Interaction between Four Half-Spin Particles and the Decay of the μ-Meson

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ABSTRACT. The general direct coupling between four fermions is studied. It is a linear combination of the five invariants used in β-decay theory. Altering the order of the particles in the Hamiltonian changes only the coefficients of the linear combination. The formalism of charge conjugation is used with the ordinary theory or the Majorana abbreviated theory for neutral particles. This is applied to the study of the decay of the μ-meson into an electron and two neutrinos.

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§ 1. INTRODUCTION

The purpose of this work is to study the most general contact interaction between four half-spin particles. The interaction Hamiltonian density used here consists of the most general linear combination of scalars which can be made up from four wave functions (or the conjugate of some of them). The interaction can be produced physically by an intermediary field; then the Hamiltonian only gives a phenomenological description.

Such an interaction between four fermions can give rise to $2^4 = 16$ kinds of processes, where i particles ($0 \leq i \leq 4$) are absorbed ($M = -1$), and j particles ($j = 4 - i$) are emitted ($M = +1$). M is a dichotomic variable having the values -1 and $+1$ for absorbed and emitted particles respectively. For example:

$i = 1$, β-decay: $N = P^+ + e^- + \nu$ with $M_N = -1$, $M_P = M_e = M_\nu = +1$;

$i = 2$, K-capture; scattering of fermions by fermions;

$i = 3$, pair annihilation near to an electron without emission of photons;

$i = 0$ or $i = 4$ (cf. Critchfield and Wigner 1941, Critchfield 1943).

The necessary energies for these processes can be supplied by external fields (e.g. $P \to N + \epsilon^{+} + \nu$).

The study of these processes will be restricted to the Schrödinger method of perturbation theory in which the Hamiltonian of the interaction does not contain the time explicitly.

I. INTERACTION BETWEEN FOUR HALF-SPIN PARTICLES

§ 1. FORMALISM OF THE CHARGE CONJUGATE THEORY

The charge conjugate formalism, which is preferable to Dirac's hole theory (see § 9), has been developed by Majorana (1937) and later by Racah (1937) and Kramers (1937). Pauli's (1941) notation ($h = c = 1$) will be followed here.

The most convenient representation of Dirac's matrices, which does not restrict the formalism, is that proposed by Majorana (1937).

$$(\gamma_\mu \partial_\mu + \kappa)\psi = 0 \qquad \ldots\ldots (1)$$

is the Dirac equation, where the γ_μ satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}. \qquad \ldots\ldots (2)$$

In the Majorana representation ($k = 1, 2, 3$)

$$\gamma_k^\dagger = \gamma_k = \tilde{\gamma}_k = \gamma_k^*; \qquad \gamma_4^\dagger = \gamma_4 = -\tilde{\gamma}_4 = -\gamma_4^*. \qquad \ldots\ldots (3)$$

or, with $\alpha_k = i\gamma_4\gamma_k$ and $\beta = \gamma_4$,

$$\alpha_k^\dagger = \alpha_k = \tilde{\alpha}_k = \alpha_k^*; \qquad \beta^\dagger = \beta = -\tilde{\beta} = -\beta^*. \qquad \ldots\ldots (3')$$

A solution of (1) can be split into plane waves normalized in a volume V:

$$\psi_\varrho = \frac{1}{(NV)^{\frac{1}{2}}} \sum_{\mathbf{k}\sigma} [u_+(\mathbf{k},\sigma) a_\varrho(\mathbf{k},\sigma) \exp\{i(\mathbf{k}\,.\,\mathbf{x} - k_0 x_0)\} + u_-^\dagger(\mathbf{k},\sigma) b_\varrho(\mathbf{k},\sigma) \times \exp\{-i(\mathbf{k}\,.\,\mathbf{x} - k_0 x_0)\}] \qquad \ldots\ldots (4)$$

($N = \Sigma N_L(\mathbf{k},\sigma)$ is the total number of particles of this field, see equation (11)), and because of the Majorana representation

$$b_\varrho^*(\mathbf{k},\sigma) = a_\varrho(\mathbf{k},\sigma); \qquad \ldots\ldots (5)$$

where $\mathbf{k}$ is the momentum, $k_0 = (\mathbf{k}^2 + \kappa^2)^{\frac{1}{2}}$ and σ is a dichotomic variable for the two states of spin. Also

$$\left.\begin{aligned} \sum_\sigma a_\mu(\mathbf{k},\sigma) a_\nu^*(\mathbf{k},\sigma) &= \tfrac{1}{2}\left(1 + \frac{\boldsymbol{\alpha}\,.\,\mathbf{k} + \beta\kappa}{k_0}\right)_{\mu\nu} = [D(+)]_{\mu\nu} \\ \sum_\sigma b_\mu(\mathbf{k},\sigma) b_\nu^*(\mathbf{k},\sigma) = \sum_\sigma a_\mu^* a_\nu &= \tfrac{1}{2}\left(1 + \frac{\boldsymbol{\alpha}\,.\,\mathbf{k} - \beta\kappa}{k_0}\right)_{\mu\nu} = [D(-)]_{\mu\nu} = [\tilde{D}(+)]_{\mu\nu} \end{aligned}\right\} \qquad \ldots\ldots (6)$$

(there is a misprint of sign in Pauli's formula (1941, equation (86)).

It is assumed (without restricting the theory) that in (u) the indices

(a) + or − denotes for charged particles, the particle of charge $+e$ or $-e$ respectively (e being the charge of the positron);

(b) for neutral particles which have charge conjugate states, the indices + or − correspond to the sign of the magnetic moment (with respect to the spin direction);

(*c*) for neutral particles which have no different charge conjugate states, $u_{+} = u_{-}$ (this could be written u_0). These particles have no interaction with the electromagnetic field and cannot be described by Dirac's hole theory. They were first treated by Majorana (1937), and they will be referred to as 'Majorana neutral particles'.

Let us call

$$\left.\begin{array}{lll} \psi^{K} = \psi, & u^{(K)} = u & \text{if} \quad K = -1 \\ \psi^{K} = \psi^{*}, & u^{(K)} = u^{*} & \text{if} \quad K = +1 \end{array}\right\}. \qquad \ldots\ldots (7)$$

The index L is either $+1$ or -1, and indicates either (*a*) the sign of the charge or (*b*) the sign of the magnetic moment, or (*c*) for Majorana neutral particles L is arbitrary (L or $-$L indicating the same state).

It is seen from (4) that ψ^* can be obtained from ψ by changing L into $-$L, so complex conjugation and charge conjugation are equivalent. Equations (4) and (6) can be written:

$$\psi_{\rho}^{K} = \frac{1}{(NV)^{\frac{1}{2}}} \sum_{\mathbf{k}\sigma} \sum_{L} u_{L}^{(LK)}(\mathbf{k}, \sigma) a_{\rho}^{LK}(\mathbf{k}, \sigma) \exp\{-LKi(\mathbf{k} \cdot \mathbf{x} - k_0 x_0)\} \quad \ldots\ldots (8)$$

and

$$\sum_{\sigma} a^{-K}(\mathbf{k}, \sigma) a^{K}(\mathbf{k}, \sigma) = D(K). \qquad \ldots\ldots (9)$$

For Majorana neutral particles $\psi^{K} = \psi^{-K}$ (ψ is real). In formula (8) the upper index LK plays the part of the index K as defined by (7), its values being the product of the K and L occurring separately in (8).

The anti-commutation rules can be written

$$[u_{L}^{(-K)}(\mathbf{k}, \sigma),\ u_{L'}^{(K')}(\mathbf{k}', \sigma')]_{+} = \delta_{KK'}\delta_{LL'}\delta(\mathbf{k}-\mathbf{k}')\delta_{\sigma\sigma'} \qquad \ldots\ldots (10)$$

and

$$u_{L}^{(K)}(\mathbf{k}, \sigma) u_{L}^{(-K)}(\mathbf{k}, \sigma) = \begin{Bmatrix} N_{L}(\mathbf{k}, \sigma) & \text{if } K = +1 \\ 1 - N_{L}(\mathbf{k}, \sigma) & \text{if } K = -1 \end{Bmatrix}. \qquad \ldots\ldots (11)$$

The operator $N_{L}(\mathbf{k}, \sigma)$ represents the number of particles in the state $\mathbf{k}, \sigma$, L; its eigenvalues are 0 or 1.

$u_{L}^{(LK)}$ is an absorption operator if LK = M = -1, and
an emission operator if LK = M = $+1$; $\qquad \ldots\ldots (12)$

for a charged particle, of charge L*e*, it changes the total charge by an amount K*e*. All the dichotomic variables used here are listed in Table 3.

§ 2. PARTICLES OF DIFFERENT KINDS

If there are two different kinds S of particles, their amplitudes $v(S)$ commute:

$$[v^{(K)}(S, L), v^{(K')}(S', L')]_{-} = 0 \quad \text{if} \quad S \neq S'. \qquad \ldots\ldots (13)$$

However, it is even possible to consider particles of different rest mass as like particles. For example, neutrons and protons are considered as nucleons, or electrons and neutrinos as leptons, by the formalism of isotopic spin, which is nothing other than writing the same v's in a more condensed form:

$$v(S) = \begin{pmatrix} v(1) \\ 0 \end{pmatrix} \text{ if } S = 1, \quad v(S) = \begin{pmatrix} 0 \\ v(2) \end{pmatrix} \text{ if } S = 2 \text{ and } [v^{(K)}(1, L), v^{(K')}(2, L')]_{-} = 0,$$

since $v^{(K)}(1, L) v^{(K')}(2, L') = 0$.

Another interesting extension which has been made is to consider a Majorana neutral particle and the two states of a charged particle as three conjugate states $(+, 0, -)$ of the same particle (Noma 1948).

Alternately there is a transformation, due to Klein (1938), which makes it possible always to have anti-commutation between the amplitudes of two like or unlike particles.

If the v's satisfy (13), putting $v^{(K)}(1, \text{L}) = u^{(K)}(1, \text{L})$ and $v^{(K)}(2, \text{L}) = \zeta(1) u^{(K)}(2, \text{L})$ with $\zeta(1) = \Pi[1 - 2N(1, \text{L}, \mathbf{k}, \sigma)]$ (Π is taken over all the states $\text{L}, \mathbf{k}, \sigma$) one gets $[u^{(K)}(1, \text{L}), u^{(K')}(2, \text{L}')]_{+} = 0$.

This can be extended to any number of kinds S of particles, so that one gets

$$[u^{-(K)}(S, \text{L}, \mathbf{k}, \sigma),\ u^{(K')}(S', \text{L}', \mathbf{k}', \sigma')]_{+} = \delta_{KK'}\,\delta_{SS'}\,\delta_{\text{LL}'}\,\delta(\mathbf{k} - \mathbf{k}')\,\delta_{\sigma\sigma'}. \qquad \ldots\ldots (14)$$

§ 3. SCALARS FORMED FROM FOUR ψ^K

The method of formation of such scalars is well known. Using only the relations (1) and (2), it has been demonstrated by Pauli (1936), corrected by Racah (1937). With the four γ_μ one can build 16 matrices $\gamma_A = \gamma_{[i, a]}$ ($A = 1$ to 16 or $i = 1$ to 5), viz.:

A	i	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$	$a=6$
1	1	1					
2 to 5	2	γ_1	γ_2	γ_3	γ_4		
6 to 11	3	$i\gamma_2\gamma_3$	$i\gamma_3\gamma_1$	$i\gamma_1\gamma_2$	$i\gamma_1\gamma_4$	$i\gamma_2\gamma_4$	$i\gamma_3\gamma_4$
12 to 15	4	$i\gamma_2\gamma_3\gamma_4$	$i\gamma_3\gamma_1\gamma_4$	$i\gamma_1\gamma_2\gamma_4$	$-i\gamma_1\gamma_2\gamma_3$		
16	5	$\gamma_1\gamma_2\gamma_3\gamma_4$					

It is easily, the γ_μ satisfying (2) and (3), that $\gamma_A{}^2 = 1$ and $\gamma_A{}^\dagger = \gamma_A$. If

$$F_{ia} = \epsilon_{ia}\gamma_4\gamma_{[i, a]} \qquad \ldots\ldots (15)$$

where ϵ_{ia} is such that $\epsilon_{ia}{}^2 = +,\ - - - +,\ + + + - - -,\ + + + -,\ -$, then

$$F_{ia}{}^\dagger = F_{ia} \qquad \ldots\ldots (16)$$

and $$\sum_{\mu\nu}[\psi_S{}^K(\mathbf{x}, x_0)]_\mu (F_{ia})_{\mu\nu} [\psi_{S'}{}^{K'}(\mathbf{x}, x_0)]_\nu = \psi_S{}^K F_{ia} \psi_{S'}{}^{K'}$$

is a scalar for $i = 1$, the four components of a vector for $i = 2$, a skew tensor for $i = 3$, a pseudovector for $i = 4$ and a pseudoscalar for $i = 5$.

The scalars we require are $J_i(\kappa, P) =$ the scalar product of $\psi_1{}^{K_1} F_{ia} \psi_2{}^{K_2}$ and $\psi_3{}^{K_3} F_{ia} \psi_4{}^{K_4}$, i.e.

$$J_i(\kappa, P) = \sum_a \epsilon'_{ia}(\psi_1{}^{K_1} F_{ia} \psi_2{}^{K_2})(\psi_3{}^{K_3} F_{ia} \psi_4{}^{K_4}), \qquad \ldots\ldots (17)$$

where $\epsilon'_{ia} = +,\ + + + -,\ + + + - - -,\ + + + -,\ +$, and (κ) is a contraction for (K_1, K_2, K_3, K_4) and can have 16 different values; P indicates that the four ψ^K in $J_i(P)$ are in a definite order: 1, 2, 3, 4.

From (15), (16) and (3) it is found that

$$\widetilde{F}_{ia} = \theta_i F_{ia} \qquad \ldots\ldots (18)$$

with $\theta_i = -1,\ +1,\ +1,\ -1,\ -1$.

§4. THE MOST GENERAL HAMILTONIAN OF THE INTERACTION

We choose the Schrödinger picture for the operator H which then does not depend explicitly on the time, and have taken the same axes for the Heisenberg and Schrödinger pictures at time $x_0 = 0$. We therefore put $x_0 = 0$ in $J_i(\kappa, P)$, given by (17).

The most general Hamiltonian of the interaction, which must be Hermitian, can be written:

$$H = \int_V h(\mathbf{x})\, d\mathbf{x} \quad \text{with} \quad h = \sum_{P=1}^{24} \sum_{\mathbf{K}=1}^{16} \sum_{i=1}^{5} g_i(\kappa, P)[J_i(\kappa, P) + J_i^{\dagger}(\kappa, P)]. \qquad \ldots\ldots (19)$$

The sum $\sum_P$ indicates a summation over all the twenty-four permutations of the four indices $S = 1, 2, 3, 4$; the sum $\sum_{\mathbf{K}}$ indicates a summation over the 16 possible sign combinations of $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$; and $g_i(\kappa, P)$ are real numbers called interaction constants.

For the following considerations we must exclude the transition in which two particles are absorbed and emitted again in their initial states: this would correspond to the self-energies of these particles due to this interaction.

We can then say that the ψ^{K} anti-commute, and from (16) it follows that

$$(\psi_S{}^{\mathrm{K}} F_{i\alpha} \psi_{S'}{}^{\mathrm{K}'})^{\dagger} = -\psi_{S'}{}^{-\mathrm{K}'} F_{i\alpha} \psi_S{}^{-\mathrm{K}}.$$

Therefore, if P_1 is the permutation (2143)

$$J_i^{\dagger}(\kappa, P) = J_i(-\kappa, P_1 P). \qquad \ldots\ldots (20)$$

From (14) and (18) one gets

$$J_i^{\dagger}(\kappa, P_1 P) = (-1)^2 \theta_i^2 J_i(\kappa, P) = J_i(\kappa, P). \qquad \ldots\ldots (21)$$

Therefore (20) can be written

$$J_i^{\dagger}(\kappa, P) = J_i(-\kappa, P)$$

and (19) becomes

$$h = \sum_P \sum_{\mathbf{K}} \sum_i g_i(\kappa, P)[J_i(\kappa, P) + J_i(-\kappa, P)] = \sum_P \sum_{\mathbf{K}} \sum_i [g_i(\kappa, P) + g_i(-\kappa, P)] J_i(\kappa, P)$$

or

$$h = \sum_P \sum_{\mathbf{K}} \sum_i g_i'(\kappa, P) J_i(\kappa, P) \qquad \ldots\ldots (22)$$

with

$$g_i'(\kappa, P) = g_i(\kappa, P) + g_i(-\kappa, P);$$

therefore one has

$$g_i'(\kappa, P) = g_i'(-\kappa, P). \qquad \ldots\ldots (23)$$

§5. ELIMINATION OF THE SUMMATION OVER THE TWENTY-FOUR PERMUTATIONS

It is well known (see for instance Speiser 1945) that the S_4 group (four object permutations) is soluble. All the permutations are given by the twenty-four terms of the symbolic expression: $(1 + P_3)(1 + P_3 P_4 + P_4 P_3)(1 + P_2)(1 + P_1)$ with $P_1 = (2143)$, $P_2 = (3412)$, $P_3 = (2134)$, $P_4 = (1432)$. Each P is a product of the form

$$P = P_a P_b \ldots P_k, \qquad \ldots\ldots (24)$$

where $a, b, \ldots k = 0$ to 4 and $P_0 = 1$. Let us write $(-1)^P = +1$ or -1 according as P is an even or an odd permutation.

We will now show that, for every P,

$$J_i(\kappa, PP_0) = \sum_j (-1)^P C_{ij}^P J_j(\kappa, P_0), \qquad \ldots\ldots (25)$$

which means that under any permutation P of the four indices S, the J_i undergo a linear substitution; later on $J_i(\kappa, P)$ will simply be written $J_i(\kappa)$.

Indeed, for $P = P_1$, P_2 or P_3 we very easily find: $C_{ij}^{P_1} = \delta_{ij}$ (see (21)), $C_{ij}^{P_2} = \delta_{ij}$ (evident), $C_{ij}^{P_3} = \theta_i \delta_{ij}$ (see (18)). The case of $P = P_4$ has been studied by Fierz (1937), and the same method can be extended to the present choice of F_{kl}; it is found that

$$C_{ij}^{P_4} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -1 & -\frac{1}{2} & 0 & -\frac{1}{2} & -1 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 0 & -\frac{3}{2} \\ 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Therefore for every permutation P, which is always a product of the form (24) of the four permutations P_1, P_2, P_3, P_4, we get

$$C_{ij}^{P} = \sum_{lm \ldots r} C_{il}^{P_a} C_{lm}^{P_b} \ldots C_{rj}^{P_k}.$$

Then (22) can be written $h = \sum_P \sum_{\mathbf{K}} \sum_{ij} g_i'(\kappa, P)(-1)^P C_{ij}^P J_j(\kappa)$.

By putting $\quad g_j(\kappa) = \sum_{i,P} g_i'(\kappa, P)(-1)^P C_{ij}^P$

one gets
$$h = \sum_{\mathbf{K}} \sum_j g_j(\kappa) J_j(\kappa) \qquad \ldots\ldots (26)$$

and (23) gives
$$g_j(\kappa) = g_j(-\kappa). \qquad \ldots\ldots (27)$$

This last relation will be studied in §7.

Changing the order of the particles in H only means 'changing the name' of the interaction constants g_j; thus H remains invariant.

§6. A REMARKABLE INTERACTION

Can we define an interaction which is independent of the order of the four ψ^K in H?

A permutation leaves H invariant:

$$h = \sum_{\mathbf{K}} \sum_j g_j(\kappa) J_j(\kappa) = \sum_{\mathbf{K}} \sum_i g_i'(\kappa) J_i'(\kappa) \quad \text{with} \quad J_i(\kappa, P) = J_i'(\kappa),$$

but changes the 'five vector' $\vec{g}(\kappa)$ into $\vec{g}'(\kappa)$ according to $\vec{g} = (-1)^P \tilde{C}^P . \vec{g}'$ because equation (25) gives $J_i'(\kappa) = \sum_j (-1)^P C_{ij}^P J_j(\kappa)$, so that one must have

$$h = \sum_{\mathbf{K}} \sum_j g_j(\kappa) J_j(\kappa) = \sum_{\mathbf{K}} \sum_{ij} g_i'(\kappa)(-1)^P C_{ij}^P J_j(\kappa),$$

which gives $\sum_i g_i'(-1)^P C_{ij}^P = g_j$ or $(-1)^P \vec{g}' . C^P = \vec{g}$ or $(-1)^P \tilde{C}^P . \vec{g}' = \vec{g}$.

The above question is thus reduced to this: are there common 'eigenvectors' to the twenty-four $(-1)^P \tilde{C}^P$ matrices? To answer this it is sufficient to look for the common 'eigenvectors' of $-\tilde{C}^{P_3}$ and $-\tilde{C}^{P_4}$, since the other $(-1)^P \tilde{C}^P$ are products of 1, $-\tilde{C}^{P_3}$ and $-\tilde{C}^{P_4}$.

One finds that *the only* common 'eigenvectors' are given by

$$g_2 = g_3 = 0, \quad g_1 = -g_4 = g_5, \qquad \ldots\ldots (28\,a)$$

They correspond to the 'eigenvalue' +1. Therefore h is symmetrical for all the ψ^K (which anti-commute); this is easy to verify, for, starting from

$$h = \sum_K \sum_i \bar{g}(\kappa) \sum_P J_i(\kappa, P) = \sum_K \sum_i \bar{g}(\kappa) \sum_i \Big(\sum_P (-1)^P C_{ij}{}^P\Big) J_j(\kappa) = \sum_K \sum_{ij} \bar{g}(\kappa) Z_{ij} J_j(\kappa),$$

with $Z = (1 - C^{P_2})(1 + C^{P_3}C^{P_4} + C^{P_4}C^{P_3})(1 + C^{P_2})(1 + C^{P_1})$, one finds

$$h = \sum_K 2(\bar{g}_1 - 4\bar{g}_4 + \bar{g}_5)(J_1 - J_4 + J_5) \quad \text{or} \quad h = \sum_K g_0(\kappa)[J_1(\kappa) - J_4(\kappa) + J_5(\kappa)]. \qquad \ldots\ldots (28\,b)$$

It is also easy to verify that there is no h completely antisymmetrical with respect to the ψ^K (which anti-commute) because

$$(1 + C^{P_3})(1 + C^{P_2}C^{P_4} + C^{P_4}C^{P_3})(1 + C^{P_2})(1 + C^{P_1}) = 0.$$

Has this remarkable interaction a physical meaning? This question has no meaning if H is only phenomenological (see introduction). However, the field theory allows us to assume a physically direct interaction between four particles (that would be unthinkable with only the classical notion of forces), and we have seen that the only direct interaction symmetrical with respect to the four particles is the remarkable one g_0 (see equation (28)).

Now, in a Dirac equation, it is well known that F_2 and F_3 correspond to electromagnetic interaction and the sign of the interaction constant (electric charge or magnetic moment) is changed by charge conjugation; F_1, F_4, F_5 would correspond to a non-electromagnetic interaction and the sign of the interaction constant (mesic charge (Okayama 1949) for instance) is not changed by charge conjugation; but the $J_i(\kappa)$, which are quadratic with respect to the F_i, are invariant by charge conjugation and $g_i(\kappa) = g_i(-\kappa)$. However, there is only one interaction built either with F_2, F_3 or F_1, F_4, F_5 independently of the order chosen for the ψ^K in writing h: it is the remarkable interaction g_0 built with the three non-magnetic operators.

The interaction g_0 has already been proposed by Critchfield (1943) and can be written as the determinant of the four components of the four ψ^K.

§7. CLASS OF REACTIONS WITH SAME $g(\kappa)$

A reaction between four particles is defined by (see Table 3) the values of the M_S and of the L_S of each particle ($S = 1$ to 4).

The conservation of the electric charge requires

$$\sum_{(S)c} M_S L_S = 0, \qquad \ldots\ldots (29)$$

where $\sum$ means the sum over the charged particles only.

We have already seen (12) that

$$M_S L_S = \kappa_S, \qquad \ldots\ldots (30)$$

so (29) can be written

$$\sum_{(S)c} \kappa_S = 0. \qquad \ldots\ldots (31)$$

This relation indicates the possible different values which can be taken for $\kappa(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and for $g(\kappa) = g(-\kappa)$ in H.

For example, in β-radioactivity the magnetic moment of the neutron is of negative sign; let m be the sign of the magnetic moment of the neutrino emitted in β^--decay. Then the reaction

		N	P	ϵ	ν
(*a*) $N_- \to P^+ + \epsilon^- + \nu_m$ corresponds to	M	−	+	+	+
	L	−	+	−	m
and (28) gives for K	K	+	+	−	m

With four Dirac neutral particles there are of course $2^4 \times 2^4 = 256$ distinct possible reactions $\text{M} = (\text{M}_1, \text{M}_2, \text{M}_3, \text{M}_4)$, $\text{L} = (\text{L}_1, \text{L}_2, \text{L}_3, \text{L}_4)$. This number is different for other kinds of particles since the charged particles have to fulfil (29) and the Majorana particles have only one state L. The result is given in the fifth line of Table 1.

Table 1

No. of charged particles	0	0	0	0	0	2	2	2	4
No. of Dirac neutral particles	4	3	2	1	0	2	1	0	0
No. of Majorana neutral particles	0	1	2	3	4	0	1	2	0
No. of different $\vec{g}$ or classes	8	4	2	1	1	4	2	1	3
No. of different reactions	256	128	64	32	16	128	64	32	96
No. of reactions per class	32	32	32	32	16	32	32	32	32

This table refers only to the cases of four distinguishable particles.

When a reaction is realized in nature, a $\vec{g}(\text{K}) = \vec{g}(-\text{K})$ must be given for describing it, and all the reactions which have the same $\pm\text{K} = (\text{K}_1, \text{K}_2, \text{K}_3, \text{K}_4$ or $-\text{K}_1, -\text{K}_2, -\text{K}_3, -\text{K}_4)$ can be associated with this $\vec{g}$. The set of all reactions which can be described by the same $\vec{g}$ will be called a 'class'. Therefore the interaction Hamiltonian density which describes all the interactions of a class is (like (26), but without summation over K)

$$h = \sum_j g_j(\text{K})[J_j(\text{K}) + J_j(-\text{K})] = \vec{g} \,.\, [\vec{J}(\text{K}) + \vec{J}(-\text{K})].$$

For example, the reaction

(*b*) $N_- + \nu_{-m} \to P^+ + \epsilon^-$ belongs to the same class of reactions as (*a*): $\text{K} = (+, +, -, \text{m})$, but the reaction

(*c*) $N_- + \nu_{+m} \to P^+ + \epsilon^-$ belongs to another class: $\text{K}(+, +, -, -\text{m})$, unless the neutrino is a Majorana particle, when the two reactions would be identical, $\nu_{-m} = \nu_m$.

This very simple remark will be useful later on.

For given kinds (charged, Dirac or Majorana neutral) of the four fermions the fourth line of Table 1 gives the number of different classes of reactions and the sixth line the number of reactions per class.

§ 8. STUDY OF ONE REACTION. DETERMINATION OF TRANSITION PROBABILITIES: SPIN AVERAGES

For this study it will be assumed that the transition can be studied by the Born approximation, i.e. the particles are described by plane waves.

The transition is given by $\text{M} = (\text{M}_1, \text{M}_2, \text{M}_3, \text{M}_4)$ and $\text{L} = (\text{L}_1, \text{L}_2, \text{L}_3, \text{L}_4)$; therefore K is known since $\text{K}_S = \text{M}_S\text{L}_S$. If H_T is the element of H which leads to the transition T, the probability per unit time of the transition is proportional to $|H_T|^2$.

First case: the four particles are distinguishable.

From (8), (12), (17) and (26) one gets: $H_{\mathrm{T}} = \int_V h_{\mathrm{T}}(\mathbf{x})\,d\mathbf{x}$, i.e.

$$H_{\mathrm{T}} = \sum_{ia} g_i(\kappa)\epsilon_{ia}' u_{L_1}^{(L_1K_1)} u_{L_2}^{(L_2K_2)} u_{L_3}^{(L_3K_3)} u_{L_4}^{(L_4K_4)}$$
$$\times (a_1^{L_1K_1} F_{ia} a_2^{L_2K_2})(a_3^{L_3K_3} F_{ia} a_4^{L_4K_4})(V^{-\frac{1}{2}})^4 \int_V [\exp\{-i(\sum_S K_S L_S \mathbf{k}_S)\,.\,\mathbf{x}\}]\,d\mathbf{x}$$

where $\sum_S L_S K_S \mathbf{k}_S = \sum_S M_S \mathbf{k} = 0$ from the conservation of momentum. Therefore the integral is equal to V and

$$|H_{\mathrm{T}}|^2 = H_{\mathrm{T}}^{\dagger} H_{\mathrm{T}} = \sum_{ia}\sum_{jb} V^{-2} g_i g_j \epsilon_{ia}' \epsilon_{jb}' U \,.\, A_{iajb} \qquad \ldots\ldots (32)$$

with $U = 1$ if the transition is allowed, i.e. the particles to be absorbed are present and the states of the particles to be created are not occupied initially (see (11)).

$$A_{iajb}(\mathbf{k}_S, \sigma_S) = (a_2^{-M_2} F_{ia} a_1^{-M_1})(a_1^{M_1} F_{jb} a_2^{M_2})(a_4^{-M_4} F_{ia} a_3^{-M_3})(a_3^{M_3} F_{jb} a_4^{M_4}) \qquad \ldots\ldots (33)$$

where LK has been replaced by M because of (30).

However, experimental physicists do not usually distinguish between the states of spin of the particles. By summing over the different states of spin of the emitted particles and taking the average over the $n = 4 \times 2^{-\frac{1}{2}\Sigma M_S}$ initial states, which are not distinguished experimentally, one calculates:

$$\frac{1}{n}\sum_{\sigma_S} |H_{\mathrm{T}}|^2 = \frac{V^{-2}}{n} \sum_{ia}\sum_{jb} g_i g_b \epsilon_{ia}' \epsilon_{jb}' \sum_{\sigma_S} A_{iajb} \qquad \ldots\ldots (34)$$

and (9) gives:

$$\sum_{\sigma_S} A = \mathrm{Trace}\, F_{ia} D_1(M_1) F_{jb} D_2(-M_2) \times \mathrm{Trace}\, F_{ia} D_3(M_3) F_{jb} D_4(-M_4) \qquad \ldots\ldots (35)$$

Because of the symmetry between 1, 2 and 3, 4 one finds that $\sum_{\sigma_S} |H_{\mathrm{T}}|^2$ is quadratic in the M_S and does not change when one changes all the M_S into $-M_S$, so that the reversibility principle of microphysics is satisfied.

One sees that the probability of the reaction M, L depends only on the M_S; it is independent of the L_S: the formalism is completely symmetrical with respect to the charge conjugate states of the particles; therefore it is also independent of K, the class of the interaction.

This shows that for the so-called 'allowed' transitions of the β-radioactivity, the life-time, the electron spectrum, etc. . . . are the same whether the neutrino is a Majorana or a Dirac particle.

Second case: some particles indistinguishable.

Firstly, let us say two particles are indistinguishable, and let us suppose them to be $0'$ and $0''$, i.e. $K_{0'} = K_{0''}$, $L_{0'} = L_{0''}$, $M_{0'} = M_{0''}$. So we get

$$\psi_{0'}^{K_0} = \psi_{0''}^{K_0} = \psi_0^{K_0} = (2V)^{-\frac{1}{2}} \sum_{R=1}^{2} u_{L_0}^{M_0}(\mathbf{k}_R, \sigma_R) a^{M_0}(\mathbf{k}_R, \sigma_R) \exp\{-M_0 i \mathbf{k}_R \,.\, \mathbf{x}\} \qquad \ldots\ldots (36)$$

J_t has two ψ^K identical; therefore H is invariant with respect to their permutation, and it follows that *H is a function of only three independent constants g_i.*

Let us put $0'=3$, $0''=4$; noting that $[u_4^{(M)}(\mathbf{k}, \sigma)]^2=0$ (see (14)), it follows from (34):

$$V\psi_3^{K_3}F_{ia}\psi_4^{K_4}=u_{L_0}^{(M_0)}(0')u_{L_0}^{(M_0)}(0'')\frac{a^{M_0}(0')}{\sqrt{2}}F_{ia}\frac{a^{M_0}(0'')}{\sqrt{2}}$$
$$+u_{L_0}^{(M_0)}(0'')u_{L_0}^{(M_0)}(0')\frac{a^{M_0}(0'')}{\sqrt{2}}F_{ia}\frac{a^{M_0}(0')}{\sqrt{2}}$$

or
$$V\psi_3^{K_3}F_{ia}\psi_4^{K_4}=u_{L_0}^{(M_0)}(0')u_{L_0}^{(M_0)}(0'')a^{M_0}(0')\left(\frac{F_{ia}-\tilde{F}_{ia}}{2}\right)a^{M_0}(0''),$$

and we see from (18) that $J_2=J_3=0$ since $\tilde{F}_{ia}=F_{ia}$ for $i=2$ or 3. In this case H depends only on g_1, g_4, g_5. By a permutation P the new g_i (let us call them g_i') are linear functions of the three g_1, g_4, g_5.

If there are three (or four) indistinguishable particles, it is easy to see that h only depends on one g, which defines the special interaction (28), $g_0=g_1=-g_4=g_5$.

It is noteworthy that the preceding conclusions are independent of the fact that the indistinguishable particles can be of Dirac's or Majorana's kind.

Of course if the states of two indistinguishable particles were identical ($\mathbf{k}_1=\mathbf{k}_2$, $\sigma_1=\sigma_2$) (14) would give $U=0$, therefore $H=0$: the formalism assumes Pauli's exclusion principle.

Table 2

	g_1^2	g_2^2	g_3^2	g_4^2	g_5^2	$2g_1g_2$	$2g_1g_3$	$2g_2g_3$	$2g_2g_4$	$2g_3g_4$	$2g_3g_5$	$2g_4g_5$
1	1	4	6	4	1							
$\mathbf{p}_1\cdot\mathbf{p}_2+\mathbf{p}_3\cdot\mathbf{p}_4$	−1		2		−1							
$\mathbf{p}_1\cdot\mathbf{p}_3+\mathbf{p}_2\cdot\mathbf{p}_4$		−2	−4	−2			1		−2		−1	
$\mathbf{p}_1\cdot\mathbf{p}_4+\mathbf{p}_2\cdot\mathbf{p}_3$		−2	−4	−2			−1		2		1	
$(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{p}_3\cdot\mathbf{p}_4)$	1		−2		1							
$(\mathbf{p}_1\cdot\mathbf{p}_3)(\mathbf{p}_2\cdot\mathbf{p}_4)$		2	4	2			−1		2		1	
$(\mathbf{p}_1\cdot\mathbf{p}_4)(\mathbf{p}_2\cdot\mathbf{p}_3)$		2	4	2			1		−2		−1	
$\mathbf{p}_1\cdot\mathbf{p}_2m_3m_4+\mathbf{p}_3\cdot\mathbf{p}_4m_1m_2$	1	−2		2	−1							
$\mathbf{p}_1\cdot\mathbf{p}_3m_2m_4+\mathbf{p}_2\cdot\mathbf{p}_4m_1m_3$						1		−3		−3		−1
$\mathbf{p}_1\cdot\mathbf{p}_4m_2m_3+\mathbf{p}_2\cdot\mathbf{p}_3m_1m_4$						−1		−3		3		−1
$m_1m_2+m_3m_4$	−1	2		−2	1							
$m_1m_3+m_2m_4$						−1		3		3		1
$m_1m_4+m_2m_3$						1		3		−3		1
$m_1m_2m_3m_4$	1	4	6	4	1							

The formula (32) can be very useful for calculations. With the notations $(\mathbf{k}_S/k_{0S})=\mathbf{p}_S$, $(\kappa_S/k_{0S})=m_S$ one has $D_S(\mathrm{M}_S)=\frac{1}{2}(1+\boldsymbol{\alpha}\cdot\mathbf{p}_S+\mathrm{M}_S\beta m_S)$.

The value of $\sum_{\sigma_S}|H_T|^2$ for $\mathrm{M}_1=\mathrm{M}_2=\mathrm{M}_3=\mathrm{M}_4=\pm 1$, i.e.

$$\sum_{\sigma_S}|H_T|^2=V^{-2}\sum_{ia}\sum_{jb}g_ig_j\epsilon_{ia}'\epsilon_{jb}'\,\mathrm{Trace}\,F_{ia}D_1(+)F_{jb}D_2(-)\times\mathrm{Trace}\,F_{ia}D_3(+)F_{jb}D_4(-),$$

is given in Table 2. The coefficients of g_ig_j are functions of the fourteen quantities listed in the first column of the Table. The figures indicate how these functions are built up from the fourteen expressions, so that, e.g., the second column of this Table means that the coefficient of g_1^2 is

$$1-(\mathbf{p}_1\cdot\mathbf{p}_2+\mathbf{p}_3\cdot\mathbf{p}_4)+(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{p}_3\cdot\mathbf{p}_4)+\mathbf{p}_1\cdot\mathbf{p}_2m_3m_4$$
$$+\mathbf{p}_3\cdot\mathbf{p}_4m_1m_2-(m_1m_2+m_3m_4)+m_1m_2m_3m_4.$$

For the remarkable interaction $g_0 = g_1 = -g_4 = g_5$ one finds

$$\frac{V^2}{g_0^2}\sum_{\sigma_S}|H_T|^2 = 6 + (m_1^2 + m_2^2 + m_3^2 + m_4^2) - (m_1 + m_2 + m_3 + m_4)^2 + 6m_1m_2m_3m_4$$
$$+ 2(\mathbf{p}_1\cdot\mathbf{p}_2 m_3m_4 + \mathbf{p}_3\cdot\mathbf{p}_4 m_1m_2) + 2(\mathbf{p}_1\cdot\mathbf{p}_3 m_2m_4 + \mathbf{p}_2\cdot\mathbf{p}_4 m_1m_3)$$
$$+ 2(\mathbf{p}_1\cdot\mathbf{p}_4 m_2m_3 + \mathbf{p}_2\cdot\mathbf{p}_3 m_1m_4) + (\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{p}_3^2 + \mathbf{p}_4^2)$$
$$- (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4)^2 + 2[(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{p}_3\cdot\mathbf{p}_4) + (\mathbf{p}_1\cdot\mathbf{p}_3)(\mathbf{p}_2\cdot\mathbf{p}_4)$$
$$+ (\mathbf{p}_1\cdot\mathbf{p}_4)(\mathbf{p}_2\cdot\mathbf{p}_3)],$$

that is, completely symmetrical in the four particles.

For another set of values of $M = (M_1, M_2, M_3, M_4)$ one can use Table 2, replacing m_S by $M_S m_S$.

This table depends of course on the F_{ia} chosen. F_{ia} is defined in (15) as a function of γ_μ. Below, the F_{ia} are given as a function of the α_k, β. The above calculations are only possible in the Majorana representation, unless $M_1 = -M_2 = M_3 = -M_4$, when they are also possible in the Dirac representation. For the latter case the F used here are given as functions of Dirac's ρ_k and σ_k.

$F_1 =$	β						$= \rho_3$	
$F_2 =$	α_1	α_2	α_3	$[1]$			$= \rho_1\sigma_k$	$[1]$
$F_3 =$	$-i\beta\alpha_2\alpha_3$	$-i\beta\alpha_3\alpha_1$	$-i\beta\alpha_1\alpha_2$	$[-i\beta\alpha_1]$	$[-i\beta\alpha_2]$	$[-i\beta\alpha_3]$	$= \rho_3\sigma_k$	$[\rho_2\sigma_k]$
$F_4 =$	$-i\alpha_2\alpha_3$	$-i\alpha_3\alpha_1$	$-i\alpha_1\alpha_2$	$[-i\alpha_1\alpha_2\alpha_3]$			$= \sigma_k$	$[\rho_1]$
$F_5 =$	$-\beta\alpha_1\alpha_2\alpha_3$						$= \rho_2$	

(time components are given in brackets; the sign of the F_{ia} is immaterial). In Dirac's representation one must choose $\mathcal{F}_{ia}$ instead of F_{ia} in $\psi_S^K F_{ia} \psi_{S'}^{K'}$ with $C = \rho_2\sigma_2$ and

$$\mathcal{F}_{ia} = F_{ia} \quad \text{for} \quad K = +1 \quad K' = -1$$
$$\mathcal{F}_{ia} = \tilde{C} F_{ia} C^{-1} \quad \text{for} \quad K = -1 \quad K' = +1$$
$$\mathcal{F}_{ia} = F_{ia} C^{-1} \quad \text{for} \quad K = +1 \quad K' = +1$$
$$\mathcal{F}_{ia} = \tilde{C} F_{ia} \quad \text{for} \quad K = -1 \quad K' = -1$$

§ 9. DIRAC'S HOLE THEORY

It is not possible to describe the Majorana neutral particles by Dirac's hole theory. If no such particles are considered, this theory is of course entirely equivalent to the formalism of charge conjugation but not so convenient. In this section we shall follow up in some detail this equivalence, which shows in particular that it does not matter which charge we attribute to the particle or to the hole.

We distinguish between the physical particle and the theoretical 'corpuscle'. Let us write the $\psi^K(\mathbf{x})$ of one 'corpuscle' of electric charge ϵe (with $\epsilon = \pm 1$) or (for a neutral particle) of magnetic moment ϵm, of momentum $\mathbf{k}$, of energy $k_0 = \eta(\mathbf{k}^2 + \kappa^2)^{\frac{1}{2}}$ with $\eta = \pm 1$, as

$$\psi_\sigma^K(\mathbf{x}) = V^{-\frac{1}{2}} u^{K}(\mathbf{k}, \sigma, \eta) a_\sigma^K(\mathbf{k}, \sigma, \eta) \exp(-iK\mathbf{k}\cdot\mathbf{x}). \quad \ldots\ldots (37)$$

$N(\mathbf{k}, \sigma, \eta) = u^\dagger(\mathbf{k}, \sigma, \eta) u(\mathbf{k}, \sigma, \eta)$ is the operator number of 'corpuscles' in the state $\mathbf{k}, \sigma, \eta$. Physically this corresponds to

$$N'_\sigma(\mathbf{k}, \sigma) = \begin{cases} N(\mathbf{k}, \sigma, \eta) & \text{for} \quad \eta = 1 \\ 1 - N(\mathbf{k}, \sigma, \eta) & \text{for} \quad \eta = -1. \end{cases}$$

$N'_{\epsilon\eta}(\mathbf{k}, \sigma)$ is the operator number of particles of charge $\eta\boldsymbol{\epsilon}e$ (or magnetic moment $\eta\boldsymbol{\epsilon}m$), of momentum $\mathbf{k} = \eta\mathbf{k}$, in the state σ, since the vacuum is defined by $N(\mathbf{k}, \sigma, \eta) = \frac{1}{2}(1-\eta)$.

This leads to two difficulties:

(i) The operator $\zeta' = \Pi[1-2N(\mathbf{k}, \sigma, \eta)]$ corresponding to ζ of §2 is not so easily defined because now an infinite number of states is occupied. In a diagonal representation ζ' is of the form $(-1)^{\infty}$, which gives rise to additional mathematical difficulties.

Table 3. Table of the Dichotomic Variables used in this Paper

x	1	−1	Physical significance	Letters used	Definition
M	Emission	Absorption	of the particle		§ 1
$u^{(x)}$	$u^{\dagger}$	u	Hermitian conjugation	K, KL = M	(7)
u_x	u_+	u_-	The two charge conjugate states (sign of the charge or of the magnetic moment)	L	(8)
a^x, ψ^x	a^*, ψ^*	a, ψ	Complex conjugation	K, KL = M	(7)
$D(x)$	$\frac{1}{2}\left(1+\frac{\boldsymbol{\alpha}.\mathbf{k}+\beta\kappa}{k_0}\right)$	$\frac{1}{2}\left(1+\frac{\boldsymbol{\alpha}.\mathbf{k}-\beta\kappa}{k_0}\right)$		M	(6)
$\mathfrak{D}(x)$	$\frac{1}{2}\left(1+\frac{\boldsymbol{\alpha}.\mathbf{k}+\beta\kappa}{\mathbf{k}_0}\right)$	$\frac{1}{2}\left(1+\frac{\boldsymbol{\alpha}.\mathbf{k}-\beta\kappa}{\mathbf{k}_0}\right)$	$\mathfrak{D}(x) = D(\eta x)$	K	(40)
η			Sign of the energy of the corpuscle		§ 9
$\boldsymbol{\epsilon}$			Sign of the charge (or the magnetic moment) of the corpuscle		§ 9
σ			The two states of spin		§ 1
$(-1)^P$	For even	For odd	permutation P		§ 5
$\epsilon_{ia}{}^2$			In $F_{ia} = \epsilon_{ia}\gamma_4\gamma_{[ia]}$ for $F_{ia}{}^{\dagger} = F_{ia}$		(15)
$\epsilon_{ia}{}'$	Space	Time	components of the invariant $\psi_S{}^K F_{ia} \psi_{S'}{}^{K'}$		(17)
θ_i	Symmetrical	Antisymmetrical	F_{ia}; $\theta_i = -1, +1, +1, -1, -1$		(18)

(ii) The ψ^K used in h cannot be the sum of all the plane waves describing the 'corpuscles' (they are infinite in number), but contain only $u^{(K)} a_{\mathbf{k}}{}^K \exp(-Ki\mathbf{k}.\mathbf{x})$ for the 'corpuscles' which will be emitted or absorbed. Then the exclusion principle is not automatically satisfied. If there are indistinguishable particles—they correspond to indistinguishable corpuscles—one must symmetrize h with respect to the corresponding ψ^K (which anti-commute). For Majorana particles this symmetrization for an expression like $\psi_S{}^K F_{ia} \psi_{S'}{}^{K'}$ is automatically carried out in the Heisenberg notation (Heisenberg 1934). One writes $\frac{1}{2}(\psi_S{}^K F_{ia} \psi_{S'}{}^{K'} - \theta_i \psi_{S'}{}^{K'} F_{ia} \psi_S{}^K)$ instead of $\psi_S{}^K F_{ia} \psi_{S'}{}^{K'}$.

Otherwise we form h as in §4 and get the same formula for $|H_T|^2$ and A (see (32) and (33)).

For the study of a transition given by M and L (with $\sum_{(S)c} \mathrm{M}_S \mathrm{L}_S = 0$) we have the two relations:

$$\mathrm{M}_S = \eta_S \mathrm{K}_S \qquad \ldots\ldots (38)$$

and

$$\mathrm{L}_S = \eta_S \epsilon_S. \qquad \ldots\ldots (39)$$

Here a new situation arises in that it is not clear how the ϵ_S (sign of the electric charge or the magnetic moment of the corpuscles) are determined. The question arises whether the ϵ_S have a physical meaning or are arbitrary; it will be shown that they are arbitrary.

Let us therefore choose the ϵ_S arbitrarily; for the study of the M, L reaction, (39) gives us the η_S and (38) the K_S. It is well known that the summations over the spin states are made by the projection operator defined by Casimir (1933):

$$\mathfrak{T}(x) = \tfrac{1}{2}\left(1 + \frac{\boldsymbol{\alpha}.\mathfrak{k} + x\beta\kappa}{\mathfrak{k}_0}\right) = \tfrac{1}{2}\left(1 + \frac{\boldsymbol{\alpha}.\mathbf{k} + x\eta\beta\kappa}{k_0}\right). \qquad \ldots\ldots (40)$$

One easily sees, in the Majorana representation, that

$$\mathfrak{D}^{\mathrm{K}}(x) = \mathfrak{D}(-\mathrm{K}x) = \tilde{\mathfrak{T}}(\mathrm{K}x) \qquad \ldots\ldots (41)$$

and from (9)

$$\mathfrak{D}(x) = D(\eta x). \qquad \ldots\ldots (42)$$

With the well-known properties $\mathfrak{D}^{\mathrm{K}}(\eta) a^{\mathrm{K}}(\mathfrak{k}, \sigma, \eta') = \delta_{\eta\eta'} a^{\mathrm{K}}(\mathfrak{k}, \sigma, \eta')$ one gets

$$\sum_\sigma a_\mu^{-\mathrm{K}}(\mathfrak{k}, \sigma, \eta) a_\nu^{\mathrm{K}}(\mathfrak{k}, \sigma, \eta) = [\tilde{\mathfrak{D}}^{\mathrm{K}}(+)]_{\mu\nu} = [D(\mathrm{K})]_{\mu\nu} \qquad \text{(cf. (41)).}$$

This formula is equivalent to (9). Therefore, by the same method, one gets, instead of (35),

$$\sum_{\sigma_S} A = \text{Trace } F_{ia}\mathfrak{D}(\mathrm{K}_1)F_{jb}\mathfrak{T}(-\mathrm{K}_2) \times \text{Trace } F_{ia}\mathfrak{D}(\mathrm{K}_3)F_{jb}\mathfrak{T}(-\mathrm{K}_4) \qquad \ldots\ldots (43)$$

and from (42) and (38) $\mathfrak{D}(\mathrm{K}) = D(\eta\mathrm{K}) = D(\mathrm{M})$; hence the formulae (35) and (43) are identical.

Thus $\sum_\sigma |H_{\mathrm{T}}|^2$ depends only on M_S, and is independent of L_S (sign of the electric charge or of the magnetic moment of the particles), of η_S (sign of the energy of the 'corpuscles') and of ϵ_S (sign of the electric charge or of the magnetic moment of the 'corpuscles')

Thus the ϵ_S are completely arbitrary* and have no physical meaning. That shows the artificial aspect of the hole theory, which, however, is more customary and, therefore, more intuitive.

Moreover, this formulation does not contain the conservation of the electric charge. From (38), (39) and (29)

$$\sum_{(S)c} \mathrm{M}_S \mathrm{L}_S = \sum_{(S)c} (\mathrm{K}_S \eta_S)(\eta_S \epsilon_S) = \sum_{(S)c} \mathrm{K}_S \epsilon_S = 0,$$

but it has been seen that the ϵ_S are arbitrary.

There are no rules like (31) for the choice of the K_S. The K_S are completely arbitrary (a transition between particles can be described by 16 different transitions between 'corpuscles'; but after their choice the ϵ_S are determined, by (38) and (39), and give rise to a charge interpretation of the representation.

* In Tiomno and Wheeler's (1949) paper on the interaction μ, ϵ, ν, ν, the authors have had to choose a sign of the charge of the μ and ϵ corpuscles. This is only due to the fact that they have limited their choice of the Hamiltonian to the kind $h = g_i J_i(-, -, -, -) = g_i(\psi^* F_i \psi) . (\psi^* F_i \psi)$, which is the simplest when one does not use the Majorana representation of Dirac's matrices (see p. 521).

II. THE DECAY OF THE μ-MESON

Some experimental results published in 1948 pointed to the decay of the μ-meson into more than two particles (Steinberger 1948, Hincks and Pontecorvo 1948). The similarity of this decay to a β-decay was immediately noticed (Klein 1948, Horowitz *et al.* 1948). The suggested scheme is $\mu^{\pm} \rightarrow \mu^0 + \epsilon^{\pm} + \nu$, where μ^0 is a neutral meson already known (Lattes, Occhialini and Powell 1947) from the π-meson decay: $\pi^{\pm} \rightarrow \mu^{\pm} + \lambda$, where λ is written for μ^0.

From the work of Part I it is easy to study this phenomenon.

§ 10. DECAY $\mu \rightarrow \lambda + \epsilon + \nu$

Here the Born approximation is used and the electromagnetic interactions are neglected (cf. Feer (1949) for photon production by charge acceleration). From the results of Part I one sees that it is not necessary to know whether μ^0 and ν are, independently, 'Dirac' or 'Majorana' neutral particles. The results are the same in any case.

Let us call the rest-masses, in energy units, of the corresponding particles μ, λ, ϵ, ν (with $\nu=0$). The frame of reference will be used in which the μ-meson is at rest. Let us denote the momenta and the energies (in energy units) of the particles by $\mathbf{p}_\mu=0$, $\mathbf{p}_\epsilon=\mathbf{p}$, $\mathbf{p}_\lambda$, $\mathbf{p}_\nu$; $E_\mu=\mu$, $E_\epsilon=E$, E_λ, $E_\nu=p_\nu$. The conservation law gives: $\mathbf{p}+\mathbf{p}_\lambda+\mathbf{p}_\nu=0$, $\mu=E+E_\lambda+E_\nu=E_t$ (total energy). Table 2 immediately gives $\sum_\sigma |H_T|^2$ for this decay; we know that $\mathrm{M}_\mu=-1$, $\mathrm{M}_\lambda=\mathrm{M}_\epsilon=\mathrm{M}_\nu=1$; the order chosen is λ, μ, ϵ, ν (corresponding to that usually chosen for β-decay: P, N, ϵ, ν). Then (33) gives* (the g_i have the dimensions of energy . volume)

$$
\begin{aligned}
V^2\sum_\sigma |H_T|^2 = {} & (g_1^2+4g_2^2+6g_3^2+4g_4^2+g_5^2)-(g_1^2-2g_3^2+g_5^2)\,.\,\frac{\mathbf{p}\cdot\mathbf{p}_\nu}{Ep_\nu} \\
& -2(g_2^2+2g_3^2+g_4^2)\left(\frac{\mathbf{p}}{E}+\frac{\mathbf{p}_\nu}{p_\nu}\right)\cdot\frac{\mathbf{p}_\lambda}{E_\lambda}+2(g_1g_3-2g_2g_4+g_3g_5)\left(\frac{\mathbf{p}}{E}-\frac{\mathbf{p}_\nu}{p_\nu}\right)\cdot\frac{\mathbf{p}_\lambda}{E_\lambda} \\
& +(g_1^2-2g_2^2+2g_4^2-g_5^2)\frac{\lambda}{E_\lambda}\left(1-\frac{\mathbf{p}\cdot\mathbf{p}_\nu}{Ep_\nu}\right) \\
& +2(-g_1g_2+3g_2g_3+3g_3g_4+g_4g_5)\frac{\lambda}{E_\lambda}\frac{\epsilon}{E} \\
& +2(-g_1g_2-3g_2g_3+3g_3g_4-g_4g_5)\frac{\epsilon}{E}\left(1-\frac{\mathbf{p}_\lambda\cdot\mathbf{p}_\nu}{E_\lambda p_\nu}\right). \qquad \ldots\ldots (44)
\end{aligned}
$$

We want to calculate the energy spectrum of the emitted electron. Let us denote the angle between $\mathbf{p}$ and $\mathbf{p}_\nu$ by θ and let $P(E)\,dE$ be the probability per unit of time of observing the electron from a disintegration with a total energy between E and $E+dE$. We have $\epsilon \leqslant E \leqslant W=(\mu^2+\epsilon^2-\lambda^2)/2\mu$,

$$P(E)\,dE=\frac{2\pi}{\hbar}\int_0^\pi \frac{V^2}{n}\sum_\sigma |H_T|^2\frac{p^2\,dp\sin\theta\,d\theta}{(2\pi\hbar c)^6}\,p_\nu^2\left(\frac{\partial p_\nu}{\partial E_t}\right)_{E_t=\mu}.\,8\pi^2;$$

$n=2$ (see (34)); $p\,dp=E\,dE$;

$$p_\nu=\frac{\mu(W-E)}{\mu-E+p\cos\theta};\quad \left(\frac{\partial p_\nu}{\partial E_t}\right)_{E_t=\mu}=\frac{(\mu-E)(\mu-E+p\cos\theta)-\mu(W-E)}{(\mu-E+p\cos\theta)^2}.$$

* This formula contains as special cases the expressions (8–12) of Tiomno and Wheeler (1949). One would therefore expect that our formulae (45) and (46) would include the corresponding formulae in that paper; unfortunately, a closer comparison reveals various discrepancies, which, however, has been cleared up in the course of a correspondence with Prof. Wheeler and Dr. Tiomno. It appears that many formulae as given in their excellent paper are marred by errors of transcription or by slight slips in calculations. They intend to publish a full list of these errata.

The calculation gives:

$$P(E)=\frac{(E^2-\epsilon^2)^{\frac{1}{2}}\mu^2(W-E)^2}{12h(2\pi h^2c^2)^3[\mu(W-E)+\frac{1}{2}\lambda^2]^3}$$

$$\times\left[\begin{array}{l}(g_1^2-g_2^2+6g_3^2+4g_4^2+g_5^2)E\{[3\mu(W-E)+E^2-\epsilon^2][2\mu(W-E)+3\lambda^2]-3\lambda^4\}\\ +(g_1^2-2g_3^2+g_5^2)(\mu-E)(E^2-\epsilon^2)[2\mu(W-E)+3\lambda^2]\\ +2(g_2^2+2g_3^2+g_4^2)\{E\mu(W-E)+(\mu-2E)(E^2-\epsilon^2)][2\mu(W-E)+3\lambda^2]\\ +4E\mu^2(W-E)^2\}+2(g_1g_3-2g_2g_4-g_3g_5)\{\mu[E(W-E)-(E^2-\epsilon^2)]\\ \times[2\mu(W-E)+3\lambda^2]+4E\mu^2(W-E)^2\}\\ +(g_1^2-2g_2^2+2g_4^2-g_5^2)6\lambda(\mu E-\epsilon^2)[\mu(W-E)+\frac{1}{2}\lambda^2]\\ +2(-g_1g_2+3g_2g_3+3g_3g_4+g_4g_5)6\lambda\epsilon(\mu-E)[\mu(W-E)+\frac{1}{2}\lambda^2]\\ -2(g_1g_2+3g_2g_3-3g_3g_4+g_4g_5)12\epsilon[\mu(W-E)+\frac{1}{2}\lambda^2]^2.\end{array}\right]$$

...... (45)

§11. DECAY $\mu^{\pm}\to\epsilon^{\pm}+2\nu$

During the last year the existence of the μ^0-meson has become doubtful, and some recent experimental data (Leighton, Anderson and Seriff 1949) show that $W=55$ Mev. i.e. about $\frac{1}{2}\mu$. Therefore the mass λ must be very small and the decay of the μ-meson into an electron and two neutrinos is very probable.

We must consider two cases:

(i) All neutrinos are Dirac neutral particles, and the two emitted neutrinos are distinguishable (their magnetic moments are of opposite sign). We can then obtain $\sum_\sigma|H_T|^2$ and $P(E)$ by putting $\lambda=0$ in (44) and (45).

We get a simpler formula by choosing the order μ, ϵ, ν, ν (of course now the g_i are not the same as in (45)):

$$P(E)=\frac{(E^2-\epsilon^2)^{\frac{1}{2}}}{3h(2\pi h^2c^2)^3}\mu[3E(W-E)K_1+2(E^2-\epsilon^2)K_2+3\epsilon(W-E)K_3]. \quad ......(46)$$

with $K_1=g_1^2+2(g_2^2+g_3^2+g_4^2)+g_5^2$, $K_2=g_2^2+2g_3^2+g_4^2$, $K_3=g_1^2-2g_2^2+2g_4^2-g_5^2$. In the preliminary note (Michel 1949) that formula was incorrect (put $K_4=0$ and correct K_3), but the errors do not change the spectral distribution by an appreciable amount.

(ii) Either all the neutrinos are Majorana neutral particles or all the neutrinos are Dirac neutral particles, and the two emitted neutrinos have the same charge conjugate state (their magnetic moments have the same sign); in both cases the two emitted neutrinos are indistinguishable.

From Part I (§8, second case) we know that then we obtain the same result as above except that there is no interaction in g_2 and g_3 (in the formula (46) put $g_2=g_3=0$).

§12 COMPARISON WITH EXPERIMENTAL RESULTS

The term proportional to K_3 does not appreciably change the curve $P(E)$, and it is neglected in the Figure.

If τ is the mean life of the μ-meson (2·15 microseconds)

$$\tau\int^{W}P(E)\,dE=1. \quad ......(47)$$

It is already well known (Tiomno and Wheeler 1949, Horowitz *et al.* 1948) that the g_i are of the order of the Fermi constant ($\sim 10^{-49}$ erg. cm³). The condition (47) gives us the scale for drawing the different curves which we want to compare.

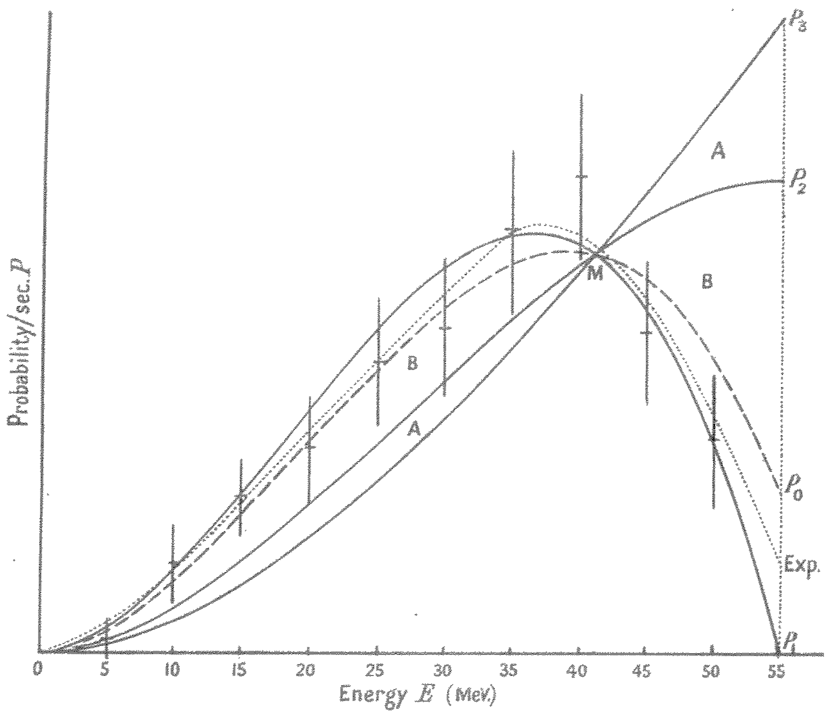
Let us denote the curve $P(E)$ by P_i when only one $g_i \neq 0$ ($\mathsf{P}_1 = \mathsf{P}_5$ and $\mathsf{P}_2 = \mathsf{P}_4$). All the possible curves P have (for $0 \leqslant \rho \leqslant 1$) the shape

case (i) $\mathsf{P}_1(E) + \rho[\mathsf{P}_3(E) - \mathsf{P}_1(E)]$ and sweep the whole areas A and B

case (ii) $\mathsf{P}_1(E) + \rho[\mathsf{P}_2(E) - \mathsf{P}_1(E)]$ and sweep only the whole area B.

All the curves pass through the same point M.

The experimental curve and the expected statistical spread given by Leighton, Anderson and Seriff (1949, Figure 5) are plotted together. The curve P_0 corresponding to the remarkable interaction g_0 (see (28)) is also shown.



——— Theoretical curves $\mathsf{P}_1 \equiv \mathsf{P}_5$, $\mathsf{P}_2 \equiv \mathsf{P}_4$, P_3, boundaries of the A and B areas.
- - - - Theoretical curve P_0, remarkable interaction.
...... Experimental curve of Anderson *et al.*

The agreement is quite satisfactory and the experimental curve also passes through the point M.

There is thus strong evidence for the decay of the μ-meson into one electron and two neutrinos.* But, since the experimental curve falls in the B area, nothing can be said about the nature of the neutrinos; furthermore, the good fit of the theoretical curves with experimental results does not prove that a 'direct interaction' *necessarily* exists between μ-mesons, electrons and neutrinos.

* Of course this conclusion rests on the assumption that the μ-meson has a spin of one-half. It would be possible, as pointed out by J. Tiomno (*Phys. Rev.*, 1949, **76**, 856), to reproduce the experimental curve on the assumption of an integral spin for the μ-meson and one of the neutral particles

§ 13. PROPERTIES OF THE NEUTRINO

(*a*) *Its magnetic moment.* From the theoretical point of view it is more exact to ask: is the neutrino a Dirac or a Majorana neutral particle ?

From the decay of the μ-meson it would have been possible to say that the neutrino is a Dirac particle if the experimental curve lay in the A area, but this is not the case. However, there is one phenomenon rather favourable to the hypothesis: the neutrino is a Majorana particle. This, as was shown by Furry (1939), is a double β-radioactivity without the emission of neutrinos. Recently experimental evidence for this phenomenon was published (Fireman 1949); it is the spontaneous decay

$$^{124}_{50}\mathrm{Sn} \rightarrow {}^{124}_{52}\mathrm{Te} + 2e^-.$$

No neutrinos are emitted, otherwise the lifetime would be 10^{10} times larger than the observed one.

Theoretically this is explained by the reactions: (*a*) $N_- \rightarrow P^+ + \epsilon^- + \nu_{III}$ and (*c*) $N_- + \nu_{III} \rightarrow P^+ + \epsilon^-$ in the following scheme:

$$\text{initial state} \rightarrow \begin{matrix}\text{intermediary} \\ \text{virtual state}\end{matrix} \rightarrow \text{final state}$$

$$N_{-1} + N_{-2} \rightarrow P_1^+ + \epsilon_1^- + \nu_{III} + N_{-2} \rightarrow P_1^+ + \epsilon_1^- + P_2^+ + \epsilon_2^-$$

but we have seen (§ 8) that the (*a*) and (*c*) reactions do not belong to the same 'class' unless the neutrino is a Majorana particle.

The simplest course is, therefore, to assume that the interaction Hamiltonian is reduced to terms corresponding to the single class embodying (*a*) and (*c*) when the neutrino is treated as a Majorana particle. However, one cannot exclude the possibility that the Hamiltonian would consist of two distinct $\vec{g}(\kappa)$ corresponding to the two classes involved in which neutrinos are Dirac particles. This possibility has been discussed by Touschek (1948); for ordinary β-decay this leads for each process to two competing transitions which differ only by the sign of the magnetic moment of the neutrino, of course without interference between them for such phenomena of first order in the $\vec{g}$. It may be observed, in particular, that if $\vec{g}(\kappa_1) = \vec{g}(\kappa_2)$ (the same $\vec{g}$ for both classes) this theory, which also maintains distinction between neutrinos and anti-neutrinos, is formally identical for all β-processes with that involving Majorana neutrinos.

Thus if the neutrino is a Majorana particle it has no electromagnetic interaction. The neutrino appears only in the spontaneous decay of elementary particles, and it is only required to preserve the conservation laws of energy, momentum and angular momentum. It therefore seems to be connected with the gravitational field; but perhaps not so simply as might be thought (Gamow and Teller 1937). The interaction between μ-meson, electron and neutrinos gives, in the second order, a very small interaction of infinite range between a μ-meson and an electron of the same charge. According to the calculation of Noma (1948), the static potential interaction is proportional to r^{-5}.

(*b*) *Its mass.* From the formulae (45) and (46) it appears that there is a large difference between the two cases $\lambda \neq 0$ and $\lambda = 0$. This is due to the fact that, in (45), $P(E)$ is proportional to $(W - E)^2$, but this factor, which disappears if $\lambda = 0$, would also have disappeared if $\nu \neq 0$ when $\lambda \neq 0$, and even if $\nu = \lambda \neq 0$, as in the well-known case of β-spectra the curve $\mathsf{P}(E, \nu) \rightarrow \mathsf{P}(E, 0)$ when $\nu \rightarrow 0$, although the same is not true for the derivative dP/dE. One cannot hope evidence of this kind will give an answer to the question: is the mass of the neutrino finite? Equation (48) gives a good example for the mass ϵ of the electron. The tangent of $\mathsf{P}(E)$ at the point $E=\epsilon$ is vertical when $\epsilon \neq 0$ and horizontal when $\epsilon = 0$, but $\mathsf{P}(E, \epsilon) \to \mathsf{P}(E, 0)$ continuously as $\epsilon \to 0$.

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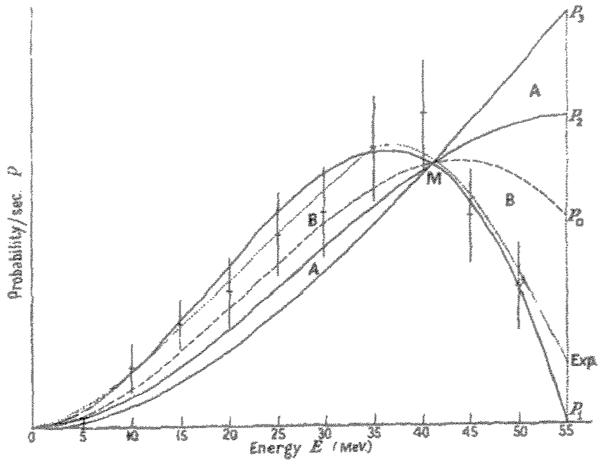
Corrigendum

"Interaction between Four Half-Spin Particles and the Decay of the μ-Meson", by L. MICHEL (*Proc. Phys. Soc.* A, 1950, **63**, 514).

By an unfortunate oversight the curve P_0 in the diagram on page 529 has not been drawn correctly to scale. Since this figure may be useful for a comparison of experimental data with theory a correct drawing is given here.



The following minor corrections to the article may also be pointed out:

Page 525, 6th line from bottom, *insert*: "For Majorana particles" before "This symmetrization";

Page 528, in formula (45), 2nd line, first factor between brackets, *replace* the term "g_2^2" by "$4g_2^2$".