INTRODUCTION TO SPONTANEOUS SYMMETRY BREAKING

SOME EXAMPLES

Louis MICHEL
Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

0. Introduction.

Let me begin this first lecture of the school by an historical reference [1]. E.P. Wigner was working as an engineer in a Budapest leather factory when at 24 he received an invitation to become assistant of the new theoretical physics professor (Becker) in Berlin. When he arrived, the assistant position was not yet established and he was advised, while waiting, to work with Dr. Weissenberg, a known crystallographer, who told him : "There is a miracle . Why in a crystal atoms are most often on a symmetry axis or on a symmetry plane. Why ?". The day after, Wigner tried to give him an answer : "On a symmetry axis or a symmetry plan the potential is more likely to have an extremum". "Well, well you seem to be right, but one needs an elegant proof" and in his interview Wigner adds : "Then I started with the idea to write a book on group theory".

You will not find this subject treated in the famous Wigner book [2], which appeared soon after (1931), or in most books on physical application of group theory. So I will deal with it in these lectures:§7.

These lectures are nearly self-containing. The first part (Sections 1 to 5) will teach the basic of group actions, illustrated by physical examples, and explain the principal mechanism for the mathematical study of spontaneous symmetry breaking (Section 6). The last three sections give the results of recent, mostly unpublished work to which I am collaborating, in three domains : 1) basic concepts of crystallography (useful for the lectures on modulated crystals), 2) renormalisation group application to Landau theory, 3) a Spin 10 grand unification scheme (sections 8,9,10).

1. Group actions. Orbits and Strata. Examples.

You all know what is a group G . If $H < G$ (< reads "subgroup") it is interesting to consider the cosets g_1H , g_2H , etc. of H . We denote by $[G:H]$ the set of these cosets. If G is finite and has $|G|$ elements, the number of cosets is $|G|/|H|$. If G and H are Lie groups, then $[G:H]$ is a manifold whose dimension is :

$$\dim[G:H] = \dim G - \dim H . \qquad (1)$$

If for every $g \in G$, $gH = Hg$, we note $H \triangleleft G$ and say that H is an invariant subgroup of G . Then there is a natural group law on the set $[G:H]$, given by $g_1H.g_2H = g_1g_2.H$. This group is denoted by G/H and it is called the quotient group of G by H .

All the books on group theory and quantum mechanics study the linear representations of a group G on a vector space E . Such a representation is a homomorphism $G \xrightarrow{f} GL(E)$ of G in the general linear group on E . The group $GL(E)$ is the automorphism group $\mathrm{Aut}\, E$ of E . The set of elements of G which are represented by the identity on E form the kernel of f , $\ker f \triangleleft G$ and $G/\ker f \sim \mathrm{Im}\, f$, the image of f , which is $< GL(E)$. In quantum physics, E is the Hilbert space of state vectors, and we need to consider only unitary G representations, i.e. $\mathrm{Im}\, f < U(E)$, where $U(E) = \mathrm{Aut}\, E$, the automorphism group of the Hilbert space.

But symmetry groups may also enter into physics through an action on a mathematical structure M (e.g. a manifold) which is defined by the group homomorphism $G \xrightarrow{f} \mathrm{Aut}\, M$. The action is effective if $\ker f = [1]$.

Given two G actions G,f,M and $Gf'M'$, by <u>definition</u> an equivariant map $M \stackrel{\theta}{\to} M'$, satisfies the commutative diagram 1 for every element of G .

$$\forall y \in G \qquad \begin{array}{ccc} M & \xrightarrow{\ \theta\ } & M' \\ \downarrow{\scriptstyle f'(y)} & & \downarrow{\scriptstyle f'(g)} \\ M & \xrightarrow{\ \theta\ } & M' \end{array}$$

Diagram 1

<u>Definition</u>. The two actions are equivalent when θ is a bijective map (and therefore an isomorphism between M and M'). When M is a vector space or a Hilbert space, this definition of equivalence coincides with the usual one for G-linear representations.

When a physical system has a symmetry group G , all functions describing physical properties of this system must be G-invariant or G-covariant. But these conditions depend only on the image $\mathrm{Im}\, f$. So a weaker definition of equivalence is often useful in physics, as J. Mozrzymas and I showed [3].

<u>Definition</u>. Two actions G,f,M and G',f',M' are weakly equivalent if there is an isomorphism $M \stackrel{\theta}{\to} M'$ such that the corresponding automorphism $\mathrm{Aut}\, M \stackrel{\theta^*}{\to} \mathrm{Aut}\, M'$ identifies the two images : $\theta^*(\mathrm{Im}\, f) = (\mathrm{Im}\, f')$. For instance when $M = M'$, $\mathrm{Im}\, f$ and $\mathrm{Im}\, f'$ are conjugate subgroups of $\mathrm{Aut}\, M$. The two non trivial inequivalent representations of Z_3 (the cyclic group with 3 elements) or the two 3-dimensional inequivalent representations of $SU(3)$ are weakly equivalent. Weak equivalence of actions (which is even defined for two different groups) will appear as a natural and important concept in the study made below of spontaneous symmetry breaking in phase transitions.

To simplify notations, we will often use $g.m$ instead of $f(g)m$, the transform of m by g . The set of all transforms of m is denoted by $G.m$ and is called the G orbit of m . The little group G_m

(mathematiciens often say the isotropy group) is the set of all elements of G such that $g.m = m$. Note that $G_{g.m} = gGg^{-1}$ so the little groups of an orbit form a conjugation class of G subgroups, that we denote by $[G_m]$. When the actions of G on two orbits are equivalent, it is easy to prove that these orbits have the same conjugate class $[H]$ of little groups. They are said to be of the same type. The sets of cosets $[G:H]$ with the G action $g.xH = gxH$ is a prototype of this type of orbits. By definition,a stratum is the union of all orbits of the same type; equivalently $m' \in S(m)$, the stratum of m , when $G_{m'}$ and G_m are conjugate.

The decomposition of a group action into strata yields a primary important information, very relevant physically. For instance : 1) in the linear representation action of the Lorentz group on Minkowski space there are three other strata outside the origin (unique fixed point) : their elements are respectively the time-like, space-like and light-like vectors. Let us choose four other examples : 2) The symmetry group of an axially symmetric ellipsoid (as a simplified model of the earth) is $D_{\infty h}$, generated by C_∞ , the group of rotations around the axis containing the two poles, the rotation by π around axes in the equatorial plane (with C_∞ , they generate the group D_∞) and finitly the symmetry h through the equatorial plane. (Note that this figures has a symmetry center; when taken as origin $-I \in D_{\infty h}$) . There are three strata : the two poles (i.e. a two-point orbit), the equator (one connected orbit) and the rest, an open dense set in which the orbits are the pair of parallel circles with the same North and South latitude. 3) Consider the n dimensional hypercube, centered at the origin. Its 2^n vertices have coordinates $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n$ with $\varepsilon_i^2 = 1$; the center of its $2n$ faces are at the tops of $\pm$ the unit vectors of the coordinate axes. Its symmetry group is generated by the diagonal matrices with ± 1 as elements: (they form an Abelian group $\sim Z_2^n$) and the permutation group S_n of the coordinate axes. This $2^n.n!$ element group is denotes B_n in the classification by Coxeter of the finite groups generated by reflections. Chemists, physicists and crystallographers also use the notations : when $n = 2$, C_{2v} or $2mm$, the element group of symmetry of the square, and when $n = 3$, O_h or m3m , the 48 element group of symmetry of the cube. Outside the origin, the 2 dimensional linear action of C_{2v} containing three strata, the 2 coordinate axes, the two diagonal axes and the rest. The three dimensional representation of O_h contains strata given in table 1. As we see the strata correspond to the symmetry elements.

Table 1. Strata of the 3 dimensional representation of O_h. The little group and its number of elements are given at the end of the lines.

(0)	the origin	O_h ,48
(1)	the 8 axes containing the vertices -(0)	C_{3v}, 6
(2)	the 12 axes containing the center of edges -(0)	C_{2v}, 2
(3)	the 6 axes containing the center of faces -(0)	C_{4v}, 4
(4)	the 3 symmetry planes (coordinate planes) -(0) - (2) -(3)	C_s , 2
(5)	the 6 other symmetry planes -(0) -(1) -(2) -(3)	C'_s , 2
(6)	the rest, open dense	1 , 1

4) In the action of the 230 crystallographic groups on the 3-dimensional space, the strata are tabulated in the international Tables for Crystallography under the name "Wyckoff positions" [4]. For each space group, there is a finite number of them. There is only one of dimension 3, and one can check that it is open dense. 5) In an n dimensional real vector space R^n a lattice is a closed subgroup Z^n generated by n basis (i.e. linearly independent vectors of R^n . The general linear group $GL(n,R)$ transforms any basis into any basis, so the set L of lattice is the orbit $[GL(n,R) : GL(n,Z)]$. Indeed the little group of a lattice transforms the set of lattice points (i.e. vectors with integral coordinates) into itself. Note that $GL(n,Z) = \text{Aut } Z^n$. The orthogonal subgroup $O(n) < GL(n,R)$ respects the space metric. The strata of its action on L correspond to crystallographic systems, the corresponding little groups P_H are called the holohedries of the lattices. For $n = 3$ there are (*) 7 crystallographic systems :

(*) These crystallographic systems were listed by Weiss in 1815. This stratum definition corresponds to the "French systems" in ref [4]. Strangely enough the International Tables have adopted an unnatural definition.

Systems	Triclinic,	Monoclinic,	Orthorhombic,	Tetragonal,	Trigonal
	Tri,	Mon,	Ort,	Tet,	Trg
P_H	$T = C_i$,	$2/m = C_{2n}$,	$mmm = D_{2h}$,	$4/mmm = D_{4h}$,	$\bar{3}m = D_{3d}$,

Hexagonal,	Cubic
Hex	Cub
$6/mmm = D_{6h}$,	$m3m = O_h$

For $n = 2$ there are 4 crystallographic systems. We will prove it from a natural description of the space of 2 dimensional lattices as an orbit space. (See below).

In all these examples, the number of strata is finite. This will be the case in most physics problems.

2. Orbit Space. Examples.

In the action of G on M the set of orbits we denoted by $M|G$ the set of orbits and by $\overline{M|G}$ the set of strata and by π, σ the canonical surjective maps

$$M \xrightarrow{\pi} M|G \xrightarrow{\sigma} \overline{M|G} \qquad (2)$$

In the five examples of the preceding section we have studied $\overline{M|G}$. Let us now study the orbit space.

1) The scalar product $S = (a,a)$ is a real number and it is an invariant of the Lorentz group. To any value of (a,a) corresponds a unique orbit except $(a,a) = 0$ which is both the length of light-like vectors and the $\vec{0}$ vector. So if we consider the Minkowski space minus the origin, the orbit space is R and the three strata are defined by $S > 0$, $S = 0$, $S < 0$.

2) The orbit space is $0 \leq \theta \leq \pi/2$ where θ is the absolute value of the latitude; the three strata are $\theta = 0$, $\theta = \frac{\pi}{2}$, $0 < \theta < \frac{\pi}{2}$.

3) The orbit space is one of the convex connected cones formed by the symmetry hyperplanes ex : for $n = 2$, $x_1 \geq 0$, $x_2 - x_1 > 0$, for $n > 2$, $x_1 \geq 0$, $x_2 - x_1 \geq 0$, $x_3 - x_2 > 0$,..., $x_n - x_{n-1} > 0$. It is called a Weyl chamber.

4) The orbit space for the translation group Z^n est $R^n | Z^n$; it has the topology of a torus $(S_1)^n$ (where S_k is the k dimensional sphere). It is what the crystallographer calls a Wigner Satz cell with its opposite faces identified. The action of G on R^n defines an action of the point group $P = G/Z^n$ on the torus $R^n | Z^n$. We leave to the reader the determination of the orbit space in general. It is not a manifold except in the case where there is one stratum only : this occurs for respectively 2 and 13 crystallographic groups for $n = 2$ and 3 . Then $R^n | G$ is a flat Riemann manifold with G as first homotopy group.

5) We will study the action of the group $O(n) \times R^x$, including the dilations for the case $n = 2$. Indeed the symmetry of a lattice is independent from its scale. We choose as first generator $\vec{a}$ of the lattice, one of the shortest vectors (*); by a dilation and rotation we

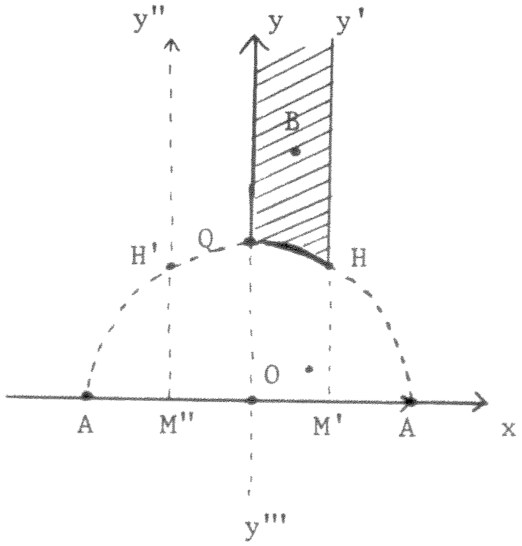

Figure 1

(*) Beware that this is not always possible for $n \geq 5$. I am grateful to Henry Bacry for advice and discussions on Fig. 1.

bring it to the unit vector of the x axis. Then we can look for the second generator $\vec{b}$ in a vertical band of width 1, for instance $-\frac{1}{2} < x \leq \frac{1}{2}$ and also require its component y to be > 0 (since $\vec{b} \in Z^2 \Rightarrow -\vec{b} \in Z^2$, $\vec{b}+m\vec{a} \in Z^2$), moreover since $|\vec{b}| \geq |\vec{a}|$, the point B, top of the vector $O\vec{B}$, must be outside the open unit circle centered in O. Finally, if the abscissa of B is negative, by the reflection through the y-axis, we change its sign. So the orbit space is the hatched domain $yQHy'$. The (open dense) inside represents the diclinic system, $P_H = \{I,-I\} = C_i$ or $\bar{1}$. The boundary minus Q, H represents the orthorhombic system $P_H = C_{2v}$ or $2mm$, Q the quadratic system $P_H = C_{4v} = 4mm$ and H the hexagonal system $P_H = C_{\ v}$ or $6mm$. The corresponding study for $n = 3$ has been done in [5].

Let us add another example : 6) Consider the decay of a particle of energy momentum $\underline{P}$ into three particles of energy momenta p_i, $i = 1,2,3$. The phase space M is defined by the relations :

$$(\underline{P},\underline{P}) = M^2 \;,\; (p_i,p_i) = m_i^2 \;,\; \underline{P} = \Sigma_i \underline{p}_i \quad . \tag{3}$$

The little group G of $\underline{P}$ in the Lorentz group is isomorphic to $O(3)$ (generally the initial particle is considered at rest; it is not relevant). It acts on M. The orbit space $M|G$ is the Dalitz plot. There are two strata, the interior, when the $3p_i$'s span a 2-plane, and the boundary, when the $3p_i$'s are colinear.

3. Action on subsets and substructures. Examples.

The action $G \xrightarrow{f} \mathrm{Aut}\, M$ of G on M defines an action of G on the set $\mathcal{P}(M)$ of subsets of M. Given such a subset $X \subset M$ one defines the centralizer in G of X as :

$$C_G(X) = \bigcap_{x \in X} G_x \quad ; \tag{4}$$

it is the largest G-subgroup which leaves fixed every element of X. Similarly one defines the stabilizer in G of X as the largest G subgroup which transforms X in itself. We denote it by $S_G(X)$. It is easy to prove that $C_G(X) \lhd S_G(X)$. We shall denote the quotient

$$W_G(X) = S_G(X)/C_G(X) \ ;$$

it acts effectively on X . If G is a compact semi-simple group, its Lie algebra $\mathcal{G}$ has an orthogonal scalar product, the Cartan Killing form. The adjoint representation of G is the natural linear representation on $\mathcal{G}$(as a vector space). If X is a Cartan subalgebra, i.e. a maximal Abelian subalgebra (they are all conjugate by G) , then $W_G(X)$ is the Weyl group; it is a Coxeter group.

Note a fundamental relation satisfies by centralizers in any group action

$$C_G(\cup_i M_i) = \cap \, C_G(M_i) \quad .$$

<u>Exercise</u>. Prove that for linear representations of finite groups or enumerable groups (i.e. crystallographic groups), interactions of little groups are little groups. (This is not true for other groups in general!). The proof is an appendix A of [6] . It is also useful to introduce the traditional notation M^g for the set of elements of M invariant by G . Similarly

$$H < G \quad , \quad M^H = \bigcap_{g \in H} M^g \quad .$$

4. Partial ordering of the strata. Compact group actions.

There is a partial ordering, by inclusion, on $\{< G\}$, the set of subgroups of G . When G is compact (this includes finite), it defines a partial order (by inclusion up to a conjugation) on the set of conjugation classes of closed subgroups of G . This is also true for crystallographic groups or for the conjugate classes of finite subgroups of an arbitrary group. This induces a partial ordering on the set of strata.

If the action of G is continuous, M^g is closed, so is M^H as an intersection of closed sets. $\bigcup_{H \in [H]} M^H$ is the union of all strata with little group conjugate class $\geq [H]$. For finite groups, as a finite union of closed sets, it is closed, so the strata for maximal isotropy groups are closed. This extends to compact group action. Similarly, it is easy to prove [3a] that for a finite group action, there is a unique minimal conjugate class of little groups (Kerf itself); the corresponding stratum is open dense. We call it "generic". This is also true for smooth compact group actions [7]. We verify these two properties on examples 2,3,5,6; they are also true for 4, but not true for 1 : The Lorentz group is not compact. For smooth compact group action G on a finite dimensional differentiable manifold M, with a finite number of strata (e.g. this is the case when M is compact), Mostow [8] proved that there exists a smooth injective equivariant Map $M \xrightarrow{\theta} E$ into a real vector space E of finite dimension, carrying a linear orthogonal representation of G. So the case of linear action is pretty general !

5. Action on a group. Action of G on itself.

We consider the actions on G preserving its group law. They are defined by $K \xrightarrow{f} \text{Aut}\, G$ and most properties of the action depend only on $\text{Im}\, f < \text{Aut}\, G$. Consider the particular case $G = K$; then, for the "natural" action of G on itself, $\text{Im}\, f = \text{In Aut}\, G$, the group of inner automorphisms. One proves that $\text{In Aut}\, G \triangle \text{Aut}\, G$ and one defines $\text{Out}\, G = \text{Aut}\, G/\text{In Aut}\, G$. Obviously $\text{Ker}\, f = C(G)$, the center of G. The orbit $G\cdot x = \{gxy^{-1}, \forall g \in G\}$ is called the conjugation class and the isotropy group G_x is the centraliser of x. The corresponding action of G on the set $\{< G\}$ of its subgroups, defines for each $H < G$, the centralizer $C_G(H)$ and the stabilizer $N_G(H)$; the latter is also called the <u>normalizer</u> : it is the largest subgroup of G which has H as invariant subgroup. Since both H and $C_G(H)$ are invariant subgroups of $N_G(H)$, this is also the case of $H.C_G(H)$ and one finds (proof left to the reader) that there is an injective homomorphism

$$\frac{N_G(H)}{H.C_G(H)} \rightarrowtail \text{Out}\, H \tag{7}$$

The G orbit of H is the conjugation class of subgroups $[H]_G$ that we have already studied. There is also a natural action of $\text{Aut}\, G$ on

G . The action of G on the orbits $[G:H_1]$ and $[G:H_2]$ when the subgroups H_1 and H_2 belong to the same orbit of Aut G are quasi-equivalent but may be non equivalent.

Given an action $Q \xrightarrow{f} \text{Aut } G$ one forms the semi-direct product $G \rtimes Q$ defined by the group law

$$(g_1,q_1)(g_2,q_2) = (g_1.f(q_1)[g_2],q_1q_2) \tag{8}$$

When f is the trivial homomorphism, the law (8) is that of the direct product. More generally one calls extension E of Q by G a group such that $G \triangleleft E$ and the action by E inner automorphisms $E \xrightarrow{\tilde{f}} \text{Aut } G$ factorizes : $\tilde{f} = f \circ s$ where s is defined by $E \xrightarrow{s} E/G = Q$. Two extensions E and E' are equivalent if there is a commutative diagram

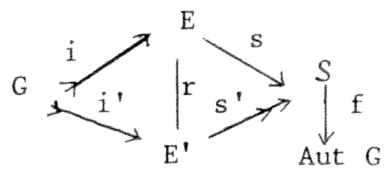

Diagram 2

(then r is isomorphism) and there is a natural group law on the set of equivalence classes of extension. This group is denoted by $H_f^2(Q,G)$ and it is called the second cohomology group of Q with value in G . (For lectures in physics Summer School, see e.g. [9] where the original mathematical literature is quoted and explained). The semi-direct product represents the unit of the cohomology group. When G is not Abelian, there is a natural action $G \xrightarrow{\phi} \text{Aut } C(G)$ on its center, and the set of equivalence classes of extension of Q by G is isomorphic to $H^2_{\phi \circ f}(Q,C(G))$.

Examples of semi-direct products are the Euclidean groups $E(n) = R^n \rtimes O(n)$ and the affine group $Aff(n) = R^n \rtimes GL(n,R)$.

6. Spontaneous symmetry breaking. Example of Landau.

When a physical problem has a symmetry group G , a solution is not necessarily G-invariant. If H is its isotropy group, then one can build an orbit $[G:H]$ of solutions. The set $\mathcal{S}$ of solutions of the problem is G invariant. As a general example assume that S is a set of stable states of the system and they depend on one (or several) G invariant parameters λ (e.g. temperature, time...). For some value of λ the state representative function $s(\lambda)$ may change of strata on S ; then the symmetry of the physical system changes. If it decreases(*), one says the symmetry is spontaneously broken.

There are half a dozen of mathematical schemes used for describing symmetry breaking in physics. We describe here the most common one, and the only one used in the other lectures. A system of equations for a G-symmetric physical problem is a smooth G-equivariant map ϕ between two functional spaces F_1, F_2 , carrying a linear representation of G . Moreover, we assume that it depends differentiably on parameters λ . Assume that at λ_o we know a unique solution $u_o \in F_1$ of the problem, i.e.

$$\phi(u_o,\lambda_o) = 0 \tag{9}$$

If the Fréchet derivative

$$\forall v \in F_1 \quad \frac{d\phi_{u_o}}{du}(v) = \lim_{\varepsilon \to o} \frac{1}{\varepsilon}\left(\phi(u_o + v,\lambda_o) - \phi(u_o,\lambda_o)\right) \tag{10}$$

is inversible in a neighbourhood of λ_o , by the implicit function theorem, we can compute a G-invariant solution $u(\lambda)$ satisfying $\phi(u,\lambda) = 0$. If, for the value λ_i of λ , the linear operator $\frac{d\phi_{u_c}}{du}$ has a non trivial kernel, there is a bifurcation and Ker $\frac{d\phi_{u_c}}{du}$ is the tangent plane to the set of possible solutions which appear; it is stable by G and carries a linear representation of this group. It is in general irreducible (otherwise we have an "accidental degeneracy"). At λ_c , the symmetry G will be broken into one of the isotropic groups of the representation on Ker$\frac{d\phi_{u_c}}{du}$. Which one? This will

(*) If the transformation is reversible, e.g. phase transitions in thermodynamics, it increases one way and decreases the other way.

be seen in the example of the Landau theory of second order phase transition which will be treated by several lecturers, mainly Prof. P. Toledano. In this theory F_1 is a space of physical functions (e.g. electron density, etc...) defined on the crystal and ϕ is the set of derivative of a thermodynamic potential, e.g. Gibbs Free energy V , when the parameters λ are the temperature, the pressure etc. The orthogonal irreducible (on the real) representation of a crystallographic group G are finite dimensional. Let E be the orthogonal vector space $\mathrm{Ker}\,\frac{d\phi_{u_c}}{du}$, carrier of an irrep of G . The simplest modelization of the restriction $V|_E$ on this kernel has been proposed by Landau nearly fifty years ago : it is a G-invariant degree four polynomial bounded below :

$$x \in E \qquad V(x) = p_4(x) + a(T)(x,x) \tag{11}$$

where p_4 is a strictly positive homogeneous G-invariant quartic polynomial, (x,x) is the invariant orthogonal scalar product and $a(T)$ is a function of temperature whose value has the same sign as $T-T_c$. When the physical signification of x is clear, it is called the order parameter; its number of component is $\dim E$. The physically stable states are represented by the minima of V as a function of T. The symmetry is broken into their isotropy group. The potential V(x) in (11) has no third degree term. As Landau pointed out, this is necessary for avoiding a first order transition (with a jump at $T = T_c$ from one minimum to another one). However, such generalized potential is useful for the study of the symmetry change in weak 1st order transitions. As we will see later (see also lecture of Prof. Ruegg) Higgs potentials are of a similar nature.

7. Minima of a G-invariant potential.

It is time to prove Prof. Wigner's answer. We first recall that the Euclidean group E(3) is the semi-direct product of the orthogonal group O(3) by the translation group R^3 :

$$E(3) = R^3 \rtimes O(3) .$$

A crystallographic group G is a discrete closed subgroup of $E(3)$ containing a translation lattice $\sim Z^3$. The little group G_x of any point x is finite. Indeed $E(3)_x \sim O(3)$ so $G_x = O(3) \cap G$ and the intersection of a compact and a discrete closed subgroup of $E(3)$ is a finite subgroup. This is also true of the point group G/Z^3. But P is a subgroup of G , and then an isotropy group P_x only when G is a semi-direct product (crystallographers say "symmorphic").

The gradient of a G-invariant potential at x is invariant by G_x . So it vanishes at a symmetry center; it is along a symmetry axis or in a symmetry plane. Consider a symmetry axis which does not carry symmetry centers or a symmetry plane which does not contain higher symmetry elements. Due to its periodicity, a continuous function reaches maxima and minima on these symmetry elements; on these points its gradient vanishes, so they are extrema for the full function. As we will see later, Morse theory can give some conditions for the localisation of the minima.

In the case of a smooth compact (and in particularly finite) group action on a manifold M (linear representations are a special case) there are classical theorems easy to prove (see e.g. [6], appendix C and references there). For any $x \in M$, there is a neighbourhood V_x such that for all $y \in V_x$, $[G_y] \leq [G_x]$. As a corollary a G equivariant differentiable tangent vector field is tangent to its stratum.

In a linear representation, smooth G-invariant functions are smooth functions of G-invariant polynomials whose ring is finitely generated. So one can establish invariant equations for localizing the zero of G equivariant vector fields. This has been done in [10] for the representations of all closed subgroups of $O(3)$; one sees easily that it is "easier" to have zero on symmetry elements (i.e. non generic strata).

A degree four polynomial on E (such as a Landau or Higgs polynomial) bounded below and maximum at the origin 0, has two radial minima on each straight line containing 0. Let M be the set of these radial minima. It is a smooth manifold, homotopic to a sphere (and with 0 as symmetry center if there are no 3rd degree terms). It has extrema on every closed strata on M (see e.g. [11] and earlier quoted references); hence every maximal conjugation class of entropy groups is that of extrema. To obtain conditions on minima one can apply The Morse theory [3a], [11]. One can also prove that for irreducible representations there are no extrema in the generic stratum [11], [12]. What can be said about the lowest minima. It has been conjectured that they occur only with maximal isotropy groups [13]. This has no meaning. Indeed, one has to make the following distinctions. Given the image G of the symmetry group, one chooses a Landau polynomial with a quartic term $p_4 \in \mathcal{P}_4^G$ where $\mathcal{P}_4$ is the vector space of quartic polynomials in n variables; $\dim \mathcal{P}_4 = \binom{n+3}{4}$ and we define $\nu_G = \dim \mathcal{P}_4^G$. One has to consider the centralizer in $O(n)$, $C_{O(n)}(\mathcal{P}_4^G) \geq G$. Finally any mathematical theorem or conjecture can be formulated only in term of the exact isotropy group $O(n)_{p_4} = \tilde{G} \gneqq G$ (see [11]). However, counter-examples have recently been found [14], [15], to the conjecture that the isotropy group of the absolute minimum of a Landau potential on E, is a maximal isotropy group on E of the isotropy group in $O(n)$, (i.e. $\tilde{G} = O(n)_{p_4}$) of the Landau potential. Professor Ruegg will give in his lectures a similar counter-example for a Higgs potential published in [16] and another example has been found in [17].

The Wigner problem for a crystal can be transformed into a problem of action of the finite point group P on the torus $R^3|Z^3$, the orbit space of the translation group (indeed, a triply periodic function is defined by its values on this torus) and the Morse theory is also applicable.

8. Basic Concepts in Crystallography.

This section is based on an unpublished manuscript with Prof. Jan Mozrzymas. It is a direct application of the concepts of group action.

It will help the participants of these school not acquainted with crystallography to follow some of the lectures. In §1, example 5 we defined the lattices and classed them into crystallographic systems. All other definitions we will give here are independent of the dimension n ; we give them here for $n = 3$. The isotropy groups of the lattices belonging to a crystallographic system from a conjugation class $[P_H]$ of $O(n)$. The conjugation classes $\leq$ that of the holohedries P_H are called geometric classes. There are 32 in $n = 3$ dimensions. Most macroscopic properties of the crystal are classified according to these classes. A group P of one of these classes is called the point group of the crystal. Its action in the lattice is an injective homomorphism $P \overset{f}{\rightarrowtail} GL(3,Z) = \mathrm{Aut}\, Z^3$. However, the equivalence chosen is neither the usual one, nor the weak one; indeed the point group is given as an $O(3)$ subgroup (up to a conjugation). For instance the holohedry group $O_n = m3m$ of the cubic system has an automorphism which exchanges the conjugation class of the 6 plane symmetry with that of the rotations by π around axes forming the middle of the edges. Such "non geometric" automorphisms are not considered. So for crystallographers, two actions f, f' of the point group P are equivalent if the images $\mathrm{Im}\, f$ and $\mathrm{Im}\, f'$ are conjugated in $GL(3,Z)$. So the 73 conjugation classes of finite subgroups of $GL(3,Z)$ correspond to all possible actions of the 32 geometric classes. They are called arithmetic classes. To the seven holohedries correspond 14 arithmetic classes; they are exactly the Bravais classes (the 1850 definition of Bravais was different!). For each of the 73 arithmetic classes one has to solve an extension problem. The equivalence of extensions defined by diagram 2 is too fine for the crystallographers. The normalizer $N_{GL(3,Z)}(\mathrm{Im}\, f)$ acts on P and also on the lattice Z^3 , so it acts on the cohomology group $H^2_f(P,Z^3)$. To each orbit of the normalizer is corresponding one crystallographic class. There are 219 of them for $n = 3$. One shows that this equivalence corresponds exactly to the following : two crystallographic space groups (i.e. closed discrete subgroups of $E(3)$ containing a lattice Z^3) belong to the same mathematical crystallographic class if they are conjugated by an element of the affine group $\mathrm{Aff}(3)$. It is a remarkable theorem of

Bieberbach [18] that isomorphic space groups (in n dimensions) are conjugated in Aff(n) . The equivalence definition used in crystallography is slightly stricter. Indeed, although the interatomic distances in a crystal phase change with temperature, the symmetry is considered the same. However, since temperature changes are continuous, two crystallographic space groups belong to the same physical crystallographic class if they are conjugated by an element of the connected affine group $Aff_+(n) = R^n \rtimes GL_+(n,R)$, where $GL_+(n,R)$ is the group of linear transformation with positive determinant. For $n = 3$, 11 mathematical classes split into a pair of "enantiomorphic" physical classes, so there are 230 of the latter. Other basic references to n dimensional crystallography are [19] and [20].

9. Renormalization of the Landau theory of second order phase transition.

As you will hear in other lectures, Landau theory of second order phase transition in crystals is rather successful for explaining symmetry changes. However, it fails completely for giving the critical exponents. So as soon as the Wilson renormalisation with $\varepsilon = 4-d$ (d is the space dimension) expansion was proposed, it was used for Landau theory by adding to the potential V of equation (11) a kinetic energy term. Here I will simply give a general but abstract formulation of the so-called renormalisation group technique and explain the main results of some of my recent papers [21], [22] and in collaboration with J.C. Toledano [23] and also with P. Toledano and Brézin [24].

The right hand side of the renormalization group equation

$$\frac{dg(\lambda)}{\lambda d\lambda} = \beta(g(\lambda)) \tag{12}$$

is a vector field β defined on the vector space P_4 of quartic polynomials. A fixed point g satisfies $\beta(\tilde{g}) = 0$. As we have noted in §7, the quartic part p_4 of the Landau potential depends only on the image $G < O(n)$ of the representation of the symmetry group. Hence weakly equivalent representations yield the same potential. The expression of p_4 depends on the orthonormal basis chosen for the n dimensional representation space. However, physics must not depend on this choice of basis, so the vector field β must be $O(n)$ equivariant, and the critical exponents are $O(n)$ invariants. As a consequence, the trajectory by equation (12) of any $g \in P_4^G$ (the subspace of G-invariant quartic polynomials) stays in this subspace and stops at a fixed point $\tilde{g} \in P_4^G$. It is stable if

$$\frac{d\beta}{dg}(\tilde{g})\Big|_{P_4^G} > 0 \qquad (13)$$

If this condition is satisfied, and $\tilde{g} > 0$ then the effective Landau potential is obtained from (11) by replacing p_4 by $\tilde{g}$. One interprets the non satisfaction of (13) by a lack of second order phase transition. The vector field β has been computed in [25]. Some successful predictions based on this scheme were made.
(P. Toledano's lectures are more critical). The recent results were obtained

(i) At the approximation computed in [25], the vector field β is a gradient.

(ii) If it exists, the stable fixed point $\tilde{g}$ is unique ([21] completed in [24]) and

(iii) its isotropy group $O(n)_{\tilde{g}}$ is equal to its normalizer in $O(n)$.

(iv) If the stabilizer $S_{O(n)}(P_4^G)$ does not leave invariant a quartic polynomial, there are no stable fixed points.

Property (i) depends essentially on the approximation. One can hope it is not the case for (ii). Property (iii) is a consequence of (ii) and (iv) is a simple corollary of (iii). Remark that (iii) is very restrictive. For $n = 2,3,4$ the number of closed strict subgroups of $O(n)$ which satisfies it are respectively $0,1,3$.

In [22] there are some remarks for arbitrary n .

10. A Spin 10 grand unification theory (*)

Let me first remind a few facts about simple compact Lie algebras $\mathcal{G}$ and groups G and their representation. A Cartan subalgebra of such a $\mathcal{G}$ is a maximal Abelian subalgebra (it corresponds phycically to a complete system of commuting observables); they are all conjugated. Their common dimension ℓ is the rank of the algebra. For $\ell = 5$ or $\ell \geq 9$ there are 4 such algebras(**), labelled by their Dynkin diagrams. For instance for $\ell = 5$:

A_5	B_5	C_5	D_5	
0-0-0-0-0	0-0-0-0=0	0-0-0-0=0	0-0-0< (0, 0)	(14)
SU(6)	Spin(11)	Sp(10)	Spin(10)	

(Sp is for symplectic).

To a simple compact Lie algebra $\mathcal{G}$ corresponds a unique simply connected compact Lie group $\widetilde{G}$. Its irreducible representations are labelled by a set of ℓ non negative integers placed at the vertices of the Dynkin diagram. The center of the group $\widetilde{G}$ is :

$$Z_{\ell+1} \text{ for } SU(\ell+1) = A_\ell \ , \quad Z_2 \text{ for } Spin(2\ell+1) = B_\ell \qquad (15)$$

$$1 \text{ for } Sp(2\ell) = C_\ell \ , \ Z_4 \text{ for } Spin(4k+2) = D_{2k+1}, \ Z_2^2 \text{ for } Spin(4k) = D_{2k}$$

The other groups G with the same Lie algebra $\mathcal{G}$ are quotient $\widetilde{G}/F$ of $\widetilde{G}$ by a finite subgroup of the center. So its irreducible representations form a subset of those of $\widetilde{G}$. More generally a compact Lie group H is of the form :

(*) See also O'Raifeartaigh and Ruegg lectures.

(**) For other dimensions one has to add the five exceptional Lie algebras : G_2, F_4, E_6, E_7, E_8 . The series B,C are defined for $\ell \geq 2$ and D for $\ell \geq 4$. Moreover, we have the group isomorphisms : $SU(2) = Spin(3)$, $SU(2) \times SU(2) = Spin(4)$, $Spin(5) = Sp(4)$, $SU(4) = Spin(6)$.

$$H = N/F \qquad N = U(1)^k \times (\times_{i=1}^{j} \tilde{K}_i) \ , \quad F \ \text{finite} < C(N) \qquad (16)$$

i.e. N is the direct product of k Abelian $U(1)$ and j simply connected compact simple groups and F is a finite subgroup of the center of N. The irreps of H are the tensor products of those of the factors of N for which F is represented trivially. Example : (e.g. see [6], [9a]).

$$U(n) = \frac{SU(n) \times U(1)}{Z_n} \ , \quad Z_n = \{(e^{2\pi ik/n} I_n, e^{-\pi ik/n}) \ , \ 0 \leq k < n\} \qquad (17)$$

so the irreps of $U(n)$ are labelled by the integers

$$a_i \geq 0 \ , \ a_1, a_2, \ldots, a_{n-1}, m \quad \text{with} \quad \sum_{k=1}^{n-1} ka_k + m \equiv 0 \bmod n \qquad (18)$$

For $SU(2)$ the tradition is to use the spin $t = \frac{1}{2} a$, where a is the Dynkin label and $2t+1 = a+1$ the dimension of the irreducible representation. It is also interesting to replace (17) by

$$U(n) = \frac{SU(n) \times R}{Z} \ , \quad Z \ \text{generated by} \ (e^{2\pi i/n} I_n, \alpha) \qquad (19)$$

because $SU(n) \times R$ is the universal covering of $U(n)$ and we wish to emphasize that α is an arbitrary real number, so there is no natural scale for the value of the real parameter; there is the relation (18) for quantum numbers. This relation is satisfied for the standard unified electroweak theory and for what we believe are exactly preserved gauge symmetry interactions, i.e. electrochromodynamics, see table 2. The gauge groups of these theories are respectively $U(2)$ and $U(3)$. Any symmetry group G of a grand unified theory (GUT) must contain $U(2)$ and $U(3)$ as subgroups; since the electromagnetic gauge is common to both, these subgroups have an intersection $U(1)$ and they generate a subgroup

$$S(U(3) \times U(2)) \quad \frac{SU(3) \times SU(2) \times U(1)}{Z_6(\xi)} \qquad (20)$$

where ξ , the generator of Z_6 is

$$\xi = (e^{2\pi i/3} I_3, -I_2, e^{-2\pi i(5(3)}) \qquad (20')$$

where $H = S(U(3) \times U(2))$ is the group of matrices :

$$H = \{ \left(\begin{array}{c|c} u & o \\ \hline o & v \end{array}\right) , \ u \in U(3) \ , \ v \in U(2) \ , \ (\det u)(\det v) = 1 \} \qquad (21)$$

This group H of rank 4 is a maximal subgroup of $SU(5)$ and it is an isotropy group of the $SU(5)$ adjoint representation $\overset{1\,0\,0\,1}{0\text{-}0\text{-}0\text{-}0}$ of dimension 24. So $SU(5)$ is the smallest possible GUT symmetry group and this model was proposed ten years ago [26]. It has very good features, but one rather inelegant : the 15 fermion states of one horizontal family (12 = 3 colors × 2 spin states × 2 quark states u,d + 2 for electron +1 for neutrinos) are in a reducible representation $\overset{0\,0\,0\,1}{0\text{-}0\text{-}0\text{-}0} \oplus \overset{0\,1\,0\,0}{0\text{-}0\text{-}0\text{-}0}$ of dimension 5+10 of $SU(5)$.

Table 2. Multiplet of particles; q is the electric charge.

U(2) symmetry: representation t,y with $2t+y = 0 \mod 2$; $q = t_z + \frac{1}{2} y$

irrep	1,0 + 0,0	$\frac{1}{2}$,−1	$\frac{1}{2}$, 1	0,2	0,−2	$\frac{1}{2}$,1
particle	$\gamma W^+ W^- Z^o$	$\nu_L \varepsilon_L^-$	ν_R, ε_R^+	ε_L^+	ε_R^-	Higgs.

U(3) symmetry: representation c_1,c_2,x ; $c_1+2c_2+x \equiv 0$ mod ; $q = \frac{x}{3}$

1,0,2	1,0,−1	0,1,−2	0,1,1
u	d	$\bar{u}$	$\bar{d}$

SU(5) symmetry, irrep $a_1 a_2 a_3 a_4$ reduces on H into $c_1,c_2;2t,m$ with $c_1+2c_2 \equiv m \mod 3$, $2t \equiv m \mod 2$, $q = t_z + \frac{m}{6}$

irrep $\bar{5}$: $(0,0,0,1) = (0,1;0,2) + (0,0;1,-3)$

$(\bar{d}_L)$ $\quad \nu_L, \varepsilon_L^-$

irrep 10 : $(0,1,0,0) = (0,1;0,-4) + (0,0;0,6) + (1,0;1,1)$

$(\bar{u})_L$ $\quad \varepsilon_L^+$ $\quad u_L+d_L$

However, if the recent possible observations of neutrino oscillations (in Bugey and CERN) or of a neutrino mass (in Moscow) are confirmed, the neutrino must also have two states, so each fermion family require a 16 dimensional representation. It is a hard problem to find a G which contains the exact number of families (presently believed to be three); provisorily the simplet extension of SU(5) symmetry is up to Spin(10) , the covering of SO(10), which has two complex conjugate irreducible spinor representations $\overset{0\;0\;0}{\circ\!-\!\circ\!-\!\circ}\!<^{\circ\;1}_{\circ\;0}$ and $\overset{0\;0\;0}{\circ\!-\!\circ\!-\!\circ}\!<^{\circ\;0}_{\circ\;1}$ of dimension 16. This is well known among high energy physicists. The adjoint representation $\overset{0\;1\;0}{\circ\!-\!\circ\!-\!\circ}\!<^{\circ\;0}_{\circ\;0}$ is of dimension 45; so this is the number of gauge bosons. In which representation should the Higgs scalar be in order to break the symmetry on the subgroup H of equation (21) ? I am working on this unsolved problem with Ömer Kaymakcalan(*), K.C. Wali, L. O'Raifeartaigh, W.D. McGlinn. I sketch here the method for solving it. On the ground of physical elegance we consider only representations of small dimension d (say $d < 100$). They are

irrep	(00000),	(10000),	(20000),	(01000),	(00010),	(00001)
d	1	10	54	45	16	$\overline{16}$

where $(a_1 a_2, a_3, a_4, a_5)$ labels the representation $\overset{a_1\;a_2\;a_3}{\circ\!-\!\circ\!-\!\circ}\!<^{\circ\;a_4}_{\circ\;a_5}$

To obtain the observed breaking on H , one has to choose a reducible representation. The isotropy subgroups of a direct sum of inequivalent irreps are the intersections of the isotropy groups of the irreps. A more efficient method to compute the isotropy subgroups of the direct sum of irreps carried by the space $E = E_1 + E_2$ is to look for the isotropy subgroups of E_1 and study their action and corresponding isotropy subgroups on E_2 .

Results added in April when these notes have been written : H is an isotropy subgroup of the representation 45 + 54 , and corres-

(*) This gifted young physicist died of illness in Syracuse, N.Y. (USA) two days after I was giving this lecture.

ponding to the absolute minimal of a Higgs potential (depending on 11 parameters). However, there are two distinct conjugate classes of Spin 10 subgroups isomorphic to H . One obtains the wrong class : this [H] is not < [SU(5)] . The right class [H] appears as isotropy subgroups of the representations : a) $54 + 16 + \overline{16}$, b) $45 + 16 + \overline{16}$. In case a), the minimum of the Higgs potential covers an infinity of orbits and there are pseudo Goldstone bosons. The case b) gives a good solution of the problem. This solution has already been found in [27], [28]. It is quite elegant. Indeed it uses only two types of representations. The adjoint one for the spin 1 gauge bosons and spin 0 Higgs bosons. The spinor representations for the Fermions (quarks and leptons) and the rest of the Higgs bosons. So this model presents some remnants of a supersymmetry.

REFERENCES

[1] Doncel, M.G., Michel, L., Six, J., Interview de Eugen P. Wigner sur sa vie scientifique, Archives internationales d'histoire des sciences, 34 (1984) 177-217 (n°112, juin 1984).

[2] Wigner, E.P., English translation "Group Theory and its application to the quantum mechanics of atomic spectra", Academic Press, 1959, New-York.

[3] Michel, L., Mozrzymas, J., a) Application of Morse Theory to the symmetry breaking in the Landau Theory of second order phase transitions, VIth International Colloquium "Group Theoretical Methods in Physics", Tübingen 1977, Lecture Notes in Physics 79, 447-461, Springer (1978). b) Weak equivalence of irreducible representations of little space groups, Match (= Communications in Mathematical Chemistry) 10 (1980) 223-226.

[4] International Tables for Crystallography, Reidel 1983, Dordrecht.

[5] Schwarzenberger, R.L.E., Proc. Camb. Phil. Soc. 72 (1972) 325-349.

[6] Michel, L., Symmetry Defects and broken Symmetry Configurations. Hidden Symmetry, Rev. Mod. Phys. 52 (1980) 617.

[7] Montgommery, D., Yang, C.T., Trans. Am. Math. Soc. 87 (1958) 284-297.

[8] Mostow, G., Ann. Math. (1957) 432 and 513.

[9] Michel, L., a) Invariance in Quantum Mechanics and Group Extensions, Istambul Summer School, 16 juillet - 4 août 1962, Gordon and Breach (New-York) 1964. b) Relativistic invariance and internal symmetries, 1965 Brandeis Summer Institute in Theoretical Physics, Vol.I, p.247, Gordon and Breach, New-York 1966. c) Relations entre symétries internes et Invariance relativiste, Cargèse 1965 lectures in theoretical Physics, p.409, Gordon and Breach, New-York 1966.

[10] Jarić, M., Michel, L., Sharp, R.T., Invariant formulation for the zeros of covariant vector fields, Proceedings, Group Theoretical Methods in Physics XI, Lecture Notes Phys. 180, 317-318, Springer (1983) and J. Physique 45 (1984) 1.

[11] Michel, L., Minima of Higgs-Landau Polynomials, p.157-203 in Regards sur la Physique Contemporaine, Edition CNRS (Paris 1980).

[12] Jaric, M., in Group Theoretical Methods in Physics (IX), Lecture Notes in Physics 135, 12.

[13] Ascher, E., J. Phys. 10 (1977) 1365.

[14] Mukamel, D., Jaric, M.V., Phys. Rev. B 29 (1984) 1465.

[15] Jaric, M.V., Phys. Rev. Lett. 51 (1983) 2073.

[16] Abud, M., Anastaze, G., Eckert, P., Ruegg, H., Phys. Lett. 142 B 1984, p.371.

[17] Burzlaff, M., O'Raifeartaigh, L., To appear in Phys. Lett. B.

[18] Bieberbach, L., Math. Ann. 72 (1912) 400.

[19] Brown, H., Bülow, R., Neubüser, J., Wondratschek, H., Zassenhaus, H., Crystallographic groups of four dimensional space, John Wiley, New-York, 1978.

[20] Schwarzenberger, R.L.E., N-dimensional crystallography, Pitman Publ., London, 1980.

[21] Michel, L., Phys. Rev. B 29 (1984) 2777.

[22] Michel, L., in Group Theoretical Methods in Physics, p.162-184, World Scientific, Singapore 1984.

[23] Michel, L., Toledano, J.C., Phys. Rev. Lett.

[24] Toledano, J.C., Michel, L., Toledano, P., Brézin, E., Fixed points and stability for anisotropic systems with 4-component order parameters, Phys. Rev. B (to appear).

[25] Brézin, E., Le Guillou, J., Zinn-Justin, J., Phys. Rev. B, 10 (1974) 892.

[26] Georgi, H., Glashow, S.L., Phys. Rev. Lett. 32 (1974) 438.

[27] Bucella, F., Ruegg, H., Savoy, I.A., Phys. Lett. 94 B (1980) 491.

[28] Yasue, M., Phys. Rev. D 24 (1981) 1005.