Representations of Compact Semi Simple Lie Groups With Meridian Sections

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ABSTRACT

As a useful mathematical tool for the study of Higgs polynomials in grand unification theories we give several geometrical properties of the simplest and "most useful" representations of semi-simple compact Lie groups. These representations fulfil Theorem 3. They are low dimensional, they have a meridian section and their ring of invariant polynomials is given by that of a finite group (the generalized Weyl group). Although most of these properties are implicit in the huge but difficult mathematical literature on algebraic group actions, the proofs given here are elementary and probably new.

For the study of the "great unification of fundamental interactions" physicists need some properties of the linear actions of compact Lie groups which are not explicitly found in the mathematical literature of the last twenty years, although they do exist. Indeed they are proven for algebraic groups, and compact Lie groups are algebraic groups on the real field (remark that often these mathematical papers assume the field to be algebraically closed: this excludes the field of real numbers, so by careful reading one has to decide if the results extend to the latter case). Direct proofs for compact groups are more geometric and simpler. They are given here, because most of them do not seem to be written down. They may help physicists.

1 The Orbit and Strate of Orthogonal Representations of Compact Lie Groups

In physics we also consider complex (unitary) linear representations of a compact Lie group G on a finite dimensional Hilbert space $\mathcal{H}$. However, by doubling its dimension, one can consider $\mathcal{H}$ as a real vector space $\mathcal{E}$, which then carries a real orthogonal representation of G. From now on we simply write "a representation of G" for "a real orthogonal linear representation of a compact Lie group G".

In such representations, the isotropy groups and the orbits are closed and therefore compact. (This is not true in general for algebraic groups; it is one important cause for the simplicity of the proofs given here.) We denote by (H) a conjugation class of closed subgroups of G. The isotropy groups of a G-orbit form a conjugation class (H). Orbits with same conjugation class (H) of isotropy groups are of the same type; their union is called a stratum. The partial ordering by inclusion on the set of the closed G-subgroups, induces a partial ordering on the set of conjugation classes $\{(H)\}$ and therefore on the subset $\mathcal{K}$ of isotropy group conjugation classes ($\sim$ to that of strata) which occurs in a representation of G.

Let $G \cdot x$ be the orbit of x and $\mathcal{N}_x$ its normal linear manifold at x; it passes through the origin and it is a subspace of $\mathcal{E}$.

Lemma 1. Any normal subspace $\mathcal{N}_x$ to an orbit $G \cdot x$ cuts any other orbit. Indeed, if $G \cdot y$ is any other orbit, the minimum of the function $d(x', y')$, distance between $x' \in G \cdot x$, $y' \in G \cdot y$ exists since $G \cdot x$ and $G \cdot y$ are compact; and the straight line $x'y'$ is *orthogonal* to both orbit $G \cdot x$ and $G \cdot y$ at x' and y' respectively. By a group element which transforms x' in x, y' is transformed into y'' so $\mathcal{N}_x$ contains $y'' \in G \cdot x$.

Lemma 2. Given any $x \in \mathcal{E}$, there is a tubular neighbourhood $\mathcal{V}_x$ of $G \cdot x$ such that $y \in \mathcal{V}_x \Rightarrow (G_y) \leq (G_x)$.

Indeed there exists $\mathcal{V}_x$ such that for any $y \in \mathcal{V}_x$ there is a *unique* point $x' = r(y) \in G \cdot x$ which is the nearest point of the orbit. Since the group action preserve the distance, for any $g \in G$, $g \cdot r(y) = r(g \cdot y)$. If $g \in G_y$ the isotropy group of y, it implies $g \in G_{r(y)}$ so $(G_y) \leq (G_x)$.

The equality occurs if $y \in S(x)$, the stratum of x (i.e. the set of points with isotropy group $\in (G_x)$). So if (G_x) is minimal in $\mathcal{K}$ (defined above), $S(x)$ contains a neighbourhood of x. Finite groups are compact Lie Groups of dimension zero. For these groups, Lemma 1 is trivial ($\mathcal{N}_x = \mathcal{E}$); from lemma 2 we obtain for the closure $\overline{S(x)}$ of the stratum $S(x)$ of x

$$\overline{S(x)} = \cup_{H \in (G_x)} \mathcal{E}^H,$$

where $\mathcal{E}^H$ is the set of points y invariant by H (i.e. $H \geq G_y$); it is a subspace of $\mathcal{E}$. So if (G_x) is minimal $\overline{S(x)} = \mathcal{E}$, i.e. the kernel of the representation is the minimal little group and the corresponding stratum is open dense. For a general compact group G, the corresponding situation is given by theorem due to[1].

Theorem 1 (Montgomery and Yang). In a representation of G, there is a unique minimal class $(H) \in \mathcal{K}$, the corresponding stratum is *open dense*. We will call this stratum, the generic stratum and also qualify by "generic" its isotropy groups and orbits (they are often qualified by "principal" in the math. literature). Moreover, $\mathcal{N}_x$, the normal at x to the *generic* orbit $G \cdot x$ is called the *global slice* at x.

The orthogonal representation of G on $\mathcal{E}$ defines a linear representation $a \mapsto L(a)$ of the Lie algebra $\mathcal{G}$ of G, by antisymmetric operators: $(x, L(a)y) = -(L(a)x, y)$ i.e. $L(a)^T = -L(a)$. In the particular case of the adjoint representation $\mathcal{E} = \mathcal{G}$, $L(a)$ is denoted by $Ad(a)$; it is defined by $Ad(a)b = a \wedge b$ where $\wedge$ is the Lie algebra law on $\mathcal{G}$.

For any representation, the tangent plane at x to the orbit $G \cdot x$ is given by $x + T_x(G \cdot x)$, this subspace being defined by

$$T_x(G \cdot x) = \{L(a)x, \quad \forall a \in \mathcal{G}\} = \mathcal{N}_x^{\perp}. \tag{1}$$

The set $\mathcal{G}_x = \{a \in G, L(a)x = 0\}$ is the Lie algebra of G_x, the isotropy group of x. The restriction of the representation of G to G_x leaves invariant each of the two orthogonal spaces $\mathcal{N}_x$ and $T_x(G \cdot x)$. The representation of G_x on $T_x(G \cdot x)$ is obtained by the restriction of the adjoint G action, on the subspace $\mathcal{G}_x^{\perp} \subseteq \mathcal{G}$. The stratum $S(x)$ is also a manifold. Let $x + T_x(S(x))$ its tangent plane at x. The representation of G_x on the subspace $\mathcal{N}_x \cap T_x(S(x))$ is trivial (as a direct application of Lemma 2). If $S(x)$ is the generic stratum, $\mathcal{N}_x \subseteq T_x(S(x)) = \mathcal{E}$. So we know completely the representation of the generic isotropy group on $\mathcal{E}$ (see e.g.[2]). For semi-simple Lie algebra, $-\mathrm{tr}(L(a)^2)$ is an invariant orthogonal scalar product of the representation. For simple Lie algebra, it is proportional to $-\mathrm{tr}(\mathrm{Ada})^2 > 0$, the Cartan-Killing metric. Indeed, if it were not the case one could find λ such that $-[(\mathrm{tr}\, L(a)^2) - \lambda\, \mathrm{tr}(\mathrm{Ada})^2] \geq 0$ and vanishes on a subspace of $\mathcal{G}$ only and it is then easy to prove that it would be an ideal of $\mathcal{G}$. So we can prove the

Theorem 2. For a representation of a simple compact Lie group G, when the value of $\lambda = \mathrm{tr}(L(a))^2/\mathrm{tr}(\mathrm{Ada})^2$ is > 1, $= 1$, < 1, the *generic* isotropy group is respectively finite, Lie Abelian, Lie non Abelian. Assume $s \in \mathcal{G}_x$ the generic isotropy Lie Algebra. Then, from our previous study

$$-\mathrm{tr}\, L(s)^2 = -\mathrm{tr}(\mathrm{Ads}\,|_{G_x^{\perp}})^2 = -[\mathrm{tr}(\mathrm{Ads})^2 - \mathrm{tr}\ (\mathrm{Ads}\,|_{\mathcal{G}_x})^2]$$

so $\lambda = \mathrm{tr}\, L(s)^2/\mathrm{tr}(\mathrm{Ads})^2 \leq 1$. The case $\lambda = 1$ correspond to $\mathrm{Ads}\,|_{\mathcal{G}_x} = 0; \quad \forall s \in \mathcal{G}_x$, i.e. $\mathcal{G}_x$ Abelian. Finally $\mathcal{G} > 1$ requires $\mathcal{G}_x = 0$.

This is the compact version of a theorem in[3]. See also P. Houston[4] for a proof.

In a neighbourhood of x, the slice $\mathcal{N}_x$ cuts every orbit in one point. This is not true globally, first the G-orbits cut the slice $\mathcal{N}_x$ in several points; moreover, when one goes further away from x on $\mathcal{N}_x$, one orbit or a family of them may even not stay transverse to $\mathcal{N}_x$. We will study sufficient conditions for this to happen. There is a more interesting situation which might occur. The slice $\mathcal{N}_x$ is orthogonal to every orbit. We then say that it is a *meridian section.* A simple example is given by the action on R^3 of $SO(2)$, i.e. the group of rotations around an axis O_z. A plane containing O_z is a slice; it is also a meridian section; indeed, it cuts any generic orbit (a circle of axis O_z) orthogonally at two points and the non generic orbits are the points of O_z.

It is easy to find sufficient conditions for the absence of meridian section when $\dim G > 0$. We recall first the equation of a slice $\mathcal{N}_x$, where $S(x)$ is the generic stratum (indeed the tangent plane to the orbit is given in (1))

$$y \in \mathcal{N}_x, \qquad \forall a \in \mathcal{G} \quad (y, L(a)x) = 0. \tag{2}$$

Lemma 3. If $d =$ dimension of a generic orbit $< 1/2 \dim \mathcal{E} = n/2$, a slice is not a meridian section.

Indeed it is sufficient to find $y \in \mathcal{N}_x$ such that the tangent plane $T_y(G \cdot y)$ cuts $\mathcal{N}_x$, i.e.

$$\exists b \in \mathcal{G}, \qquad \forall a \in \mathcal{G} \quad (L(b)y, L(a)x) = 0. \tag{3}$$

The set of Equations (2) and (3) is equivalent to:

$$\forall a \in \mathcal{G}(y, L(a)x) = 0, \quad (y, L(a)L(b)x) = 0.$$

This system of $2d$ linear equations in the n components of y have solutions $\neq 0$ if $2d < n$. Since $d \leq \dim \mathcal{G}$, we have the weaker:

Corollary 2. There are no meridian sections when $2 \dim \mathcal{G} < \dim \mathcal{E} = n$.

When G is semi-simple, there is only a finite number of orthogonal representations of dimension smaller than $2 \dim G$. For a given G this number is rather limited: see e.g. the tables of McKay and Patera[5] and we can list the representation which might have a meridian section (Lemma 3 is stricter and allows some eliminations). We can now study sufficient conditions for the existence of a meridian section. We first introduce some notation. As we have shown G_x acts trivially on $\mathcal{N}_x$ so it is the centralizer of $\mathcal{N}_x$. We denote by $S(\mathcal{N}_x)$ the stabilizer of $\mathcal{N}_x$, i.e. the largest subgroup of G which transforms $\mathcal{N}_x$ in itself. It is easy to check that G_x is an invariant subgroup of $S(\mathcal{N}_x)$ and we denote the quotient $W_x = S(\mathcal{N}_x)/G_x$; we will explain later why we call it the generalized Weyl group. Note that if W_x is not trivial, it acts effectively on $\mathcal{N}_x$. Since in a neighbourhood of x, every G orbit is cut by $\mathcal{N}_x$ into a unique point, W_x has to be a discrete group, so, as a continuous image of a closed compact group, it is finite. We denote by $\mathcal{E}^{G_x}$ the subspace of $\mathcal{E}$ whose points are invariant by G_x and we have seen that $\mathcal{N}_x \subseteq \mathcal{E}^{G_x}$. We denote by $N_G(G_x)$ the normalizer of G_x into G i.g. the largest G-subgroup which has G_x as invariant subgroup.

Theorem 3. If x is a generic point of the representation of G in $\mathcal{E}$, the following conditions are equivalent

a) $\mathcal{N}_x = \mathcal{E}^{G_x}$;

b) $\mathcal{S}(\mathcal{N}_x) = N_G(G_x)$;

c) the representation of G_x in $T_x(G \cdot x)$ (i.e. the restriction of the adjoint representation of G to G_x for the space $G_x^\perp \subseteq \mathcal{G}$) does not contain the trivial representation.

They imply the existence of a meridian section.

c $\Rightarrow$ a). Indeed c) implies that $\mathcal{E}^{G_x} \cap T_x(G \cdot x) = 0$; since $T_x(G \cdot x)$ is an invariant subspace of the orthogonal representation of G_x so $\mathcal{E}^{G_x} \subseteq T_G(G \cdot x)^\perp = \mathcal{N}_x$. We have also seen $\mathcal{N}_x \subseteq \mathcal{E}^{G_x}$ so $\mathcal{N}_x = \mathcal{E}^{G_x}$.

a $\Rightarrow$ b). By definition of $N_G(G_x)$, $\mathcal{S}(\mathcal{N}_x) < \mathcal{N}_G(G_x)$ since G_x is an invariant subgroup of $\mathcal{S}(\mathcal{N}_x)$. Let $n \in N_G(G_x)$ and $c \in G_x$ so $n^{-1}cn \in G_x$. Given any $y \in \mathcal{N}_x$, $n^{-1}cny = y$ so $cny \in ny$, i.e. $ny \in \mathcal{E}^{O_x}$. From a, $ny \in \mathcal{N}_x$ so $n \in \mathcal{S}(\mathcal{N}_a)$ and $N_G(G_x) < \mathcal{S}(\mathcal{N}_x)$.

b $\Rightarrow$ c). The orbit $N_G(G_x)$. x is the set of points of the G orbit $G \cdot x$ which have G_x as isotropy group and by b, they are in $\mathcal{N}_x$. So there are no point in $\mathcal{N}_x^\perp = T_x(G \cdot x)$ invariant by G_x.

These equivalent conditions show that the slice $\mathcal{N}_x$ is a meridian section. Indeed at any point $y \in \mathcal{N}_x \cap S(x)$, $G_y = G_x$ so from c there are no points (outside the origin) invariant by G_y in $T_y(G \cdot y)$; hence $T_y(G \cdot y)^\perp = \mathcal{E}^{G_y} = \mathcal{E}^{G_x} = \mathcal{N}_y$ and $\mathcal{N}_y = \mathcal{N}_x$. In other words, for any $y \in \mathcal{N}_x \cap S(x)$ (dense in $\mathcal{N}_x$), the G orbit $G \cdot y$ is $\perp$ to $\mathcal{N}_x$.

Corollary 3. When the equivalent conditions a, b, c are satisfied, the orbits and strata of the orthogonal representation of W_x on $\mathcal{N}_x$ are the intersections by $\mathcal{N}_x$ of the orbits and strata of G on $\mathcal{E}$.

From Theorem 3 this is true for $\mathcal{N}_x \cap S(x)$ dense in $\mathcal{N}_x$. For the other points of $\mathcal{N}_x$, apply the arguments of Lemma 2 both to the action of G on $\mathcal{E}$ and W_x on $\mathcal{N}_x$. As a direct consequence of the correspondence established in Corollary 3 we obtain:

Corollary 4. When the equivalent conditions a, b, c are satisfied, the rings $\mathcal{P}^G(\mathcal{E})$ of G-invariant polynomials on $\mathcal{E}$ and $\mathcal{P}^{W_x}(\mathcal{N}_x)$, the W_x invariant polynomials on $\mathcal{N}_x$ are isomorphic.

It is important to note that the equivalent conditions a, b, c are sufficient for the existence of a meridian section, but they are not necessary. Indeed, they are not satisfied for the simple example of the 3 dimensional representation of $SO(2)$ on R^3

that we have chosen for the existence of a meridian section. The well known example for the validity of the a $\Leftrightarrow$ b $\Leftrightarrow$ c condition is the adjoint representation $\mathcal{E} = \mathcal{G}$ for semi simple Lie groups. In that case $\mathcal{G}_x$ is a Cartan subalgebra i.e. a maximal Abelian subalgebra of $\mathcal{G}$. (Note that $\mathcal{N}_x = \mathcal{G}_x$.) Hence there are no trivial representation of $\mathcal{G}_x$ on $\mathcal{G}_x^\perp$ and condition c is fulfilled. For the adjoint representation W_x was called the Weyl group. It is generated by reflection and therefore[6], the ring of invariant polynomial $\mathcal{P}^{W_x}(\mathcal{N}_x)$ is a polynomial ring, i.e. the set of polynomials with r variables ($r = \dim \mathcal{G}_x =$ rank of the group G) and these variables are themselves r invariant homogeneous polynomials of well defined degree (determined by Coxeter; they are equal to $b_k + 1$ where the b_k's are the Betti numbers of the simply connected G). The representations of G on $\mathcal{E}$, such that $\mathcal{P}^G(\mathcal{E})$ is a polynomial ring, have been determined for algebraic groups by[7] for irreducible representations and independently by[8] and[9] for reducible representations.

Looking at these lists, it seems that these representations satisfy a $\Leftrightarrow$ b $\Leftrightarrow$ c, so the generalized Weyl group is generated by reflections and its invariants (and therefore the G invariants on $\mathcal{E}$) are completely known. But I have not yet been able to establish a theorem on that point.

In any case, Theorem 3 and its corollary give a powerful way for the representations which satisfy conditions a $\Leftrightarrow$ b $\Leftrightarrow$ c to find the ring of invariant polynomials. Indeed this problem is more easy to solve for finite groups (for reviews see for instance[10,11].

The study of the corresponding Higgs polynomial and their minimization is then greatly simplified (see e.g.[12]).

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