MINIMA OF HIGGS - LANDAU POLYNOMIALS

L. Michel [*]

CERN - Geneva

A B S T R A C T

These (non-degenerate) fourth degree, bounded below polynomials, invariant by a compact or a crystallographic group G, are used to produce a spontaneous breaking of the symmetry G. They have extrema for each maximal little group which appears in the representation of G on the vector space E, but they may have their minimum anywhere when this representation is reducible. In the opposite case such a polynomial cannot have an extremum on the open dense subset of E whose points have a minimal (up to a conjugation in G) little group. Although not every maximal little group can be the little group G_m of a minimum of P, it is known for dim $E \leq 3$ that G_m is a maximal little group. It has been conjectured that this is true for any dimension if G is the effective invariance group of the polynomial ; the invariance group H provided by the physics might be only a subgroup of G. A counter-example, found by J.C. and P. Toledano, is given to the conjecture applied to H. Some sufficient conditions on the G invariants are also given for the truth of the conjecture.

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[*] Permanent address : IHES, F-91440 Bures sur Yvette.

INTRODUCTION

A physical problem with a symmetry group G may have a solution s with a smaller symmetry group $H = G_s$. Then, from the group invariance of the problem, one can build with the elements of G not in H a family of other solutions that we call an orbit of G. Of course the whole set S of solutions is invariant by G and can be decomposed into a disjoint union of G orbits. If the solution chosen by nature has a symmetry group strictly smaller than G, we speak of a broken symmetry state.

The solution s may depend on external parameters (such as the temperature T, the pressure $p,\ldots$) and the symmetry of s might change suddenly for a set of critical values T_o, $p_o,\ldots$ of the parameters. There is great interest to predict the nature of these symmetry changes without actually solving the full physical problem. This was first attempted by Landau [1] in 1937, in his theory of second order phase transitions. There are several other types of mechanisms [*] for describing spontaneous symmetry breaking ; the Landau model, which obtains the lower symmetry by minimizing a G invariant fourth degree polynomial is the simplest and it is quite successful. Exactly the same mechanism is used in non-Abelian gauge field theory for obtaining spontaneously broken symmetry states from a G invariant Lagrangian : this is a part of the Higgs mechanism.

For the last ten years I have been fascinated by spontaneous symmetry breakings. I am surely not the only one to have noticed that several mechanisms seem to prefer a principle of "minimal symmetry breaking" [Ref. 2b) p. 133 Th. 2 ; 2e) p. 45], i.e., the preserved symmetry group H is maximal among the possible candidate subgroups appearing in the mechanism ; the critical orbits I introduced with Radicati [3] have this property and many high energy physicists believe that the minima of Higgs polynomials are on critical orbits. Indeed this is often true, but it is time to make a precise review of what is proved and what is still a conjecture about the minima of Higgs-Landau polynomials. I found the colloquium in the honour of Tony Visconti to be an excellent opportunity for such a review. I present it as a small token of the admiration I have for Tony and my gratefulness for thirty years of friendship.

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[*] I had several occasions to review the mechanisms of spontaneous symmetry breaking : cf. Refs. 2a-f).

1. - GROUP ACTIONS

I first recall a few basic concepts concerning group actions. Every mathematical collective object M (a set, a manifold, a vector space,...) as well as every physical theory, has a group of automorphisms ; we denote it by Aut M. An action of the group G on M is given by a homomorphism :

$$G \xrightarrow{\ f\ } Aut\ M \qquad (1)$$

The action is effective if Ker f, the kernel of f, is trivial [i.e., any two different elements g and g' of G induce on M different automorphisms, f(g) and f(g')]. We will use generally g.m as a short for f(g)m, the transform of $m \in M$ by $g \in G$. The little group $G_m = \{g \in G, g.m = m\}$ is the subgroup of G which leaves m invariant. The orbit $G(m) = \{m' \in M,$ $\exists g \in G, m' = g.m\}$ is the set of the transforms of m by the group G. Points of the same orbit have conjugated little groups :

$$G_{g.m} = g\ G_m\ g^{-1} \qquad (2)$$

Conversely, points with conjugated little groups may not be on the same orbit, but, by definition, they are on the same stratum [*]) ; we denote by S(m) the stratum of m.

Given two G actions $G \xrightarrow{f} Aut\ M$, $G \xrightarrow{f'} Aut\ M'$, a map $M \xrightarrow{\phi} M'$ commuting with the two group actions is said to be equivariant when :

$$\forall g \in G \qquad f'(g) \circ \phi = \phi \circ f(g) \qquad (3)$$

If the equivariant map is an isomorphism, the two actions are equivalent $f' \approx f$ (this is the usual definition of equivalence for linear representations of G). Then one proves easily that the actions of G on two orbits are equivalent if and only if the two orbits have the same little groups ; hence equivalence classes of G orbits, the G orbit types, are classified by the conjugation classes of the subgroups of G and, therefore, a stratum

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[*]) Although it is not often expressed explicitly, the concept of stratum is much used by physicists : e.g., in the action of the Lorentz group on Minkowski space, the four strata are : the space-like vectors, the light-like vectors, the time-like vectors and the null vector.

is the union of all orbits of the same type in an action of G on M.
One often denotes by $[G:H]$ the type of G orbits which have H as
little group.

Mozrzymas and I have pointed out [4] [see also 5] that a weaker
type of equivalence is also useful in physics ; it applies even to different
groups acting on different objects : $G \xrightarrow{f} \text{Aut } M$ and $G' \xrightarrow{f'} \text{Aut } M'$; the
actions f and f' are equivalent in the weak form : $f \sim f'$, if there is
an isomorphism $M \xrightarrow{\phi} M'$ and also an isomorphism $\text{Aut } M \xrightarrow{\tilde{\phi}} \text{Aut } M'$ such that

$$\text{Im } f' = \tilde{\phi} \left(\text{Im } f \right)$$

(4)

["Im" is for Image ; (4) means that the two groups f(G) and f'(G') are
conjugated in $\text{Aut } M = \text{Aut } M'$]. Indeed two weakly equivalent representations
of the groups G and G' have a same set of possible Higgs-Landau polyno-
mials.

From very general principles of quantum physics one can already
obtain in a G invariant physical theory the possible symmetry groups (sub-
groups of G) of the equilibrium states [e.g., Ref. 6) where relevant previous
references are given]; applied to the Euclidean invariance this yields a na-
tural classification of the mesomorphic states of matter (crystals, liquid
crystals,...) and of their symmetry defects [7]. Here we are concerned only
in the case where the quotient group $\text{Im } f = G/\text{Ker } f$ is compact ; moreover
we assume that the action is smooth enough [*] so that the little groups are
closed (this is the case for a continuous linear representation). Since
finite groups are compact, our assumption includes the case where Im f is
a finite group.

There is a natural strict partial order relation on the set of
conjugated classes of closed subgroups of a compact group G ; if we denote
by (H) and (H') respectively the conjugation classes of the subgroups

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*) Another definition of the action of G on M (equivalent to the defini-
tion used in the paper when M is an abstract set) is given by a map
$G \times M \xrightarrow{\phi} M$ satisfying $\phi(1,m) = m$, $\phi(g'g,m) = \phi(g', \phi(g,m))$ in short
$\phi(g,m) = g \circ m$. This definition is well adapted when G and M have in
common a mathematical structure : e.g., M is a smooth (= indefinitely
differentiable) manifold and G is a Lie group ; the smooth map ϕ
(between manifolds) defines a smooth action of G on M.

H and H' of G, we say that (H) < (H') if there is a subgroup of (H) which is a strict subgroup of a given group of (H') [*]. This induces a partial ordering on the subset $\mathcal{K}$ of <u>conjugation classes of little groups</u> appearing in the smooth action $G \xrightarrow{f} \text{Aut } M$ with Im f compact. One then proves [9] that $\mathcal{K}$ has a minimal element and the corresponding stratum is open dense ; we call it the <u>generic stratum</u>. If there are no fixed points there might be several maximal elements of $\mathcal{K}$, i.e., several maximal little groups, defined up to a conjugation, in the group action.

With these concepts we can explain what is presently known on the extrema and the minima of Higgs-Landau polynomials.

2. - <u>HIGGS-LANDAU POLYNOMIALS</u>

<u>Definition</u> : Given a continuous linear representation $g \to \Delta(g)$ of a compact (or finite) group G on a real m dimensional vector space E with no invariant vectors $\neq 0$ (i.e. : $G_x = G \Rightarrow x = 0$) a Higgs-Landau polynomial is a fourth degree polynomial $P(x)$ defined on E, G invariant $(P(\Delta(g)x) = P(x))$, bounded below, and with its lowest value not at the origin.

So the lowest value of $P(x)$ is reached at least on one orbit of points that we shall call the absolute minima of P, and their little groups are strict subgroups of G. Hence if the solution of a G symmetric problem is given by the minimization of a Higgs-Landau polynomial, the nature of the symmetry breaking is given by the little groups of the absolute minima. Note that a polynomial must have at least degree four in order to produce such a symmetry breaking mechanism.

In quantum field theory the limitation to fourth degree Higgs polynomial is generally required by the renormalizability of the theory. The vector space E is the space of values of the multi-component spin

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[*] Then it is not possible that a subgroup of (H') be a strict subgroup of H in opposition to the case where G is not compact ; for instance, if G = Aff(n), the affine group in n dimensions and if (H) and (H') are two conjugate classes of crystallographic groups in n dimensions (so H and H' are not isomorphic) one can have two pairs of subgroups : H, $H_1 \in$ (H) and H', $H_1' \in$ (H') such that H < H' and $H_1' < H_1$. A systematic study is given for n = 2 in Ref. 8), Table 4.

zero Higgs field ; if this field is complex, one can consider separately
its real and its pure imaginary components : indeed, given a complex linear
representation $g \to \Delta'(g)$ of G on the complex vector space E', one can
consider the direct sum $\Delta' \oplus \overline{\Delta'}$ which is equivalent to a real representa-
tion on the real vector space E whose dimension is twice the (complex)
dimension of E'.

As we said these polynomials were introduced by Landau for pre-
dicting symmetry changes in second order phase transitions in crystals. A
crystallographic space group G' is an infinite discrete subgroup of the
Euclidean group $E(3)$ such that the orbit $[E(3):G']$ be compact ; so G'
is not compact. However, the physically interesting unitary representations
of G', for instance those which satisfy the Lifschitz criterion [1b], cor-
respond to wave vectors $\vec{k}$ on the surface of the Brillouin zone with a
higher symmetry so the image $\Delta(G')$ of the representation is finite[*].
The polynomial $P(x)$ is then obtained by an expansion of the free energy
thermodynamic potential, limited usually to the fourth degree, although
that some phase transitions require to continue the expansion further. We
shall call generalized Landau polynomials those which satisfy the definition
of Higgs-Landau polynomials except the limitation to fourth degree. In the
next section we will prove that when these polynomials have a maximum at the
origin, they have extrema with every maximal little group appearing in the
action of G on $E - \{0\}$. It will be more difficult to give general results
on the little groups of the minima of these polynomials outside the trivial
case in which $E - \{0\}$ has only one stratum[**] (i.e. : $\mathcal{K}$ has only one
element) ; in the following we exclude this case by assumption, so there is
the generic open dense stratum and one or several exceptional strata. After
giving, in Section 4, some results on the structure of G invariant poly-
nomials, we will prove, in Section 5, that for generalized Landau polynomials
and even for Landau polynomials the little group of an absolute minimum can
be any element of $\mathcal{K}$. On the contrary, when the representation of G on

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[*] A crystallographic space group G' has a translation group $T \simeq Z^3$ as
invariant subgroup and the quotient $P = G'/T$, the point group, is iso-
morphic to a finite subgroup of $O(3)$. The action of P on T defines
an action of P on the dual T^* (which is the Brillouin zone). The
elements $k \in T^*$ which are on "critical orbits" (these are defined later
in the paper) are among these higher symmetry wave vectors of special
physical interest.

[**] For instance, $G = C_n$ the cyclic group of $n > 2$ elements with a
faithful, real two-dimensional, representation.

E <u>is irreducible on the real, a Higgs-Landau polynomial cannot have an</u> <u>extremum in the generic stratum</u> : this excludes a "maximal symmetry breaking" for the representation.

I am still unable to complete the proof or find a counter example to my

<u>Conjecture</u> : <u>If the representation of the symmetry group</u> $\tilde{G}$ <u>of a</u> <u>Higgs-Landau polynomial</u> $P(x)$ <u>on</u> E <u>is irreducible (on the real),</u> <u>its minima have little groups maximal in</u> $\mathcal{K}$ (the set of conjugation classes of little groups on $E - \{0\}$).

Mozrzymas and I [4] proved it only for the dimensions $m \leq 3$. We also showed by examples that not every maximal little group in $\mathcal{K}$ can be the little group of a minimum of $P(x)$. Our proof will be recalled in Section 7. I hope that the material of this paper will help some reader to prove or disprove the conjecture. Meanwhile I think that this paper presents an efficient method for minimizing Higgs-Landau polynomials.

To conclude this section I have to explain a very important point. Of course, any mathematical theorem on Higgs-Landau polynomials can be valid only for the effective symmetry group $\tilde{G}$ of the polynomial $P(x)$. However, this group might be larger than the group $\text{Im } f$ given by physics. Indeed it will often happen that the polynomials on E whose exact symmetry group is $\text{Im } f$ must have a degree higher than four [*]. So the proof of the conjecture for the invariance group $\tilde{G}$ of $P(x)$ will not prove it for the physical group $\text{Im } f$, when the latter is a strict subgroup of $\tilde{G}$. As a matter of fact, the conjecture is false for the physical group $\text{Im } f$. J.C. and P. Toledano have given to me a counter example that I explain in Section 8.

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[*] The physical consequence of this fact has been well pointed out by Weinberg [10]. If there is a subgroup H of the little group of $\tilde{G}$ which is not a subgroup of $\text{Im } f$, the corresponding Higgs bosons have a small mass and are "pseudo-Goldstone bosons" so H is an approximate symmetry of the theory.

3. - ORBITS AND STRATA IN AN ORTHOGONAL REPRESENTATION OF A COMPACT (OR FINITE) GROUP

Let $g \to \Delta(g)$ be an orthogonal representation [*] of G on E, and (x,x) an invariant scalar product [**]. The orbits $G(\lambda x)$ for $\lambda \neq 0$ are all of the same type and have parallel tangent planes at the points λx. Let $\mathcal{M}(x) \subset \mathcal{S}(x)$ the vector subspaces parallel to all tangent planes, at x, to the orbits $G(\lambda x)$ and the strata $S(\lambda x)$, respectively. Let $\mathcal{N}(x)$ be the orthogonal subspace to $\mathcal{M}(x)$ it is the normal plane at λx of the orbit $G(\lambda x)$. For each $x \neq 0$ there is a neighbourhood V_x of the orbit $G(x)$ such that $\mathcal{N}(x) \cap V_x$ cuts $G(x)$; at x only [and also any other orbit $G(x') \subset V_x$ in one point only when x is in the generic stratum] [***]. The map $V_x \xrightarrow{r} G(x)$ which sends $\mathcal{N}(x) \cap V_x$ on x is an equivariant retraction : $r(\Delta(g)x') = \Delta(g)r(x')$ so $g \in G_{x'}$ implies $g \in G_{r(x')} = G_x$, i.e., for every $x' \in V_x$, $(G_{x'}) \leq (G_x)$. Then it is easy to deduce [***] that for all points $x' \in V_x \cap \mathcal{S}(x)^{\perp}$ we have $(G_{x'})$ strictly smaller than G_x ; incidentally this shows that the generic stratum is open. Since the gradient of a G invariant function is invariant by G_x at x, it must be in $\mathcal{S}(x)$, i.e., the gradient of a G invariant function is tangent at x to the stratum $S(x)$. So on a one-dimensional stratum the gradient df/dx of a G invariant function is collinear to x. Hence, a generalized Landau polynomial with a maximum at 0 has an extremum on each half-ray $(\{\lambda x\}, \lambda > 0)$ of a one-dimensional stratum [****].

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[*] This is not a restriction since every finite dimensional real linear representation of a compact group is equivalent to an orthogonal representation.

[**] This scalar product is unique up to a factor when the real representation of G is irreducible on the real.

[***] The results of this section are proved in Refs. 11a,b) for the smooth action of a compact group G on a finite dimensional manifold M. This is not more general than the situation of the paper ; according to a beautiful theorem of Mostow [12a], when the number of strata is finite, such an action can be embedded equivariantly into an orthogonal action. Mostow had also proved [12b] that when M is compact (it includes the case where M is the unit sphere of a finite dimensional vector space E carrying an orthogonal representation of G, as in this paper) the number of strata is finite. The subspace $\mathcal{N}(x)$ that we define is generally called the global "slice" at x and $\mathcal{N}(x) \cap V_x$ is the local slice.

[****] Let $p(\hat{x})$ be the restriction of the polynomial $P(x)$ to the unit sphere $(\hat{x}, \hat{x}) = 1$ (we denote by $\hat{x}$ a unit vector). The one-dimensional strata of E cut the unit sphere into isolated points ; these points belong to critical orbits (= orbits isolated in their stratum). Each such critical orbit is an orbit of extrema of any G invariant function on the unit sphere.

We have proved that in a neighbourhood of $E - \{0\}$ of each half-ray the conjugate class of little groups of points outside the half-ray are strictly smaller ; this means that the conjugation class of little groups of a one-dimensional stratum is maximal in $\mathcal{K}$. The converse may not be true : strata corresponding to maximal elements of $\mathcal{K}$ may be of dimension larger than one ; however, we will extend to them the preceding result.

We denote by ε^g the eigenspace with eigenvalue 1 of $\Delta(g)$ and we define for any subgroup $H < G$

$$E^H = \bigcap_{g \in H} E^g \quad , \quad c_H = \dim E^H \tag{5}$$

The dimension c_H of E^H is sometimes called the subduction degree of H for the representation of G on E. If $H < H' \leq G$ and $c_H = c_{H'}$, then H is not a little group of this representation. If

$$\text{all } H' > H \quad , \quad c_{H'} < c_H \tag{6}$$

it does not necessarily imply that H is a little group ; however, it is the case when the set of subgroups of G larger than H is finite (e.g., G finite) or enumerable [e.g., Ref. 2f), Appendix A] ; then c_H is the dimension of the stratum with little groups in the conjugation class (H).

Let H be a maximal little group on $E - \{0\}$; one then shows that [see footnote ***) on p. 7]

$$E^{(H)} = \bigcup_{H \in (H)} E^H = S(x) \bigcup \{0\} \quad \text{where} \quad H = G_x \tag{7}$$

is closed [it contains the stratum corresponding to (H) and the origin]. Let $\tilde{E}$ be the compactification of E obtained by adding the point at infinity *), that we shall denote by ω. Multiplying $P(x)$ by a convenient smooth function vanishing outside a small compact containing ω, we transform $P(x)$ into a smooth function $\hat{P}$ on $\tilde{E}$ with a maximum at ω and with the same extrema as $P(x)$ elsewhere. Since $\tilde{E}^{(H)}$ is closed, it is compact and the restriction $\hat{P}|_H$ of $\hat{P}$ to $\tilde{E}^{(H)}$ must have a minimum somewhere

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*) The topology of $\hat{E}$ is that of a sphere S_m ; one can realize this compactification by a stereographic projection ; 0 and ω are the two poles of S_m ; rays in E become meridians in S_m.

outside O and ω if P is maximum at O. Since the gradient of P is tangent to $E^{(H)}$ it vanishes at this minimum of $\hat{P}|_H$. Hence, <u>a generalized Higgs-Landau polynomial maximal at O has an extremum on every connected component of a stratum corresponding to a maximal little group in $\mathcal{K}$</u>.

In Appendix A, we give some relations between the orbit and strata structure for the orthogonal representations of compact and finite groups. Obviously, in the latter case $\mathcal{M}(x) = 0$ and $\mathcal{N}(x) = E$.

4. - G INVARIANT POLYNOMIALS ON E

The polynomials on E form an infinite dimensional vector space $\mathcal{G}$ which is also a ring and an algebra. The homogeneous polynomials of degree n are defined by

$$\theta_n(\lambda u) = \lambda^n \theta_n(u) \tag{8a}$$

they form a vector subspace $\mathcal{G}_n$ of dimension $\binom{n+m-1}{n}$. The differential of $\theta_n(u)$ is the linear form on E defined by

$$x \longmapsto \mathcal{D}_{xu}\theta_n(u) = \lim_{\lambda \to 0} \frac{1}{\lambda}\left(\theta_n(u+\lambda x) - \theta_n(u)\right) \tag{8b}$$

The "operators" $\mathcal{D}_{xu}$ commute $\mathcal{D}_{xu}\mathcal{D}_{yu}\theta_n = \mathcal{D}_{yu}\mathcal{D}_{xu}\theta_n$. One has also :

$$\mathcal{D}_{xu}\theta_n(u)\Big|_{x=u} = n\,\theta_n(u) \tag{8c}$$

By "polarization" [13] one transforms a homogeneous polynomial θ_n of degree n into a multilinear form completely symmetrical in n variables [see also Refs. 14), 15]

$$\tilde{\theta}_n(x_1, x_2, \ldots, x_n) = \frac{1}{n!}\mathcal{D}_{x_1 u}\mathcal{D}_{x_2 u}\cdots\mathcal{D}_{x_n u}\,\theta_n(u) \tag{9a}$$

For example :

$$\tilde{\theta}_2(x,y) = \frac{1}{2!} \left[\theta_2(x+y) - \theta_2(x) - \theta_2(y) \right] \tag{9b}$$

$$\tilde{\theta}_3(x,y,z) = \frac{1}{3!} \left[\theta_3(x+y+z) - \theta_3(y+z) - \theta_3(z+x) - \theta_3(x+y) - \theta_3(x) - \theta_3(y) - \theta_3(z) \right] \tag{9c}$$

Note that :

$$\tilde{\theta}_n(u,u,\dots,u) = \theta_n(u) \tag{9d}$$

The action of the orthogonal group $SO(m)$ on E leaves invariant the scalar product (x,x). It induces an action on any $\mathcal{P}_n$. We give it explicitly for $m = 2,3,4$. We denote by (j), $j > 0$, the two-dimensional, irreducible on the real, representations of $SO(2)$ and by (0) its trivial representation. We denote by $\{j\}$ the $2j+1$ dimensional representation of $SO(3)$; since $SO(4) \simeq (SU(2) \times SU(2))/Z_2$ its irreducible representations can be labelled (j,j') with $2j$, $2j'$, $j+j'$ non-negative integers. With these notations, the representations of $SO(m)$ on $\mathcal{P}_n$ are the direct sums :

	n even	n odd	
$SO(2)$ on $\mathcal{P}_n$	$\bigoplus_{k=0}^{\frac{n}{2}} (2k)$	$\bigoplus_{k=0}^{\frac{n-1}{2}} (2k+1)$	(10a)
$SO(3)$ on $\mathcal{P}_n$	$\bigoplus_{j=0}^{\frac{n}{2}} \{2j\}$	$\bigoplus_{j=0}^{\frac{n-1}{2}} \{2j+1\}$	(10b)
$SO(4)$ on $\mathcal{P}_n$	$\bigoplus_{j=0}^{\frac{n}{2}} \{j,j\}$	$\bigoplus_{j=0}^{\frac{n-1}{2}} \{j+\frac{1}{2}, j+\frac{1}{2}\}$	(10c)

Note that for $m > 3$ all the representations of $SO(m)$ do not appear in its action on $\mathcal{P}$. There is a finite number of strata in the action of $O(m)$ on $\mathcal{P}_n$. Here we are interested by those corresponding to little groups H such that the restriction to such a subgroup of the m dimensional representation of $O(m)$ be irreducible on the real. For $O(3)$ there are only seven such subgroups : T, T_h, T_d, O, O_h, Y, Y_h in Schönflies notation (T_d and O_h are, respectively, the symmetry group of a regular tetrahedron and of a cube). For $n = 3$ there is only a one-dimensional stratum of polynomials invariant by an irreducible subgroup of $O(3)$. In a convenient co-ordinate system this polynomial can be written :

$$\theta_3(x) = 3x_1 x_2 x_3 \; ; \; \text{invariance group } T_d \tag{11}$$

Similarly, for $n = 4$, there is, outside the $O(3)$ invariants $\lambda(x,x)^2$, only a two-dimensional stratum of polynomials invariant by an irreducible subgroup of $O(3)$; in the same co-ordinate system as in (11) a representative of these polynomials can be chosen as

$$\theta_4(x) = (x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2) = \frac{1}{2}\left[(x_1^2 + x_2^2 + x_3^2)^2 - (x_1^4 + x_2^4 + x_3^4)\right] \tag{12}$$

It is invariant by O_h which is obtained by adding to T_d the inversion -1 through the origin *).

Now we consider a third degree homogeneous polynomial invariant by a group H :

$$g \in H \qquad \theta_3(\Delta(g)x) = \theta_3(x) = \lambda^{-3}\theta_3(\lambda x) \tag{13}$$

If we fix x and y in the corresponding trilinear form $\widetilde{\theta}_3(x,y,z)$, we obtain a linear form in z ; it must be the scalar product of z with a vector of E which we denote $x_\vee y$ since it is bilinear in x and y. So

$$\widetilde{\theta}_3(x,y,z) = (x_\vee y, z) = (x, y_\vee z) = (y_\vee x, z) \tag{14}$$

(using the complete symmetry of the trilinear form). The correspondence :

$$x \otimes y \longmapsto x_\vee y = y_\vee x \quad , \quad E \otimes E \xrightarrow{\ \vee\ } E \tag{15}$$

defines a symmetric, in general non-associative, algebra on E which has H as an automorphism group :

$$g \in H \qquad \Delta(g)x_\vee \Delta(g)y = \Delta(g)(x_\vee y) \tag{16}$$

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*) A complete table of the invariant polynomials for $O(2)$ and all n is given in Ref. 16).

One can also define a linear operator D_x, linear in x, by

$$y \longmapsto x \vee y = D_x y \quad ; \quad E \xrightarrow{\ D\ } \mathcal{L}(E) \tag{17}$$

It is a symmetrical covariant operator :

$$D_x^T = D_x \quad , \quad g \in H \ , \ D_{\Delta(g)x} = \Delta(g) \, D_x \, \Delta(g)^{-1} \tag{18}$$

These operators have been introduced by Biedenharn [17] for the adjoint representation of $SU(n)$ and for $n = 3$ by Gell-Mann [18] ; for a systematic study of these operators and of the v algebras, see Refs. 2b) and 14). Remark that the trace of D_x is an H invariant linear form on E. If the orthogonal representation of H on E has no invariant vector, then

$$tr \, D_x = 0 \tag{19}$$

Similarly, from each degree n homogeneous polynomials Θ_n, we can build a multi-algebra $\overset{n-1}{\otimes} E \to E$ and introduce an operator-valued map $\overset{n-2}{\otimes} E \to \mathcal{L}(E)$. For instance, for $n = 4$, one defines :

$$E \otimes E \xrightarrow{\ T\ } \mathcal{L}(E) \quad , \quad T(x \otimes y) = T_{(x,y)} \tag{20a}$$

such that

$$\tilde{\Theta}_4(v, x, y, z) = (v, T_{x,y} \, z) = (v, T_{y,z} \, z) = (x, T_{y,z} \, v) \tag{20b}$$

this operator satisfies

$$T_{x,y}^T = T_{x,y} = T_{y,x} = \Delta(g)^{-1} \, T_{\Delta(g)x, \Delta(g)y} \, \Delta(g) \tag{21}$$

The simplest example of such an operator is given by :

$$(x,x)^2 = (x, T_{x,x} \, x) \quad \text{with} \quad T_{x,x} = \tfrac{1}{3}(x,x)(I + 2 P_x) \tag{22}$$

where P_x is the orthogonal projector on x.

Given a homogeneous H invariant third degree polynomial $(x_v x, x) = (x, D_x x)$, one can form a fourth degree polynomial

$$(x, D_x^2 x) = (x_v x, x_v x) = (x, T_{x,x} x) \; ; \; T_{x,y} = \frac{1}{3}(D_x D_y + D_y D_x + D_{x v y}) \qquad (23)$$

which is invariant by $H \times Z_2(-1)$. For example, for Θ_3 given in (11)

$$D_x = \frac{1}{2} \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix} \qquad (24)$$

and $(x, D_x^2 x)$ is the polynomial of (12). More generally, given H invariant polynomials $\Theta_3(x)$ and $\Theta_4(x)$ of degrees 3 and 4, one can form H invariant homogeneous polynomials $(x, D_x^q x)$ and $(x, T_{x,x}^q x)$ of degree $q+2$ and $2q+2$, respectively. Of course, there cannot be more than m which are algebraically independent so for $q \geq m'$ these polynomials are algebraic functions of those with q, $0 \leq q \leq m' \leq m$.

Given an orthogonal representation of G on the m dimensional vector space E, we denote by $\mathcal{J}^G$ the algebra of G invariant polynomials. References 19)-22) give information on the structure of $\mathcal{J}^G$, mainly when G is finite, but it can be extended to the case G compact. The Molien generating function $M(t)$ yields the dimension :

$$b_n = \dim \mathcal{J}_n^G \qquad , \qquad \mathcal{J}_n^G = \mathcal{J}^G \cap \mathcal{J}_n \qquad (25)$$

of the vector space $\mathcal{J}_n^G$ of degree n homogeneous G invariant polynomials on E. This function $M(t)$ can be computed by :

$$M(t) = \sum_{n=0}^{\infty} b_n t^n = \int_G \frac{d\mu(g)}{\det(I - t A(g))} \qquad (26)$$

[for finite groups $\int_G d\mu(g)$ is to be replaced by $|G|^{-1} \sum_{g \in G}$]. It is a rational fraction *) :

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*) The trouble is that $N(t)$ and $D(t)$ may not necessarily be relatively prime polynomials and it is not always easy to know the form of $M(t)$ corresponding to the structure of $\mathcal{J}^G$.

$$M(t) = \frac{1 + \sum_{\alpha=1}^{\mu-1} t^{s_\alpha}}{\prod_{i=1}^{m'} (1 - t^{d_i})} = \frac{N(t)}{D(t)} \qquad , \quad m' \le m \tag{27}$$

where $m' = m$ for finite groups. This fraction reflects the structure of $\mathcal{J}^G$: it is a μ dimensional free module on an m' variable polynomial ring, i.e., there exist among the homogeneous G invariant polynomials m' of them, θ_i of degree d_i, and $\mu-1$ of them, φ_α of degree δ_α such that every polynomial of $\mathcal{J}^G$ is of the form :

$$P(x) = Q_0(\theta_1(x), \ldots, \theta_m(x)) + \sum_{\alpha=1}^{\mu-1} \varphi_\alpha(x) \, Q_\alpha(\theta_1(x), \ldots, \theta_{m'}(x)) \tag{28}$$

Moreover the powers and the products of the φ_α's are either other φ_α's or polynomials in θ_i ; so for each φ_α there is a positive integer ν_α such that :

$$\varphi_\alpha^{\nu_\alpha} = \phi_\alpha(\theta_1(x), \ldots, \theta_m(x)) \tag{29}$$

A reflection through the hyperplane orthogonal to u is the operator $I - 2P_u$ on E, where P_u is the orthogonal projector on u. For groups G generated by reflections, $N(t) = 1$ in Eq. (27) and moreover when G is finite $(m' = m)$, if $\{x_i\}$ is an orthogonal basis of E

$$\det \frac{\partial \theta_i}{\partial x_j} = \kappa \prod_{k=1}^{r} \ell_k(x) \qquad , \quad r = \sum_{i=1}^{m} (d_i - 1) \tag{30}$$

where $l_k(x) = 0$ are the linear equations of the hyperplanes of the reflections of G, the generic stratum is given by the equation :

$$\det \frac{\partial \theta_i}{\partial x_j} \ne 0 \tag{31}$$

There are strata of any dimension between 1 and m ; the union of the strata of dimension d is the set of points in $\dot{E}$ belonging to $m-d$ reflection planes ; the corresponding little groups are generated by $m-d$ reflections. For any compact group G the generalization of (31) is the algebraic independence of the m' polynomials $\theta_i(x)$ on the generic stratum.

5. - MINIMA OF GENERALIZED LANDAU POLYNOMIALS

Equation (28) gives the form of the most general G invariant polynomial on E. Using (29) we find for its gradient :

$$\frac{dP}{dx} = \sum_{\iota=1}^{m'} F_{\iota}(x) \frac{d\theta_{\iota}(x)}{dx} \quad \text{with} \quad F_{\iota}(x) = \frac{\partial Q_{0}}{\partial \theta_{\iota}} + \sum_{\alpha=1}^{\mu-1} \left(\varphi_{\alpha} \frac{\partial Q_{\alpha}}{\partial \theta_{\iota}} + \frac{1}{\lambda_{\alpha}} \varphi_{\alpha}^{1-\gamma_{\alpha}} \frac{\partial \phi_{\alpha}}{\partial \theta_{\iota}} \right) \quad (32)$$

The condition $dP/dx = 0$ for an extremum at x requires either that the m' gradients $d\theta_i/dx$ are not linearly independent or that x is a root of the equations $F_i(x) = 0$. Remark that the algebraic independence of the θ_i on the generic stratum implies the linear independence of their gradients. Since the Q_α ($0 \le \alpha \le \mu-1$) are arbitrary, there are no conditions on the G invariant F_i. So any G orbit on E can appear as a solution of the equation $F_i(x) = 0$. For example :

1) $P(x)$ is a Higgs-Landau polynomial on the space E of a reducible G representation. So we decompose E into a direct sum of irreducible subspaces :

$$E = \bigoplus_{\alpha} E_{\alpha} \quad , \quad x = \bigoplus_{\alpha} x_{\alpha} \quad \quad x_{\alpha} \in E_{\alpha} \quad (33)$$

The partial scalar products (x_α, x_α) are G invariant ; we form $P(x)$ from them :

$$P(x) = \frac{1}{4} \sum_{\alpha, \beta} K_{\alpha\beta} (x_\alpha, x_\alpha)(x_\beta, x_\beta) - \frac{\mu^2}{2} \sum_{\gamma} \beta_\gamma (x_\gamma, x_\gamma) \quad K^T = K > 0 \; , \; \beta_\gamma \ge 0 \cdot (23)$$

The condition of positivity of the matrix K means that

$$\sum_{\alpha, \beta} \lambda_\alpha K_{\alpha\beta} \lambda_\beta > 0$$

for any set of real numbers λ_α. Then :

$$\frac{dP}{dx} = \sum_{\alpha\beta} \left(K_{\alpha\beta} (x_\alpha, x_\alpha) - \mu^2 \beta_\beta \right) x_\beta \quad (35)$$

$$\frac{d^2P}{dx^2} = \sum_{\alpha\beta} \left[(K_{\alpha\beta} (x_\alpha, x_\alpha) - \mu^2 \beta_\beta) I_\beta + 2 K_{\alpha\beta} x_\alpha)(x_\beta \right] \quad (36)$$

where $x_\alpha)(x_\beta$ is the dyadic operator defined by $x_\alpha)(x_\beta y = x_\alpha \Sigma_\beta (x_\beta, y_\beta)$.
Let K^{-1} be the inverse matrix of K. Choosing the length of the component vectors x_α such that $(x_\alpha, x_\alpha) = \mu^2 \Sigma_\beta K_{\alpha\beta}^{-1} \beta_\beta$ implies $dP/dx = 0$ and
$d^2P/dx^2 = 2\Sigma_{\alpha\beta} K_{\alpha\beta} x_\alpha)(x_\beta > 0$. Hence to form a Higgs-Landau polynomial with
a minimum at a given x we just choose $\beta_\alpha = (x_\alpha, x_\alpha)\mu^{-2}$ and $K = I$.

2) The representation of G on E is irreducible. Since the Hessian
d^2P/dx^2 of any G invariant polynomial commutes with G at the origin,
in that case it is a multiple of I, so P has either a maximum or a minimum at O. We remark that the G invariant smooth functions on E separate the orbits, i.e., there is a set of such functions so that their two
sets of values are different on two different orbits. It has been shown [23]
that any smooth function on E, invariant by the compact group G, is a
smooth function of G invariant polynomials ; hence those also separate
the orbits, and this is also true of the set of generators Θ_i, φ_α of the
ring $\mathcal{G}^G$. So there exists a G invariant polynomial $P(x)$ which vanishes
on a given orbit and is positive elsewhere. We build explicitly such a
polynomial when G is finite ; let the orbit be the set of points
$\{\xi_\alpha\}$, $1 \le \alpha \le k$. We define $B(x) = \Pi_{\alpha=1}^k (x-\xi_\alpha, x-\xi_\alpha)$; when $m > 2$, $B(x)$ is
minimum at the origin but $P(x) = B(x)(1/3\ B(x)^2 - bB(x) + 4/5\ b^2)$ with
$b = B(0)$ is maximum at O and takes its lowest value on the orbit $\{\xi_\alpha\}$.

6. - COMPUTATION OF THE MINIMA OF A HIGGS-LANDAU POLYNOMIAL

Since any linear combination of degree n homogeneous G invariant polynomials is again one of them, we can write the most general Higgs-
Landau polynomial, when the G representation is irreducible :

$$P(x) = \frac{1}{4}\left[\lambda (x,x)^2 + \alpha\, \omega(x)\right] + \frac{\beta\mu}{3}\,\sigma(x) - \frac{\mu^2}{2}(x,x) \tag{37a}$$

with

$$x \neq 0 \quad \lambda(x,x)^2 + \alpha\,\omega(x) > 0 \; ; \; \omega(\lambda x) = \lambda^4 \omega(x) \neq (x,x)^2, \; \sigma(\lambda x) = \lambda^3 \sigma(x) \tag{37b}$$

The gradient and the Hessian of P are [*]

$$\frac{dP}{dx} = \left[\alpha\, T_{x,x} + \beta\mu\, D_x + (\lambda(x,x) - \mu^2)\, I \right] x \tag{38}$$

$$\frac{d^2P}{dx^2} = 3\, \alpha\, T_{x,x} + 2\, \beta\mu\, D_x + [\lambda(x,x) - \mu^2]\, 1 + 2\, \lambda(x,x)\, P_x \tag{39}$$

where $T_{x,x}$, D_x and P_x have been defined in Section 4. We exclude the degenerate case $\alpha = \beta = 0$. Then $\omega(x)$ and/or $\sigma(x)$ are algebraically independent from (x,x) on the generic stratum, so their gradients $T_{x,x} x$, $D_x = x_v x$ and x are linearly independent. Hence, <u>when the representation</u> <u>of G on E is irreducible, a Higgs-Landau polynomial has no extrema on</u> <u>the (open dense) generic stratum.</u>

A very important case is the absence of third degree invariants ; this is required in Landau theory for the existence of a second order phase transition. Then Eqs. (37)-(39) can be simply written :

$$P(x) = \tfrac{1}{4} \omega(x) - \tfrac{\mu^2}{2}(x,x) \;,\; \frac{dP}{dx} = (T_{x,x} - \mu^2)\, x \;,\; \frac{d^2P}{dx^2} = 3\, T_{x,x} - \mu^2\, I \;,\; \omega(\hat{x}) > 0 \tag{40}$$

We recall that the invariance group of the physical theory might be only a subgroup of the invariance group $\tilde{G}$ of $\omega(x)$; of course, any stated mathematical result is relative to $\tilde{G}$ only. Consider the case where the G invariant polynomials Θ_i represented by the denominator of the Molien function $M(t)$, are given by :

$$q > 1 \;,\; \omega_{2q} = (x, T_{x,x}^{q-1}\, x) \;,\; \left(so\;\; \omega_2(x) = (x,x) \right) \tag{41}$$

Then $dP/dx = 0$ implies that $d\omega_{2q}/dx = 2q\mu^{2(q-1)}x$, so with Eq. (29) the gradient of all $\tilde{G}$ invariant polynomials are collinear to x at the extrema of $P(x)$; hence <u>the extrema of an even Higgs-Landau polynomial are on one</u> <u>dimensional strata, i.e., have maximal little groups, when the generators</u> <u>$\Theta_q(x)$ of $\mathcal{I}^{\tilde{G}}$ are of the form $(x, T_{x,x}^{q-1}x)$, $q \geq 1$.</u> This is a sufficient condition for the truth of our conjecture. A particular case of this sufficient condition is that the denominator $D(t)$ of the Molien function be :

---

[*] Since the Hessian at the origin, $(d^2P/dx^2)(0)$, is a symmetric operator commuting with G, it must be a multiple of the identity I ; indeed it is $-\mu^2 I$ and $P(0)$ is a maximum of P.

$(1-t^2)(1-t^4)$. A similar sufficient condition is $D(t) = (1-t^2)(1-t^3)$ as is the case for the adjoint representation of $SU(3)$ [applications in Ref. 3] and the representation $j = 2$ of $SO(3)$ [applications in Ref. 2e].

We conclude this section by some remarks on the operators $T_{x,y}$. For the action of G on E the definition (20a), (20b) tells that T is a tensor operator of the variance $\epsilon \otimes \epsilon$; it is reducible. One can define

$$\mathcal{E} \otimes \mathcal{E} = \mathcal{E}_\wedge \mathcal{E} \;\oplus\; \mathcal{E}_\vee \mathcal{E} \tag{42}$$

the direct sum of the antisymmetric and the symmetric parts of the tensor product. $E_\wedge E \subset \operatorname{Ker} T$ since $T_{x,y} = T_{y,x}$ and correspondingly $\operatorname{Im} T \subset \{$set of symmetric linear operators$\}$ $(T_{x,y}^T = T_{x,y})$. When the representation of G on E is irreducible, $E_\vee E$ has a unique invariant, the tensor $\Sigma_i \, e_i \otimes e_i$ where $\{e_i\}$ is an orthonormal basis of E. The G equivariance requires

$$T\left(\Sigma_i \, e_i \otimes e_i\right) = \Sigma_i \, T_{e_i, e_i} = t \, I \tag{43a}$$

and it is easy to compute that

$$\operatorname{tr} T_{x,y} = (x,y) \, t \tag{43b}$$

It would be worth studying systematically these tensor operators. For instance, by considering the polynomial $P(x)$ defined in (40), we have proved :

<u>Theorem</u> : If $(x, T_{x,x} x) > 0$ for $x \neq 0$, there exists an eigenvector y, $T_{y,y} y = \mu^2 y$ such that $3T - \mu^2 I \geq 0$.

In Appendix B, we give an example of application of this method for computing the minima of Higgs-Landau polynomials : we deal with the general case when $\dim E = 3$; it also corresponds to $G = SU(4)$ and E carries its adjoint representation, as we will show in Appendix A.

7. - APPLICATION OF THE MORSE THEORY

Up to now, all the results we have obtained apply to the _extrema_ at $x \neq 0$ of $P(x)$, a Higgs-Landau polynomial ; indeed it is not easy to distinguish between minima : $dP/dx = 0$ and $d^2P/dx^2 > 0$ and other types of extrema (saddle points) [*]. We consider here only the case in which the orthogonal representation of G on E is irreducible on the real ; so $P(x)$ is maximum (see footnote on p. 17) at the origin $x = 0$. This is also true of the degree 4 polynomial in λ, $P(\lambda x)$ for any $x \neq 0$; since such a polynomial has no other maximum this is also true of $P(x)$. When we compactify E, as in the end of Section 3, the smooth function $\hat{P}(x)$ has therefore exactly _two maxima_ on $\hat{E} = S_m$, at the two poles of the m dimensional stereographic sphere.

The Morse theory deals with the nature of the extrema of functions defined on a compact topological space. It is true that it deals only with Morse functions, i.e., functions whose extrema are not degenerate : $\det(d^2P/dx^2) \neq 0$; this is not the case for G invariant functions when the G orbit of an extremum has dimension > 0. But an extension of the theory to a G equivariant Morse theory has been made by Wasserman [24]. If we consider only the case Im f finite [**], then for an open dense domain of its coefficients a Landau polynomial is a Morse function.

We just recall here the application of the Morse theory made in Ref. 4). Indeed it proves our conjecture for $m \geq 3$. Let c_k be the number of extrema of (the Morse function) $\hat{P}(x)$ with k positive eigenvalues ; so c_0 is the number of minima and c_m is the number of maxima. We have shown at the beginning of this section that

$$c_m = 2 \tag{44}$$

----

[*] It is easy to show that a degree 3, G invariant polynomial which has not a minimum at $x = 0$, has no minima when there is no one-dimensional G invariant subspace of E (so $m > 1$). Indeed from (19) and from (39), with $\lambda = \alpha = 0$, one obtains for the trace of the Hessian at any extremum $-\mu^2 m < 0$, so the Hessian cannot be a positive operator.

[**] In Appendix A, one explains when the action of a compact Lie group can be translated in terms of the action of a finite group.

The Morse relations are for a m dimensional manifold

$$\sum_{k=0}^{\ell} (-1)^{\ell-k} (c_k - b_k) \quad \begin{array}{l} \geq 0 \quad \text{for} \quad 0 \leq \ell < m \\ = 0 \quad \text{for} \quad \ell = m \end{array} \tag{45a}$$

and for the sphere S_m

$$b_k = 0 \quad \text{except} \quad b_0 = b_m = 1 \tag{45b}$$

Let n be the number of points of the smallest G orbit ; from (45) and (44), we obtain :

$$c_0 \geq n \quad , \quad c_1 \geq n \quad , \quad c_{m-1} \geq n \tag{46}$$

Moreover the total number of extrema of $\hat{P}(x)$ satisfy

$$\sum_{k=0}^{m} c_k \leq 3^m + 1 \tag{47}$$

since $dP/dx = 0$ is a system of m equations of degree 3.

We refer to Ref. 4) for the cases $m = 1$ and $m = 2$; here we study the case $m = 3$. Then, as we have seen in Section 4, for an open dense subdomain of its coefficients, the general Higgs-Landau polynomial is invariant by $\tilde{G} = T_d$ the symmetry group of the regular tetrahedron ; it is generated by reflections so all the little groups have also this property. We summarize the relevant properties of this action in Table I. Then Eqs. (44) to (47) imply :

$$c_0 \geq 4 \quad , \quad c_1 \geq 4 \quad , \quad c_2 \geq 4 \quad , \quad c_3 = 2 \tag{48}$$

and they have five possible types of solutions :

$$
\begin{array}{lllllll}
c_0 = & 4 & , & 6 & , & 4+4 & , & 4 & , & 4+6 \\
c_1 = & 6 & , & 12 & , & 12 & , & 12 & , & 12 \\
c_2 = & 4 & , & 4+4 & , & 6 & , & 4+6 & , & 4
\end{array}
\tag{49}
$$

As we show in Appendix B, all these possible solutions do occur. We also explain there how these results can be applied to the adjoint representation of SU(4).

Through Eq. (49), Morse theory predicts, for $m = 3$, that minima of Higgs-Landau polynomials can occur only on orbits with maximal little groups $(C_{3v}, C_{2v}$; the strata are of dimension 1). This happens also to be true for saddle points of type ++-. There might be extrema on the two-dimensional stratum (C_s) ; they are all saddle points of type +--. Finally we verify our prediction (for any m) of the absence of extrema in the generic stratum.

For crystals, one considers mainly even Landau polynomials ; for $m = 3$, $\widetilde{G} = O_h$, the symmetry group of the cube. It is a group generated by reflections : besides the six reflections generating T_d, there are three reflections through the three symmetry planes parallel to two opposite faces of the cube. Relevant properties of this action are given in Table II.

The only solutions of Eqs. (44) to (47) are :

$$c_0 = 6 \text{ or } 8 \quad , \quad c_2 = 8 \text{ or } 6 \quad , \quad c_1 = 12 \tag{50}$$

The little groups of all extrema are maximal little groups as we predicted in Section 6 for even Higgs-Landau polynomials when all invariants are generated by those of Eq. (41). Remark, however, that some maximal little groups (here C_{2v}) cannot be those of minima.

8. - THE TOLEDANOS' COUNTER-EXAMPLE

In this section we consider the simplest case, most commonly used in solid state physics of even, degree four [*], Landau polynomial on E with a finite irreducible symmetry group G. Then Eqs. (40) describe completely the situation. Even so, there seems to be a great confusion in the

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[*] So we exclude explicitly the use of degree 6 Landau potentials which are, for instance, necessary when a tri-critical point appears, due to the change of sign of the coefficient of the term of degree 4. Remark that in this case, after the second order phase transition to the less ordered phase, going away from the origin of E, the influence of the degree 6 term must be felt and there may appear a shift of the minimum to a lesser symmetry stratum. This would be only an apparent violation of the conjecture since it does not apply to degree 6 polynomials.

relevant solid state literature. To avoid the computation of minima (most often made by a cumbersome method) criteria have been proposed for selecting a list of candidate subgroups $H < G$ onto which the symmetry can be broken. The least selective one is the criterion of Birman [25] ; indeed it is essentially our Eq. (6) and for finite groups it is therefore the selection of all little groups in the action of G on E. This first selection generally has to be made and Birman's method is usually the most efficient for doing it. The most restrictive and recent criterion in Ascher's [26] maximality principle : it is essentially our conjecture [*] but applied to the physical symmetry group G. As we pointed out, the effective symmetry group $\tilde{G}$ of the Landau polynomial might be strictly larger and our conjecture applies to $\tilde{G}$ only. J.C. and P. Toledano have a realistic counterexample to this maximality principle applied to G, which is not a counterexample to our conjecture applied to $\tilde{G}$. To propagandize this yet unpublished Toledanos' work we sketch it here. At the same time it provides an illustration for our computational techniques.

Let τ_i be the three Pauli matrices. The set

$$\Delta (C_{4v}) = \{ \pm I, \pm \tau_1, \pm \tau_3, \pm i \tau_2 \}$$ (51)

of eight matrices form a faithful representation (generated by reflections) of the group C_{4v}. The 4×4 orthogonal real matrices :

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \qquad A, B \in \Delta (C_{4v})$$ (52)

generate an irreducible (on the real) representation of the 128 element group :

$$\tilde{G} = (C_{4v} \times C_{4v}) \square Z_2$$ (53)

----

[*] In 25 little groups (or isotropy groups, stabilizers,... which are synonymous in the mathematical literature) are introduced implicitly by Eq. (6) with the restriction that the set of G subgroups larger than H are finite.

which is the semi-direct product defined by the action of the non-trivial element of Z_2 : it exchanges the two factors of the direct product $C_{4v} \times C_{4v}$. The Molien function is

$$\tilde{M}(t) = \frac{1+t^6}{(1-t^2)(1-t^4)^2(1-t^6)} \tag{54}$$

The most general degree 4, G invariant polynomial is a linear combination :

$$\omega = \sum_{i=1}^{3} \alpha_i \, \omega^{(i)} \tag{55a}$$

$$\omega^{(1)} = x_1^4 + x_2^4 + x_3^4 + x_4^4 \geq 0 \, , \, \omega^{(2)} = 2(x_1^2 x_2^2 + x_3^2 x_4^2) \geq 0 \, , \, \omega^{(3)} = 2(x_1^2 + x_2^2)(x_3^2 + x_4^2) \geq 0 \tag{55b}$$

The positivity condition on ω defines the domain [*] :

$$\omega(x) \geq 0 \iff \alpha_i \in \mathcal{D} = \{ \alpha_1 > 0 \, , \, \alpha_1 + \alpha_2 \geq 0 \, , \, \alpha_1 + \alpha_3 \geq 0 \, , \, \alpha_1 + \alpha_2 + 2\alpha_3 \geq 0 \} \tag{56}$$

Then

$$T_{x,x}^{(1)} = \begin{pmatrix} x_1^2 & 0 & 0 & 0 \\ 0 & x_2^2 & 0 & 0 \\ 0 & 0 & x_3^2 & 0 \\ 0 & 0 & 0 & x_4^2 \end{pmatrix} \, , \quad T_{x,x}^{(2)} = \frac{1}{3} \begin{pmatrix} x_2^2 & 2x_1 x_2 & 0 & 0 \\ 2x_1 x_2 & x_1^2 & 0 & 0 \\ 0 & 0 & x_4^2 & 2x_3 x_4 \\ 0 & 0 & 2x_3 x_4 & x_3^2 \end{pmatrix} \tag{57a}$$

$$T_{x,x}^{(3)} = \frac{1}{3} \begin{pmatrix} x_3^2 + x_4^2 & 0 & 2x_1 x_3 & 2x_1 x_4 \\ 0 & x_3^2 + x_4^2 & 2x_2 x_3 & 2x_2 x_4 \\ 2x_3 x_1 & 2x_3 x_2 & x_1^2 + x_2^2 & 0 \\ 2x_4 x_1 & 2x_4 x_2 & 0 & x_1^2 + x_2^2 \end{pmatrix} \tag{57b}$$

$$T_{x,x} = \sum_{i=1}^{3} \alpha_i \, T_{x,x}^{(i)} \tag{57c}$$

---

[*] Although it is not essential to our argument, we have not proved that for an open dense subdomain of (56) the effective group of invariance of ω is not strictly larger than $\tilde{G}$.

Remark that

$$\sum_{i=1}^{3} w^{(i)}(x) = (x,x)^2 \tag{58a}$$

so

$$\sum_{i=1}^{3} T^{(i)}_{i,x} = \tfrac{1}{3}(x,x)\left(I + 2 P_x\right) \tag{58b}$$

Looking for the eigenspaces with the eigenvalue 1 of the $\Delta(g)$'s, one finds the structure of $\widetilde{\mathcal{X}}$, the set of conjugation classes of little groups = the set of strata. $\widetilde{\mathcal{X}}$ has four maximal elements ; each one corresponds to a little group H_i of minima of $P(x)$ of (40). Table III specifies the little groups H_i, a typical vector of the corresponding orbit of minima and the corresponding sub-domain $\mathcal{D}_i$ of coefficients. Note that $\cup_i \mathcal{D}_i = \mathcal{D}$ and there are no other minima of $P(x)$.
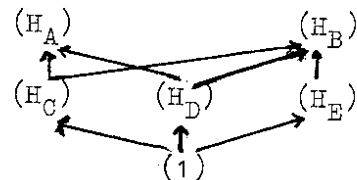
J.C. and P. Toledano have considered the unique real irreducible representation of the space group $I4_1 = C_4^6$ for the wave-vector $\vec{k}$ corresponding to the point N of the Brillouin zone : the little point group is trivial and the representation of the space group obtained by induction has a four-dimensional, 32 element image G which is generated by the matrices

$$c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad a = \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & i\tau_2 \\ I & 0 \end{pmatrix} \tag{59}$$

It is a subgroup of G defined in (52) and (53). Its Molien function is

$$M(t) = \frac{1 + 3t^6 + 3t^8 + t^{14}}{(1-t^2)(1-t^4)^2(1-t^8)} \tag{60}$$

so it has the same three-dimensional vector space of degree 4 invariant homogeneous polynomials as $\widetilde{G}$. The partially ordered set $\mathcal{X}$ of conjugation classes of little groups of G is given by this diagram. One has the isomorphism $Z_2 \times Z_2 \approx H_A \approx H_B$ $Z_2 \approx H_C \approx H_D \approx H_E$. The two groups of (H_C) are generated by c and $-c$; each other class contains four subgroups obtained by :

$$H_A = G \cap H_1 \quad , \quad H_B = G \cap H_2 \quad , \quad H_D = G \cap H_3 \quad , H_E = G \cap H_4 \tag{61}$$

where H_i, $i = 1,2,3,4$, is the conjugate class of little groups (subgroups of $\widetilde{G}$) of the orbit of minima in the domain $\mathcal{D}_i$; these groups are given in Table III. Since any G invariant Landau polynomial is invariant by $\widetilde{G}$, the subgroups of G, little groups of minima are the intersection by G of H_i. Only H_A and H_B (domain $\mathcal{D}_1$ and $\mathcal{D}_2$) are maximal little groups of G ; those of the classes H_D and H_E are not, as the preceding diagram shows. They correspond to the domains $\mathcal{D}_3$ and $\mathcal{D}_4$.

9. - CONCLUSION

Although the Higgs mechanism in gauge theories and the Landau theory of second order phase transitions apply to different domains of physics, they are based on the same mathematical formalism : minimization of a G invariant polynomial of degree 4, with G respectively a compact Lie group and a finite group. In Appendix A we study the relations between these two types of group actions and show that they can be strictly related in some cases. Obviously, any mathematical theorem on the nature of the symmetry breaking can be based only on the effective symmetry group $\widetilde{G}$ of the polynomial although the symmetry group G introduced by the physics might be strictly smaller. The conjecture that the little groups of minima are maximal among the little groups of the $\widetilde{G}$ action has only been partially proved while a counter-example is given to the usual similar conjecture with respect to G.

In high energy physics, any grand unification scheme requires a final symmetry breaking to a relatively small subgroup $[SU(3)_c \times U(1)]$. This can be accomplished only with several irreducible Higgs multiplets used successively or simultaneously.

APPENDIX A - LINEAR ACTION OF COMPACT LIE GROUPS

We recall in Section 2 that every finite dimensional representation of a compact Lie group is equivalent to a unitary representation and it can also be considered as a real, therefore orthogonal, representation $g \to \Delta(g)$ on the real vector space E. This defines a representation $a \to \Delta(g)$ of the (real) Lie algebra $\mathcal{G}$ of G. Indeed every $a \in \mathcal{G}$ defines a one-parameter subgroup $g(\theta)$ of G :

$$g(\theta) = e^{\theta a} \quad , \quad L(a) = \frac{d}{d\theta} \Delta(e^{\theta a}) \Big|_{\theta = 0} \tag{A.1}$$

It is easy to verify that :

$$L(\lambda a) = \lambda L(a) \quad , \quad L(a+b) = L(a) + L(b) \quad , \quad [L(a), L(b)] = L(a_\wedge b) \tag{A.2}$$

where $\wedge$ denotes the Lie algebra law. By differentiation of $\Delta(g)\Delta(g)^T = I$, we find :

$$L(a)^T = -L(a) \tag{A.3}$$

The Lie algebra $\mathcal{G}_x$ of the little group G_x, $x \in E$, is characterized by :

$$\mathcal{G}_x = \{ a \in \mathcal{G} , \ L(a)x = 0 \} \tag{A.4}$$

In Section 3, we have defined $\mathcal{M}(x)$ the vector subspace of E such that in the affine space of E :

$$x + \mathcal{M}(x) = T_x(G(x)) \tag{A.5}$$

is the tangent plane at x to the orbit $G(x)$; here $\mathcal{M}(x)$ is the set of vectors :

$$\mathcal{M}(x) = \{ L(a)x , \ \forall a \in \mathcal{G} \} \tag{A.6}$$

Then the global slice (defined in Section 3) is

$$\mathcal{N}(x) = \mathcal{M}(x)^{\perp} = \{ y \in E, \forall a \in \mathcal{G}, (y, L(a)x) = 0 \} \tag{A.7}$$

Note that (A.3) implies that $x \in \mathcal{N}(x)$.

From the Euclidean metric on E, one defines a metric on the orbit space (generally denoted by E/G) : the distance between the two orbits $G(x)$ and $G(y)$ is :

$$\hat{d}(G(x), G(y)) = \inf_{x' \in G(x), y' \in G(y)} d(x', y') \tag{A.8}$$

This is the minimum of the distance between all pairs of points, one on each orbit. Since the orbits are compact this minimum does exist ; moreover, if $d(x',y')$ is such a minimum, we obtain equivalent pairs of minima $\Delta(g)x'$, $\Delta(g)y'$ from the group action. The line carrying any such a pair is orthogonal to both orbits, hence it is in the intersection of the slices $\mathcal{N}(\Delta(g)x')$ and $\mathcal{N}(\Delta(g)y')$. By choosing g such that $x = \Delta(g)x'$, we have proved :

Lemma A.1 : Any global slice $\mathcal{N}(x)$ cuts every orbit of G on E.

Consider now a generic slice, i.e., the slice of a point x of the generic stratum (corresponding to the minimal little group). As we saw in Section 3, there is a neighbourhood V_x such that each point of $V_x \cap \mathcal{N}(x)$ has the same little group G_x and is on a different orbit. By linearization we deduce that G_x leaves fixed every point of $\mathcal{N}(x)$, i.e. :

$$\forall g \in G_x, \quad \Delta(g)\Big|_{\mathcal{N}(x)} = \mathbb{1}_{\mathcal{N}(x)} ; \tag{A.9}$$

the representation Δ of G restricted to G_x is trivial on $\mathcal{N}(x)$. Consider the subgroup $\hat{G}_x$ of G which transforms $\mathcal{N}(x)$ into itself ; $G_x \lhd \hat{G}_x$ ($\lhd$ reads "invariant subgroup") ; indeed for every $g \in \hat{G}_x$ and every $y \in \mathcal{N}(x)$, $\Delta(G_x)\Delta(g)y = \Delta(g)y = \Delta(g)\Delta(G_x)y$, i.e., $\Delta(G_x) = \Delta(g)\Delta(G_x)\Delta(g)^{-1}$. Hence the quotient group $W_x = \hat{G}_x/G_x$ acts effectively on $\mathcal{N}(x)$; since points of W_x orbits are isolated in $V_x \cap \mathcal{N}(x)$, W_x is a discrete group.

We want to study when the action of W_x on the generic slice $\mathcal{N}(x)$ has the same decomposition into orbits and strata that the action of G on E has, i.e., when the orbits and strata of W_x on $\mathcal{N}(x)$ are the intersection by $\mathcal{N}(x)$ of the orbits and strata of G on E. This will not be the case if the tangent plane to the orbit $G(y)$, $y \in \mathcal{N}(x)$, cuts along a line the slice $\mathcal{N}(x)$; this is the case if the next equation holds [see Eqs. (A.6) and (A.7)] :

$$\forall a \in \mathcal{G}, \quad (y, L(a)x) = 0 \quad, \quad \exists b \in \mathcal{G} \quad (L(b)y, L(a)x) = 0 \tag{A.10}$$

With the use of (A.2) we obtain the equivalent relations :

$$\forall a \in \mathcal{G}, \exists b \in \mathcal{G} \quad (y, L(a)x) = 0 \quad, \quad (y, L(a)L(b)x) = 0 \tag{A.11}$$

So the vector y of the m dimensional space E is a solution of a system of $d+d'$ linear equation where d and d' are respectively the dimension of the G orbits of the point x and $L(b)x$. There exists always non-trivial solutions when $d+d' < m$. Hence we have proved the lemma :

Lemma A.2 : The generic slice cuts at least one G orbit along a line
if the dimension d of the generic orbit satisfies
$2d < \dim E$.

This applies to the irreducible representations of $SO(3)$ with $\ell \geq 3$ since $d = \dim(SO(3)) = 3$ and $m = 2\ell+1$. Of course, the same phenomenon may occur with weaker dimension conditions. Let us give an example : E carries two copies of the same representation Δ of G. Note that $y \oplus x \in \mathcal{N}(x \oplus y)$; indeed :

$$\forall a \in \mathcal{G}, \quad (y \oplus x, L(a)x \oplus L(a)y) = (y, L(a)x) + (x, L(a)y) = 0 \tag{A.12}$$

as a consequence of (A.3). One can always find a $c \in \mathcal{G}$ such that $-L(c)^2$ has an eigenvector, with positive eigenvalue, in the generic stratum :

$$-L(c)^2 x = \gamma x \quad, \gamma > 0 \quad, \quad G_x \text{ minimal} \quad; \quad y = L(c)x \neq 0 \tag{A.13}$$

and we define y by (A.13). Then

so $(x \oplus y)$ cuts the G orbit of $y \oplus x$ along a curve tangent to the direction $L(c)y \oplus L(c)x$. For instance, if E carries twice the adjoint representation of G, $E = \mathcal{G} \oplus \mathcal{G}$, when G is semi-simple $\mathcal{G}_{x \oplus y} = 0$ so the dimension d of the generic orbit is $d = \dim G$ and in that case $2d = \dim E$. Such an example is realized by the action of $SO(3)$ on the non-relativistic phase space of three distinct particles in their rest system ; the phase space has the topology $S_5 \subset E_6 =$ twice the adjoint representation of $SO(3)$.

Now we look for sufficient conditions that the orbits of W_x be the intersection by $\mathcal{N}(x)$ of the G orbits. We shall give two of them :

a) $\mathcal{N}(x) = E^{G_x}$ the subspace of vectors invariant by G_x (A.14a)

b) $\hat{G}_x = N_G(G_x)$, the normalizer of G_x in G.   (A.14b)

We recall that $N_G(G_x)$ is the largest subgroup of G which has G_x as invariant subgroup. Note also that a) is a converse of (A.9).

Lemma A.3 : Conditions a) and b) are equivalent.

Take $g \in N_G(G_x)$, so $\Delta(G_x)\Delta(g) = \Delta(g)\Delta(G_x)$ applied to any $y \in \mathcal{N}(x)$ shows that $\Delta(N_g(G_x)) \mathcal{N}(x) \subset E^{G_x}$. From a) it shows that $N_G(G_x) < \hat{G}_x$ and the equality holds by the definition of the normalizer. Conversely, all points of a generic orbit with the same little group G_x form an orbit of the normalizer $N_G(G_x)$ and by topological completion, since the generic stratum is open dense, from $\Delta(N_G(G_x)) \mathcal{N}(x) = \mathcal{N}(x)$, one obtains $E^{G_x} = \mathcal{N}(x)$.

Theorem A : If a) or b) holds, for any x of the generic stratum, the orbits of W_x are the intersection by $\mathcal{N}(x)$ of the G orbits.

That for any $y \in \mathcal{N}(x)$, $W_x(y) \subset \mathcal{N}(x) \cap G(y)$ is evident. Consider first y in the generic stratum ; then $G_y = G_x$ so $G(y) \cap \mathcal{N}(x)$ is an orbit of the normalizer $N_G(G_x)$ and by b) it is an orbit of W_x. By topological completion this is also true for the $y \in \mathcal{N}(x)$ not in the generic stratum.

This "good situation" occurs for the representations $\ell = 1$, $\ell = 2$ of $SO(3)$ and for the adjoint representations of the semi-simple compact Lie groups. Then

$$L(a)x = a_\wedge x \qquad (A.15)$$

so $\mathcal{N}(x) = \mathcal{G}_x$. The generic global slice is called "Cartan sub-algebra" ; it is an Abelian Lie algebra ; its dimension is called the rank of G. The corresponding group W_x is called the Weyl group of G ; it is generated by reflections on $\mathcal{G}_x = \mathcal{N}(x)$. Remark that

$$(\Delta(g)x, \, L(\alpha) \, L(\beta) \, \Delta(g)x) = (\Delta(g)x, \, L(\beta) \, L(\alpha) \, \Delta(g)x)$$

and $(\Delta(g)x, a_\wedge b)$ define respectively a non-degenerate Riemannian metric and a symplectic structure on any orbit of the adjoint representation of the semi-simple compact Lie group G. So every G orbit has even dimension.

Polynomials on $\mathcal{N}(x)$ can be considered as polynomials on E, which are constant on $\mathcal{M}(x) = \mathcal{N}(x)^\perp$. So in this "good situation" the G invariant polynomials on E are given by the W_x invariant polynomials on $\mathcal{N}(x)$: the study of Higgs polynomials is then completely equivalent to the study of Landau polynomials.

To end this Appendix, we consider the case of the adjoint representation of $SU(n)$. The vector space of the Lie algebra can be realized by the n×n Hermitian traceless matrices with the Euclidean scalar product :

$$x^* = x \quad , \; \text{tr}\,x = 0 \, , \quad (x,x) = \tfrac{1}{2}\,\text{tr}\,x^2 \qquad (A.16)$$

The Lie algebra and the $\vee$ algebra laws are $\big[$see Ref. 14) for details$\big]$:

$$x_\wedge y = -\tfrac{i}{2}(xy - yx) = y_\wedge x \quad , \; x_\vee y = \tfrac{\sqrt{n}}{2}(xy + yx) - I\,\tfrac{2}{\sqrt{n}}\,(x,x) \qquad (A.17)$$

The $SU(n)$ invariant polynomials are polynomials in $(x, D_x^{q-2}x)$ $2 \le q \le n$. A Cartan subalgebra is realized by the diagonal matrices so its dimension is $r = n-1$. The Weyl group is $\mathcal{S}_n$, the group of permutations of the n

eigenvalues of these diagonal matrices satisfying (A.16). The different strata on E are defined by equality relations among the eigenvalues of the matrices ; the generic stratum is made of the matrices with all eigenvalues distinct.

For any direction of an idempotent of the algebra :

$$(u,u)=1, \quad u \vee u = \eta u \tag{A.18}$$

there is a vector ξu extremal for any Higgs polynomial. Equation (A.18) requires that u has only two distinct eigenvalues ; we denote by p and q their multiplicities : $p+q=n$ and by I_p and I_p the orthogonal projectors on the corresponding eigenspace. Then solutions of (A.18) are :

$$u_{pq} = \sqrt{\frac{2q}{pn}} \, I_p - \sqrt{\frac{2p}{qn}} \, I_q = -u_{qp}$$

The spectrum of $D_{u_{pq}}$ is easily obtained from (A.17)

eigenvalues :	$\eta = \sqrt{\frac{2}{pq}}(q-p)$	$\sqrt{\frac{2q}{p}}$	$-\sqrt{\frac{2p}{q}}$	$\frac{q-p}{\sqrt{2pq}}$
multiplicity :	1	p^2-1	q^2-1	$2pq$
eigenvectors :	u_{pq}	$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$	$\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}$

(a and c are Hermitean traceless $p \times p$ and $q \times q$ matrices, respectively ; b is an arbitrary $p \times q$ matrix).

In Appendix B, we treat in detail the case $SU(4)$. Indeed the rank is 3 and the action of the Weyl group $\mathcal{S}_4$ is equivalent to that of the group T_d on the three-dimensional Euclidean space. Afterwards we sketch the generalization to any $SU(n)$. We note in this case that the extrema correspond to matrices x with 2 or 3 distinct eigenvalues since they must satisfy a third degree equation. This excludes many strata. Indeed the little groups of the adjoint representation of $SU(n)$ are of the form $S(U(p_1) \times U(p_2) \times ... \times U(p_k))$ where n is the sum of the positive integers p_i, $p_1 \geq p_2 \geq ... \geq p_k$; when $p_k = 1$ the little group is simply $U(p_1) \times ... \times U(p_{k-1})$. The minimum little group, that of the generic stratum is given by $p_i = 1$. So, when $n > 4$, not only the generic stratum, but many others do not carry extrema. I have also proved that only the maximal little groups (those little groups with $p_1 + p_2 = n$) are those of minima.

APPENDIX B - HIGGS-LANDAU POLYNOMIALS IN THREE VARIABLES

For an irreducible group action, we have shown that any Higgs-Landau polynomial in three dimensions is of the form of Eq. (37) where $(x,x) = \Theta_2(x) = \sum_{i=1}^{3} x_i^2$, $\sigma(x) = \Theta_3(x)$ of Eq. (11), $\omega(x) = \Theta_4(x)$ of Eq. (12). This is also true, as we have shown in Appendix A, for the $SU(4)$ invariant polynomials on the 15 dimensional adjoint representation ; they depend only on the three variables of a Cartan subalgebra.

Since for some vectors x, $\omega(x) = 0$, we must have $\lambda > 0$. By a rescaling :

$$\frac{\alpha}{\lambda} \to \alpha \;,\; \frac{\beta}{\sqrt{\lambda}} \to \beta \;,\; \frac{\mu}{\sqrt{\lambda}} \to \mu > 0 \tag{B.1}$$

the polynomial becomes

$$P(x) \simeq \frac{1}{4}(x,x)^2 + \frac{\alpha}{4}(x \vee x, x \vee x) + \frac{1}{3}\beta\mu(x \vee x, x) - \frac{1}{2}\mu^2(x,x) \tag{B.2}$$

The parameter μ has the dimension of x and gives the scale of the phenomenon (e.g., the unit mass for the mass breaking). When $x \neq 0$, the degree four term is always positive if and only if

$$\mathcal{D} : \quad \alpha > -3 \tag{B.3}$$

Using (B.2), Eqs. (38) and (39) become :

$$\frac{dP}{dx} = \alpha\, x \vee x \vee x + \beta\mu\, x \vee x + [(x,x) - \mu^2]x \tag{B.4a}$$

$$= [\frac{\alpha}{3}(2 D_x^2 + D_{x \vee x}) + \beta\mu D_x + ((x,x) - \mu^2)I]x \tag{B.4b}$$

$$\frac{d^2P}{dx^2} = 2\alpha D_x^2 + \alpha D_{x \vee x} + 2\beta\mu D_x + ((x,x) - \mu^2)I + 2(x,x)P_x \tag{B.5}$$

From (18), $\mathrm{tr}D_x^2$ is an invariant quadratic form in x, so it must be proportional to (x,x) ; a direct computation from (24) gives :

$$t_\nu \, D_x^2 = \tfrac{1}{2} (x,x) \tag{B.6}$$

Using (19) we find :

$$tr \frac{d^2 P}{d x^2} = (x,x)(5+\alpha) - 3\mu^2 \tag{B.7}$$

If a is an idempotent unit vector of the ∨ algebra :

$$(a,a) = a \qquad a \vee a = \eta a \tag{B.8}$$

the vector ξa is an extremum of P if it satisfies $\dfrac{dP}{dx}(\xi a) = 0$:

$$(1 + \alpha \eta^2) \, \xi^2 + \xi \beta \mu \eta - \mu^2 = 0 \tag{B.9}$$

and the nature of this extremum is given by the sign of the eigenvalues of :

$$\frac{d^2 P}{dx^2}(\xi a) = 2\alpha \xi^2 D_a^2 + (\alpha \eta \xi^2 + 2\beta \mu \xi) D_a + (\xi^2 - \mu^2) I + 2\xi^2 P_x \tag{B.10}$$

We remark that a is an eigenvector with the eigenvalue :

$$3(1 + \alpha \eta^2) \, \xi^2 + 2\xi \beta \mu \eta - \mu^2 = (1 + \alpha \eta^2) \xi^2 + \mu^2 \tag{B.11}$$

This eigenvalue is indeed positive since the polynomial in ξ, $P(\xi a)$ has a minimum for $\xi = \xi_0$ solution of (B.9) [so, from (B.3) $\eta^2 \le 1/3$].

For an open dense domain of $\mathcal{D}$, i.e., for $\beta \ne 0$, the symmetry group of this polynomial is T_d. This is the symmetry group of a regular tetrahedron ; it is generated by six reflections through the symmetry planes of equation $\ell_\alpha = 0$; each reflection plane contains an edge of the tetrahedron and cuts the opposite edge in its middle. Note that Eq. (30) is in this case :

$$\frac{1}{4!} \det \frac{\partial \theta_{i-1}}{\partial x_r} = \prod_{\alpha=1}^{6} \ell_\alpha = (x_1^2 - x_2^2)(x_2^2 - x_3^2)(x_3^2 - x_1^2) \tag{B.12}$$

If we normalize the tetrahedron such that it is inscribed in the unit sphere, its vertices are given by

$$S^{(0)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \quad S^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} , \quad S^{(2)} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} , \quad S^{(3)} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \tag{B.13}$$

The centres of the faces are at $-s^{(q)}/3$ and the middles of the edges are at $\pm q^{(k)}/2\sqrt{3}$, with

$$q^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad q^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad q^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{B.14}$$

The one-dimensional strata are at the intersections of the symmetry planes ; at the intersection of :

three of them for the stratum $\{\lambda s^{(\alpha)}\}$, little group C_{3v}, four element orbits,
  two of them for the stratum $\{\lambda q^{(h)}\}$, little group C_{2v}, six element orbits.

From the covariance (16) of the $\vee$ algebra, these vectors satisfy Eq. (B.8) with the value of η respectively $1/\sqrt{3}$ and 0 :

$$s \vee s = \frac{1}{\sqrt{3}} s \quad , \quad q \vee q = 0 \tag{B.15}$$

When $a = q$, from (B.10) and (B.15) $\xi^2 = \mu^2$, i.e. :

$$\text{"the six } \pm \mu q^{(k)} \text{ form an orbit of extrema"} \tag{B.16}$$

The corresponding value of the polynomial P is :

$$P(\mu q) = -\frac{\mu^4}{4} \tag{B.17}$$

A direct computation yields for the eigenvectors and the eigenvalues of the Hessian :

$$\frac{1}{\mu^2} \frac{d^2 P}{dx^2}(\mu q) \quad \begin{matrix} \text{eigen values} \\ \text{eigen vectors} \end{matrix} \quad \begin{matrix} 2 & \frac{\alpha}{2}+\beta & \frac{\alpha}{2}-\beta \\ q & r^{(+)} & r^{(-)} \end{matrix} \tag{B.18}$$

with

$$r^{(\pm)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm 1 \end{pmatrix} \tag{B.19a}$$

So the $q^{(k)}$ are minima when

$$\alpha > 0 \quad , \quad \alpha^2 - 4\beta^2 > 0 \tag{B.19b}$$

Note that $r^{(\pm)}$ are unit vectors normal to the symmetry planes containing q ; they are respectively the only (up to a factor) odd vectors for the corresponding plane reflection, so they are eigenvectors of D_q and therefore of the Hessian $(d^2 P/dx^2)(\mu q)$.

Similarly, if we denote by $w^{(k)}$, $k = 1,2,3$, unit normal vectors to the three symmetry planes containing $s = s^{(0)}$, we must have $s_v w^{(k)} = \chi w^{(k)}$ with χ the same for all three vectors since the little group C_{3v} of s exchanges them ; indeed a direct computation yields $\chi = -1/2\sqrt{3}$. By linearization of the $w^{(k)}$ one obtains

$$(w, s) = 0 \quad \Rightarrow \quad s_v w = -\frac{1}{2\sqrt{3}} w \tag{B.20}$$

So the eigenvectors of the Hessian $(d^2 P/dx^2)(\xi s)$ are s and any orthogonal w ; we can deduce then the Hessian eigenvalues from (B.7) and (B.11).

$$\text{Spectrum } \frac{d^2 P}{dx^2}(\xi s) = \left\{ \xi^2(1+\frac{\alpha}{3}) + \mu^2 , \text{ twice } \xi^2(2+\frac{\alpha}{3}) - 2\mu^2 \right\} \tag{B.21}$$

The values of ξ at the extrema are the solutions of (B.9) with $\eta^2 = \frac{1}{3}$:

$$\xi_\epsilon = \frac{\mu}{\sqrt{1+\frac{\alpha}{3}}} \sigma_\epsilon \quad , \quad \sigma_\epsilon = \rho + \epsilon\sqrt{1+\rho^2} , \rho = \frac{-\beta}{2\sqrt{3}+\alpha} \quad , \quad \epsilon = \pm 1 \tag{B.22}$$

Note that

$$\rho_\varepsilon = \frac{\sigma_\varepsilon^2 - 1}{2\sigma_\varepsilon} \quad , \quad \sigma_+^2 - \sigma_-^2 = 4\rho\sqrt{1+\rho^2} \qquad \text{sign } \sigma_\varepsilon = \varepsilon \tag{B.23}$$

At the extrema the Hessian spectrum becomes

$$\text{Spectrum} \frac{1}{\mu^2}\frac{d^2P}{dx^2}(\xi_\varepsilon s) = \left\{ \sigma_\varepsilon^2, \quad \text{twice } \frac{6+\alpha}{3+\alpha}\sigma_\varepsilon^2 - 2 \right\} \tag{B.24a}$$

This Hessian is a positive operator when

$$\varepsilon = 1 \quad \beta < \frac{-\alpha}{\sqrt{2(6+\alpha)}} \quad , \quad \varepsilon = -1 \quad \beta > \frac{\alpha}{\sqrt{2(6+\alpha)}} \tag{B.24b}$$

The value of the polynomial at these minima is [*]

$$P(\xi_\varepsilon s) = -\frac{\mu^4}{4}\frac{\sigma_\varepsilon^2(\sigma_\varepsilon^2+2)}{3+\alpha} \tag{B.25}$$

In the Figure we have drawn (broken lines) the curves of (B.19b) and (B.24b)

$$\beta^2 = \frac{\alpha^2}{4} \quad , \quad \beta^2 = \frac{\alpha^2}{2(6+\alpha)} \tag{B.26}$$

They divide the plane α,β in eight regions, A,B,C,D,E,D',C',B', each one corresponding to a different set of extrema ; their corresponding Morse indices are given in Table IV. The Morse theory implies the existence of twelve other extrema of type ++- in B,C,D,B',C',D'. We have proved generally that these extrema are not in the generic stratum ; so we can look for them in the symmetry planes, for instance that spanned by q and $r^{(+)}$. The unit vectors of this plane are :

$$u = q\cos\varphi + r^{(+)}\sin\varphi \tag{B.27}$$

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[*] There is a misprint in Eq. (A.21) of Ref. 15). The expression of V_{min} should be divided by four, as in Eq. (A.24) which is obtained from Eq. (A.32) for the particular case $\rho = 0$.

The general relation

$$\forall x \qquad D_x^2 - D_{x \vee x} = \tfrac{1}{4}\,(n,x)\,(\,I - P_x\,) \tag{B.28}$$

simplifies the computation of the Hessian in the basis $q, r^{(+)}, r^{(-)}$

$$\frac{d^2 P}{dx^2}(\xi u) = \begin{pmatrix} \xi^2(3 + (\tfrac{\alpha}{2}-2)\sin^2\varphi) - \mu^2 & \sin\varphi\,(\xi^2(\varepsilon+\alpha)\cos\varphi + \beta\mu\xi) & 0 \\ \sin\varphi(\xi^2(\varepsilon+\alpha)\cos\varphi + \beta\mu\xi) & \xi^2(1+\tfrac{\alpha}{2}+(2+\tfrac{\alpha}{4})\sin^2\varphi)+\beta\mu\xi\sin^2\varphi - \mu^2 & 0 \\ 0 & 0 & \xi^2(1+\tfrac{\alpha}{2}-\tfrac{3\alpha}{4}\sin^2\varphi)-\beta\mu\xi\cos\varphi - \mu^2 \end{pmatrix} \tag{B.29}$$

These new extrema are obtained when

$$\alpha^2 - 4\beta^2 > 0 \quad, \quad \cos\varphi = \sqrt{\frac{(\alpha+4)\beta^2}{\alpha(\alpha+\beta^2)}} \quad, \quad \sin\varphi = \pm\sqrt{\frac{\alpha^2-4\beta^2}{\alpha(\alpha+\beta^2)}} \tag{B.30a}$$

$$\xi = \frac{-2\mu\beta}{\alpha\cos\varphi} = -2\,\mathrm{sign}\,\alpha\,\mathrm{sign}\,\beta\sqrt{\frac{\alpha+\beta^2}{\alpha(4+\alpha)}} \tag{B.30b}$$

Then one obtains easily

$$\mathrm{tr}\,H = \frac{\alpha(8+\alpha)+4\beta^2(5+\alpha)}{\alpha(4+\alpha)} \quad, \quad \det H = -(\alpha^2-4\beta^2)\left(\frac{2[\alpha^2-2\beta^2(5+\alpha)]}{\alpha^2(4+\alpha)}\right)^2 \tag{B.31}$$

so $\alpha^2 - 4\beta^2 > 0 \Rightarrow \mathrm{tr}\,H > 0$ and $\det H < 0$. This shows that the extremum ξu, when it exists, is of type $++-$.

When there are two orbits of minima, the stable state is described by the lowest minima ; this information is given in Table IV. The curves in the plane α, β, along which the polynomial takes the same value on these two orbits of minima, are curves of first order phase transitions. They are the solid lines of the Figure. Their equations are obtained from (B.19b), (B.25) and (B.23) :

$$\alpha < 0, \beta = 0 \quad, \quad \alpha > 0 \qquad \frac{\sigma_\varepsilon^2(\sigma_\varepsilon^2+2)}{3+\alpha} = \tfrac{1}{4} \iff \beta^2 = (4+\alpha)^{3/2} - (8+3\alpha) \tag{B.32}$$

The Morse index of an extremum with non-degenerated Hessian $\left[\text{i.e.,}\right.$ $\det(d^2P/dx^2) \neq \left.\overline{0}\right]$ is the number of its negative eigenvalues. From Table IV we see that all possible sets of values of c_0, c_1, c_2 found in (49) do occur in this example.

The Appendix could be generalized to any irreducible group representation with compact or finite image when there is a unique, up to a factor, cubic invariant (which defines the $\vee$ algebra) and two linearly independent quartic invariants : $(x,x)^2$ and $(x_\vee x, x_\vee x)$. Then the most general Higgs-Landau polynomial is given by (B.2). I have not been able to prove in that case that all the minima are idempotents of the $\vee$ algebra. I have proved it in the example of the adjoint representation of $SU(n)$, generalizing the case $n = 4$ treated here. Then the minima are in strata with maximal little groups. Moreover, when $\beta \neq 0$, the invariance group of the polynomial is exactly $SU(n)$ since the $(x, D_x^{q-2} x)$ invariants generate all invariants. Depending on the values of α and β all the directions of idempotents can become those of minima. For instance :

for $\alpha = 0$ or $\beta = 0$, $\alpha < 0$, p or q = 1, little group $U(n-1)$,

for $\beta = 0$, $\alpha > 0$ for even n, p = q = n/2, little group $S(U(n/2)\times U(n/2))$,

for odd n, $|p-q| = 1$, little group $S(U(n+1/2)\times U(n-1/2))$.

Dimension of strata	1	1	2	3 (generic)
Little groups	C_{3v}	C_{2v}	C_s	1
No of points in an orbit	4	6	12	24
Types of orbit	V,F	E	e	

V = vertex, F = centre of face, E = middle of edge, of the regular tetrahedron (up to a positive dilation).

Table I - Action of T_d on its three-dimensional vector representation.

Dimension of strata	1	1	1	2	2	3
Little groups	C_{4v}	C_{3v}	C_{2v}	C_s	C'_s	1
No of points in an orbit	6	8	12	24	24	48
Types of orbit	F	V	E			

V = vertex, F = centre of face, E = middle of edge, of the cube (up to a dilation).

Table II - Action of O_h on its three-dimensional vector representation.

Subdomain $\mathcal{D}_i$ of coefficients α_i	(H_i)	Little group H_i up to a conjugation	Number of minima in the orbit	Vector of the orbit
$\mathcal{D}_1$: $\alpha_2-\alpha_1>0$, $\alpha_3-\alpha_1>0$	4	$Z_2(\tau_3)\times C_{4v}$	8	$(\xi,0,0,0)$ $\quad \xi=\dfrac{\mu^2}{\alpha_1}$
$\mathcal{D}_2$: $\alpha_1+\alpha_2>0$, $\alpha_3-\alpha_2>0$, $2\alpha_3-\alpha_1-\alpha_2>0$	4	$Z_2(\tau_1)\times C_{4v}$	8	$(\xi,\xi,0,0)$ $\quad \xi=\dfrac{\mu^2}{\alpha_1+\alpha_2}$
$\mathcal{D}_3$: $\alpha_1+\alpha_3>0$, $\alpha_1-\alpha_3>0$, $\alpha_2-\alpha_1>0$	4	$\left(Z_2(\tau_3)\times Z_2(\tau_3)\right)_\square Z_2$	16	$(\xi,0,\xi,0)$ $\quad \xi=\dfrac{\mu^2}{\alpha_1+\alpha_3}$
$\mathcal{D}_4$: $\alpha_1-\alpha_2>0$, $\alpha_1+\alpha_2+2\alpha_3>0$, $\alpha_1+\alpha_2-2\alpha_3>0$	4	$\left(Z_2(\tau_1)\times Z_2(\tau_1)\right)_\square Z_2$	16	(ξ,ξ,ξ,ξ) $\quad \xi=\dfrac{\mu^2}{\alpha_1+\alpha_2+2\alpha_3}$

The little groups of minima are maximal little groups.

Table III - Minima of $\frac{1}{4}\omega(x)-\frac{1}{2}\mu^2(x,x)$, with $\omega(x)$ defined in (55), and the domain of coefficients α_i defined in (56).

Orbit	Extrema	A	B	C	D	E	D'	C'	B'
6	$\pm\mu\,q$	1	0	0	0	1	2	2	2
4	$\xi_+ s$	2	2	2	0	0	0	0	2
4	$\xi_- s$	0	0	2	2	2	2	0	0
12	$\pm\xi\,r$		1	1	1		1	1	1

Lowest minima	A	B_1	B_2	C	D_1	D_2	E	D'	C'_1	C'_2	B'
	$\xi_- s$		$\pm\mu\,q$			$\xi_+ s$				$\xi_- s$	

Table IV — Morse index of extrema and phases for the plane α, β.

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FIGURE CAPTION

The domain $\mathcal{D}$: $\alpha > -3$ in the plane of the parameters α, β, of the Higgs-Landau polynomial (B.2) is divided into subdomains $A, B_1, B_2, C, D_1, D_2,$ E, D', C_1', C_2', B', each one corresponding to a different set of extrema, as explained in Table IV. The equations of the broken line and solid line curves are given, respectively, in (B.26) and (B.32). The solid line curves indicate first order phase transitions. The three phases correspond to three different families of minima :

$\xi_+ s$ for $C_2' \cup D' \cup E \cup D_2$, $\pm \xi q$ for $D_1 \cup C \cup B_2$, $\xi_- s$ for $B_1 \cup A \cup B' \cup C_2'$.