quad m_\alpha = \dim E_\alpha \quad , \quad 1 < m_\alpha < m_{\alpha+1} \quad . \tag{30}$$

Let (x_α, x_α) be an orthogonal scalar product in the space E_α and

$$(\phi, \phi) = \sum_\alpha (\phi_\alpha, \phi_\alpha) \quad , \quad \phi = \bigoplus_\alpha \phi_\alpha \quad . \tag{30'}$$

We study the Higgs polynomial.

$$P(\phi) = \sum_\alpha \frac{1}{4K_\alpha} (\phi_\alpha, \phi_\alpha)^2 - \frac{\mu^2}{2} (\phi, \phi) \quad , \quad K_\alpha > 0 \quad . \tag{31}$$

Using that $m_\alpha < m_{\alpha+1}$ we find for that its invariance group is the direct product

$$G = \underset{\alpha=1}{\times} O(m_\alpha) \quad , \tag{32}$$

whose dimension is

$$\dim G = \sum_{\alpha=1}^{n} \frac{m_\alpha (m_\alpha - 1)}{2} \quad . \tag{32'}$$

Vectors ϕ of the form $\phi_\alpha = 0$ except for one α have maximal isotropy groups (in $E - \{0\}$) . Indeed

$$G_\phi = O(m_1) \times O(m_2) \times \ldots \times O(m_\alpha - 1) \times \ldots \times O(m_n) \quad . \tag{33}$$

On the opposite, the vectors whose all summands x_α are different from the nul vector form the generic stratum with the minimal isotropy group $\underset{\alpha=1}{\overset{n}{\times}} O(m_\alpha^{-1})$. From (31) we obtain

$$\frac{dP}{d\phi} = \bigoplus_\alpha \phi_\alpha ((\phi_\alpha, \phi_\alpha) K_\alpha^{-1} - \mu^2) \tag{34}$$

and

$$\frac{d^2 P}{d\phi^2} = -\mu^2 I \quad \bigoplus_\alpha \quad (P_\alpha + 2P_{\phi_\alpha})(\phi_\alpha, \phi_\alpha) K_\alpha^{-1} \tag{35}$$

where P_α and P_{ϕ_α} are respectively the orthogonal projectors on E_α and ϕ_α. Equation (34) shows that for an extremum

$$\text{either} \quad \phi_\alpha = 0 \quad \text{or} \quad (\phi_\alpha, \phi_\alpha) = \mu^2 K_\alpha \quad . \tag{36}$$

The restriction of the corresponding Hessian on E_α is then

$$\frac{d^2 P}{d^2 \phi} \bigg|_{E_\alpha} = -\mu^2 I_\alpha \quad \text{or} \quad 2\mu^2 P_{\phi_\alpha} \quad . \tag{36'}$$

We have a minimum when $d^2 P / d\phi^2 > 0$ and from $1 < m_\alpha$ this can occur only when all summands ϕ_α are nonvanishing so the minimum is in the generic stratum. We remark that it is in agreement with Lemma 2. Indeed

$$\tag{37}$$

$$\frac{\partial P}{\partial (\phi_\alpha, \phi_\alpha)} = \frac{1}{2K_\alpha} ((\phi_\alpha, \phi_\alpha) - K_\alpha \mu^2)$$

which is not constant ; the annulation of all this partial derivatives yield the extrema of the generic stratum and we even showed that they are minima.

Now we will impose conditions (16) and (17) : The orthogonal action $g \longmapsto \Delta(g) = \Delta(g^{-1})$ of G on E is irreducible on the real. The G-invariant polynomial is

$$P(\phi) = \frac{1}{4} \omega(\phi) + \frac{\mu}{3} \sigma(\phi) - \frac{\mu^2}{2}(\phi,\phi) \qquad (34a)$$

with

$$\omega(\Delta(g)\phi) = \omega(\phi) = \lambda^4 \omega(x^{-1}\phi) \quad ; \quad \phi \neq 0 \quad \Rightarrow \quad \omega(\phi) > 0 \qquad (34b)$$

$$\sigma(\Delta(g)\phi) = \sigma(\phi) = \lambda^3 \sigma(\lambda^{-1}\phi) \qquad (34c)$$

If $S(E)$ is the vector space of symmetric operators on E , it is easy to compute (see e.g. [2]) the existence of linear maps

$$E \otimes E \xrightarrow{\quad T \quad} S(E) \qquad , \qquad E \xrightarrow{\quad D \quad} S(E) \qquad (35)$$

which are G-equivariant (i.e. they commute with the respective actions of G on the domain space and the image space) and which satisfy

$$\omega(\phi) = (\phi, T(\phi \otimes \phi)\phi) \quad , \quad \sigma(\phi) = (\phi, D(\phi)\phi) \qquad . \qquad (36)$$

So the gradient and Hessian of P at ϕ are

$$\frac{dP}{d\phi} = (T(\phi \otimes \phi) + \mu D(\phi) - \mu^2 I)\phi \qquad (37)$$

$$\frac{d^2P}{d\phi^2} = 3T(\phi \otimes \phi) + 2\mu D(\phi) - \mu^2 I \qquad . \qquad (38)$$

Moreover, from 17 ;

$$\operatorname{tr} D(\phi) = 0 \qquad , \qquad \operatorname{tr} T(\phi \otimes \psi) = \lambda(\phi,\psi) \qquad . \qquad (39)$$

Note that the conditions for a minimum are

$$\phi \neq 0 \quad , \quad \frac{dP}{d\phi} = 0 \quad , \quad \frac{d^2P}{d\phi^2} = T(\phi \otimes \phi) + \mu^2 I \qquad . \qquad (40)$$

If $\sigma(\phi)$ is not identically zero, i.e. there exists a third degree G-invariant on E , then the assumption of Theorem 1 are satisfied and $P(\phi)$ has no extrema on the generic stratum.

If $\sigma(\phi) \equiv 0$ and $\omega(\phi) = K(\phi,\phi)^2$, the invariance group $\tilde{G}$ is $O(m)$, m dim E , $T(\phi \otimes \phi) = (\phi,\phi)(I+2P_\phi)$, (P_ϕ is the orthogonal projector in ϕ) ; this case is well known to physicists. Theorem 1 does not apply, there is only one stratum in $E - \{0\}$, the generic one and the isotropy group of any point, and therefore of the minima, is $\tilde{H} = O(m-1)$. The interesting case occurs when $\omega(\phi)$ and $(\phi,\phi)^2$ are

linearly independent. The structure (26) of the ring of invariant polynomial and the powers ν_α must be known in order to see if theorem 1 applies. It does applies if, in the Mobius function (see 25), $\alpha > 0$ implies $\delta_\alpha > 4$ or does not exist.

Physicists working in high energy physics have computed the minima of many examples of polynomials (34). Physicists working in solid state physics have computed even more examples ! We recall that K is the set of conjugation classes of isotropy groups for the action of G on $E - \{0\}$. The following is known.

There are physical cases when the isotropy groups of a minimum of $P(\phi)$ does not belong to a maximal element of K . J.-C. and P. Toledano (1980), have given such an example for the irreducible representation of the crystallographic group $I4_1$, obtained from the point N of the Brillouin zone and of dimension $m = 4$. However, we have emphasized that it is the symmetry group $\widetilde{G} < O(m)$ of the polynomial which must be considered. In that case $\widetilde{G}$ is larger than the image of G . As shown in L. Michel (1979) for the action of $\widetilde{G}$ on $E - \{0\}$ the isotropy groups of minima belong to maximal elements of K . Remark that there are extrema whose isotropy groups belong to smaller elements of K .

This is a general situation. In all computed examples, the isotropy groups of the minima belong to the maximal elements of the set K defined by the action of $\widetilde{G}$ (and not necessarily the physical G) although there may exist extrema with smaller isotropy groups. It is tempting to make the conjecture that these results are always true, although they have been proved only for families of $\widetilde{G}$ actions ; for instance when condition (A10) of the appendix is satisfied and w_x is a Coxeter group (i.e. generated by reflections). This include more than the adjoint representation of simple Lie groups ; for instance it includes the $\ell = 2$, five-dimensional representation of $O(3)$.

In L. Michel et. al. (1978), Mozrzymas and I used Morse theory to prove this conjecture for $m \leq 3$ when $\widetilde{G}$ is finite, so in the compact case for $\dim N_n \leq 3$ when (A10) is satisfied. Our proof could be improved by applying Morse theory to the manifold M defined in (20). With conditions (16) and 17(a) (fourth degree polynomials) one proves that M is homeomorphic to the sphere S_{m-1} . Then many cases with $m = 4$ could then be proven but not all. The complete list of fourth degree four variable polynomials with irreducible isotropy group $\widetilde{G}$ in the action of $O(4)$ on E is given in L. Michel et. al (1981). I believe it is not yet known if the conjecture holds for the smallest $\widetilde{G}$ (which has eight elements).

Appendix

Correspondence between the orthogonal action of a compact group and the finite action of its finite Weyl group on the generic global slice.

We explicit in this appendix some relations between linear actions of compact groups and finite groups. Let us first recall some general and simple facts about general actions. We assume that G acts on the set E and that $N \subset E$ (i.e. N is a subset of E).

Definition of the centralizer of N in G .

$$C_G(N) = \bigcap_{\nu \in N} G_\nu \quad , \tag{A1}$$

it is the subgroup of G whose elements have fixed every point of N .

Definition of the normalizer of N in G .

$$N_G(N) = \{g \in G , \quad \forall \nu \in N , g \cdot \nu \in N \} \tag{A2}$$

it is the subgroup of G whose elements transform N into N . It is convenient to introduce the notation

$$g \cdot N = \{g \cdot \nu , \nu \in N\} \quad . \tag{A3}$$

Then

$$g \cdot N = N \iff g \in N_G(N) \quad . \tag{A4}$$

Theorem A1 : $C_G(N) \triangle N_G(N)$, ($\triangle$ reads : invariant subgroup).

Proof : Let $c \in C_G(N)$, $n \in N_G(N)$, $\nu \in N$. Then

$$(ncn^{-1}) \cdot \nu = n \cdot (c \cdot (n^{-1}\nu)) = n \cdot (n^{-1} \nu) = \nu \quad .$$

When G acts on itself by inner automorphism i.e. $g \cdot h = ghg^{-1}$ if $H < G$ (H subgroup of G) then $N_G(H)$ is the largest subgroup of G which has H as invariant subgroup. Hence

$$N_G(N) \leq N_G(C_G(N)) \quad . \tag{A5}$$

We have already introduced the notation :

$$E^G = \{\mu \in E , \quad \forall g \in G , g \cdot \mu = \mu\} \quad . \tag{A7}$$

By definition of the centralizer

$$N \subset E^{C_G(N)} \quad . \tag{A8}$$

If we assume the equality we have the easy lemma :

<u>Lemma A1</u> : $N = E^{C_G(N)}$ $\Rightarrow$ $N_G(N) = N_G(C_G(N))$.

We know (A5) $N_G(N) \le N_G(C_G(N))$. We now prove $\ge$. Let $n \in N_G(C_a(N))$ and $c \in C_G(N)$; then $n \, c \, n^{-1} \in C_G(N)$; so for any $\nu \in N$, $ncn^{-1} \cdot \nu = \nu$ i.e. $c \cdot n^{-1} \nu = n^{-1} \nu$ which means that $n^{-1} \nu \in E^{C_G(N)}$; this is N by assumption and therefore any n^{-1} , or any $n \in N_G(C_G(N)$ is also in $N_G(N)$. This lemma can be written on a simpler form :

$$N_G(E^H) = N_G(H) \qquad .$$

In general its converse is not true. As we shall see it is true for the orthogonal actions of a compact group G on a real vector space E when N_x is the global slice (= normal plane to the orbit $G(x)$) at a point x of the generic stratum. As we have seen N_x is a linear subspace of E which cuts every orbit of G (theorem 1) . We denote by W_x the quotient

$$W_x = N_G(N_x) / C_G(N_x) \qquad . \tag{A9}$$

By definition of the centralizer and the normalizer this quotient group acts effectively (i.e. without kernel) by an orthogonal representation on N_x . Each orbit of W_x is contained in the intersection of a G-orbit by N_x . Since in the neighborhood of x this intersection is an isolated point, W_x is discrete. We remark that $C_G(N_x)$ and $N_G(N_x)$, from their definition, are closed, so W_x is closed hence compact and therefore finite.

The best known example of the situation we study is the adjoint representation of the compact Lie group G . In that case the generic global slice is a Cartan subalgebra and W_x is the Weyl group. In that case W_x is generated by reflections.

We try now to prove the converse of Lemma A1 when N_x is the global generic slice of the orthogonal representation of the compact group G . The essential new fact is that $G_x = C_G(N_x)$ since the conjugate class of G_x , $x \in$ generic stratum, is minimal in the set K of conjugation classes of the isotropy groups appearing the G-action. Note also that $E^{C_G(N_x)} \equiv E^{G_x}$ is a linear subspace of E . Consider $y \in E^{G_x}$ and assume first that $G_y = G_x$, i.e. y is in the generic stratum. From theorem 1, N_x cuts $G(y)$, the G-orbit of y at least in a point x' and $G_{x'} = G_y = G_y = C_G(N_x)$. Since isotropy groups of an orbit are conjugated when two of them are identical $G_y = G_{x'}$, the elements of G which transform x' into y must belong to the normalizer of the isotropy group : $G_y \in N_G(G_x)(x')$. With the assumption $N_G(N_x) = N_G(G_x)$ we obtain that $y \in N_x$. This also show that for the generic stratum, the intersection of a G orbit $G(y)$ with N_x is the orbit of $N_G(G_x)$ and therefore of the group W_x . We are left in the case $y \in E^{G_x}$ with G_y strictly larger than G_x . Since the generic stratum is open dense we can consider all points, such as y' in the intersection of E^{G_x} and the generic stratum. We have shown that they are all in N_x and by continuity this is also true for y . Hence not only we have proven :

Lemma A2 : If N_x is the global slice of a point x of the generic stratum of an orthogonal action of a compact group G , then $C_G(N_x) = G_x$ and

$$N_x = E^{G_x} \Longleftrightarrow N_G(N_x) = N_G(G_x) \qquad . \qquad (A10)$$

We also obtain the

Theorem A2 : When (A10) holds for points of the generic stratum, the orbits of W_x in the slice N_x are the intersection of the G-orbits with N_x .

We have proven it for the points of the generic stratum and by continuity it extends to the closure, i.e. to the space E .

From this theorem, we deduce that the ring of $N_G(N)$ invariant polynomials on N_x is obtained from T^G by restriction of the domain of each polynomial from E to N_x . Note however that different groups G_1 , G_2 may have the same orbits, hence the same invariant polynomial ring T^G and the same W_x ; such an example is given by the adjointation of $SO(3)$ and $O(Z)$; then $W_x = Z_2$.

So, when (A10) holds, the Higgs problem for a compact group is identical to that for a finite group. Of course, (A10) may not be true. We give sufficient condition for that in L. Michel (1979). For instance

$$2 \times \dim \text{ (generic orbit) } < \dim E \quad \Rightarrow \text{ (A10) not true.} \qquad (A11)$$

We also show in L. Michel (1979) that (A10) does not hold when E carries a direct sum of two equivalent G representations.

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