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Nonlinear Group Action. Smooth Action of Compact Lie Groups on Manifolds

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Abstract

Linear representations of groups is a particular case of group action which is well known to physicists. The aim of these two lectures is to give the basic concepts for general group action.

In section 1 we consider the action of (abstract) groups on sets. In section 2 we study the homogeneous spaces of a given group. Section 3 is devoted to continuous actions of topological groups on topological spaces. The last section gives some results on the important particular case of smooth action of compact Lie groups on manifolds. For lack of time, no physical applications are given, but some references are provided.

Contents

1. GROUP ACTION ON SETS

1.1. Definitions

Let $\mathscr{P}(E)$ be the group of permutations of the elements of a set E. Each action of a group G on E is given by a group homomorphism

$$G \xrightarrow{f} \mathscr{P}(E). \tag{1}$$

Often, instead of $f(g)[x]$, we shall simply denote by $g.x$ the transform of $x \in E$ by $g \in G$.

An action of G on E can also be given by a map

$$G \times E \xrightarrow{\Phi} E \tag{2}$$

which satisfies

$$\Phi(e, x) = x \qquad \Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x). \tag{2'}$$

Indeed, $x \overset{\varphi_g}{\rightsquigarrow} \Phi(g, x)$ is a permutation of E whose inverse is $\varphi_{g^{-1}}$ and $g \rightsquigarrow \varphi_g$ is the homomorphism f in (1).

Let us introduce some vocabulary.

G acts *effectively* on E if $\operatorname{Ker} f = \{e\}$ (trivial) $\Leftrightarrow g \neq e$ implies $g.x \neq x$ for some $x \in E$.

G acts *freely* on E if $g \neq e$ implies $g.x \neq x$ for all $x \in E$.

G acts *transitively* on E if for all $x, y \in E$, $\exists\, g \in G$ such that $y = g.x$. In this case E is called a homogeneous space of G.

The set of fixed points, which we denote by E^G, is defined by

$$E^G = \{x \in E,\ \forall\, g \in G,\ g.x = x\}.$$

Of course, if $E^G = E$, G acts *trivially* on $E (\Leftrightarrow \operatorname{Ker} f = G)$.

1.2. Natural Transfer of Actions

Given G actions $f_1, \ldots, f_n$ on sets $E_1, \ldots, E_n$, then there are well-defined "natural" actions of G on the sets one can form from the E_i:

(a) G acts on $E_1 \times E_2$

$$g \,.\, (x_1, x_2) = (g.x_1, g.x_2). \tag{3}$$

(b) Let $\mathrm{Maps}(E_1, E_2)$ be the set of maps from E_1 to E_2. Then for any $\theta \in \mathrm{Maps}(E_1, E_2)$ and $g \in G$, the transform $g.\theta$ must satisfy the commutative diagram of maps

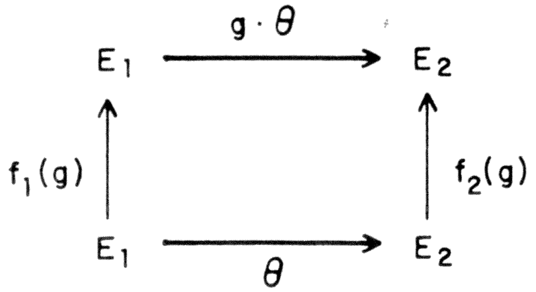

Diagram 1

Thus it is defined by

$$(g.\theta)(x_1) = g.(\theta(g^{-1}.x_1)). \tag{3'}$$

(c) If G acts on disjoint (i.e. $E_i \cap E_j = \varnothing$) sets, its natural action on the union $\cup_i E_i$ is obvious.

1.3. Equivariant and Equivalent G-Actions

Given G, we define the "morphisms" between its actions and, as a particular case, the isomorphisms, and consider isomorphic actions as equivalent. The action of G on E_1 is equivariant to the action of G on E_2 if $\mathrm{Maps}(E_1, E_2)^G$ is not empty, i.e. if there exists $\theta \in \mathrm{Maps}(E_1, E_2)$ invariant (such that $g.\theta = \theta$) for any $g \in G$. Then Diagram 2 of maps is commutative for all $g \in G$:

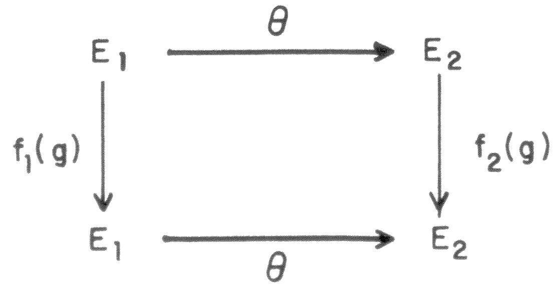

Diagram 2

$$f_2(g) \circ \theta = \theta \circ f_1(g). \tag{4}$$

We say that θ is an equivariant map. If furthermore θ is a bijective map (i.e. one-to-one and onto) the two actions of G are isomorphic. We shall also say that E_1 and E_2 are isomorphic G-spaces.

1.4. Little Group = Isotropy Group = Stabilizer of $m \in E$

This is the set $\{g \in G, g.m = m\}$. It is a subgroup of G which we denote by G_m.

1.5. The Orbit of $m \in E$

This is the set $\{g.m,\ g \in G\}$. We denote it by $G.m$. We shall denote by ψ_m the map $G \xrightarrow{\psi_m} E$, $\psi_m(g) = g.m$: Orbit of m = Image of ψ_m. When the group acts transitively there is only one orbit.

Lemma

If m' and m are on the same orbit, their little groups are conjugate.

More precisely: If

$$m' = g.m, \quad \text{then} \quad G_{m'} = gG_mg^{-1}. \tag{5}$$

1.6. Strata

If two points m' and m have conjugate little groups, they need not be on the same orbit. By definition they are on the same *stratum*. When the group acts freely, there is only one stratum ($\forall\, m \in E, G_m = \{e\}$). When E^G is not empty it is a stratum. We denote the stratum of m by $S(m)$.

1.7. Orbit Space

Being on the same orbit is an equivalence relation between the elements of E. Thus E is partitioned into orbits. The set of orbits is called the orbit space. It is usually denoted by E/G. We shall use π for the canonical map $E \xrightarrow{\pi} E/G = \pi(E)$.

1.8. Space of Strata

Having conjugate little groups is an equivalence relation for the elements of E. Thus E is partitioned into strata. We denote by σ the canonical map of E into the space of strata. This map factorizes:

$$\sigma = \tau \circ \pi. \tag{6}$$

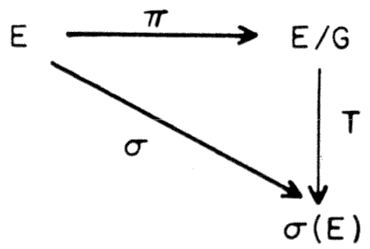

Diagram 3

1.9. Examples

(a) *SO*(2) *action on* S_2 (The 2-dimensional sphere)

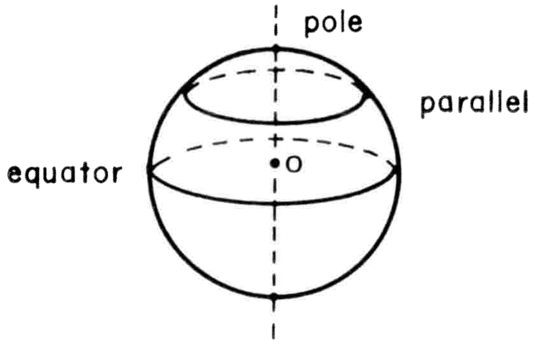

Figure 1

The orbits are the parallel circles (little group = $\{1\}$) and the two poles (= fixed point, little group = $SO(2)$). So there are two strata.

(a′) $SO(2) \times Z_2$ *action on* S_2. The group of (a) is enlarged with the symmetry through the center 0. There are no fixed points. There are now three strata:

(i) the generic one, open, dense: each orbit consists of the two parallels with the same N and S latitude.

(ii) one with one orbit: the 2 poles (little group $SO(2)$).

(iii) one with one orbit: the equator (the little group has two elements).

(a″) $SO(2)$ *action on* P_2. The real projective plane: action obtained by the equivariant map $S_2 \longrightarrow P_2$ which identifies points symmetric through 0. 3 strata, images of the 3 strata of (a′) (not of (a)).

(b) $SU(3)$ *action on* S_7. The unit sphere of the adjoint representation (= octet space). Note that $SU(3)$ does not act effectively, but the adjoint group $SU(3)/Z_3$ does. Same image as (a) but the 2 poles are four-dimensional orbits (little group $U(2)$) and the generic stratum has 6-dimensional orbits (little group $U_1 \times U_1$).

(b′) Aut $(SU(3)/Z_3)$ *action on* S_7. $SU(3)$ or $SU(3)/Z_3$ has only one class of outer automorphism (it corresponds in physics to charge conjugation). This example is similar to (a′); the "equator" becomes the 6-dimensional orbit of the "roots" of the Lie algebra. The poles are replaced by the 4-dimensional set of "pseudo-roots" (= roots of the symmetric algebra whose structure constants are the d_{ijk} of Gell–Mann); see L. Michel and L. A. Radicati, *Ann. Phys.*, **66**, 758 (1971).

(c) *Action of* $GL(n, \mathbb{C})$ *on* $\mathscr{L}(\mathscr{H}_n)$. Let $\mathscr{H}_n$ be the n-dimensional complex vector space, $\mathscr{L}(\mathscr{H}_n)$ the set of linear operators (= $n \times n$ complex matrices) on $\mathscr{H}_n$: that is an n^2-dimensional vector space. The invertible operators of $\mathscr{L}(\mathscr{H}_n)$ form a group, the general linear complex group in n dimensions: $GL(n, \mathbb{C})$. This group acts (linearly) on the vector space $\mathscr{L}(\mathscr{H}_n)$ according to the law

$$g \in GL(n, \mathbb{C}), \quad \mathscr{L}(\mathscr{H}_n) \ni x \rightsquigarrow gxg^*$$

(where * is the Hermitian conjugation of matrices). We leave as an exercise the study of the strata. We note that the Hermitian operators $x = x^*$, which form a real vector space $\mathscr{E}_{n^2}$ of dimension n^2, are transformed into themselves. We also leave as an exercise the study of the action of $GL(n, \mathbb{C})$ on $\mathscr{E}_{n^2}$. We just point out that the strictly positive (Hermitian) matrices x form one orbit which contains the unit matrix I, whose little group is $U(n)$, the group of unitary matrices of rank n (i.e. $n \times n$ matrices). Another interesting action is $x \rightsquigarrow gxg^{-1}$.

(c′) *Action of* $SL(n, \mathbb{C})$ *on* $\mathscr{L}(\mathscr{H}_n)$, and $\mathscr{E}_{n^2}$ = Hermitian of $\mathscr{L}(\mathscr{H}_n)$. The group $SL(n, \mathbb{C})$ is the subgroup of $GL(n, \mathbb{C})$ formed by the matrices of determinant unity.

(d) *Action of the connected Lorentz group on space–time.* This action is well known to the audience. The orbit space is given by Figure 2a. There are four strata, the origin 0, the light cone $-O$, its inside, its outside. This action is identical to that studied in (c′) for $n = 2$: action of $SL(2, \mathbb{C})$ on $\mathscr{E}_{n^2} = \mathscr{E}$. The center Z_2 of $SL(2, \mathbb{C})$ acts trivially and the quotient group $SL(2, \mathbb{C})/Z_2$, which acts effectively, is isomorphic to the connected Lorentz group.

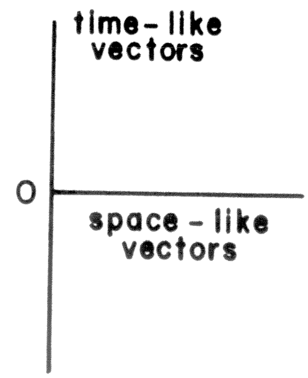

Figure 2a

The strictly positive and strictly negative matrices form the stratum of time-like vectors. The corresponding little groups are conjugates of $SU(2)$.

(d′) *Action of the complete group* $\mathscr{L}$ *on space–time.* The same four strata, but the orbit space is reduced to that of Figure 2b. Note that the little group $\mathscr{L}(p)$ of a time-like vector p is conjugate to $O(3)$, the three-dimensional orthogonal group, which is the little group for the time axis.

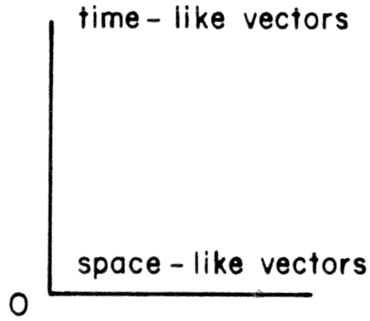

Figure 2b

(e) *Action of the connected Poincaré group of space–time.* The group generated by the translations and the connected Lorentz group transformations is the connected Poincaré group. The space–time is one orbit of this group. The stabilizer of any point 0 is the Lorentz group leaving 0 fixed.

(f) *Action of* $\mathscr{L}(\boldsymbol{p})$, *the little group of* $\boldsymbol{p}$ *on three-particle phase space* $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3$. As we have seen in (d′), $\mathscr{L}(\boldsymbol{p})$ is conjugate to $O(3)$. There are two strata: The generic stratum S; $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ are linearly independent, the 2-element little group is generated by the symmetry through the plane defined by $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$. The exceptional stratum S' : when $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ are linearly dependent; the little group is conjugate to $O(2)$. The orbit space is the Dalitz plot; $\pi(S')$ is its boundary.

We leave the case when two (or three) particles are identical to be handled as an exercise.

(g) *Any group G acts on itself* (= its set of points) *by different actions*:

(i) by left translations $x \overset{g}{\rightsquigarrow} gx$, so G acts freely and transitively on itself.

(ii) by conjugation $x \overset{g}{\rightsquigarrow} gxg^{-1}$. In this case the orbits are the conjugation classes. This action of G on itself is by automorphisms: $G \longrightarrow \text{Aut } G$, where Ker f is the center of G (it is also the stratum of fixed points) and image of f is the group of inner automorphisms.

(g′) *H, subgroup of G acts on G by left translation.* The orbits are the right cosets of H. There is only one stratum and H acts freely on G.

1.10. Exercises

(i) If G does not act effectively on E, then Ker $f = \bigcap_{m \in E} G_m$. It is the largest invariant subgroup contained in any little group G_m.

(ii) If H is a subgroup of G, the action of G on E defines an action of H on E. If, furthermore, H is an invariant subgroup of G, this defines an action of G on the orbit space E/H and also of the quotient group G/H on the orbit space E/H. Note that $\pi_H : E \xrightarrow{\pi_H} E/H$ defined by the action of H on E is an equivariant map of G actions.

The reader has already twice applied this exercise in very simple cases: to pass from example (a′) to (a″), where $G = SO(2) \times Z_2$, $H = Z_2$, $E = S_2$, $E/H = P_2$; and in (d), $G = SL(2, \mathbb{C})$, H is its center Z_2 which acts trivially on space–time.

2. HOMOGENEOUS SPACES OF (ABSTRACT) GROUPS

2.1. Classification of G-Homogeneous Spaces

Given a subgroup H of G, the relation $a^{-1}b \in H (\Leftrightarrow b \in aH)$ is an equivalence relation, the equivalence classes are the left cosets of H, and we will denote the quotient space (= coset space) by $[G:H]$.

By left translation G acts on $[G:H] \ni gH \rightsquigarrow gaH$. This action is transitive, so $[G:H]$ is an orbit of G: the little group of $H \in [G:H]$ is H itself.

Let E be a G-homogeneous space with little-group conjugate to H, $\ni m \in E$, $G_m = H$. Let $m' = g_1.m = g_2.m$; then $g_2^{-1}g_1 \in H$, so g_2 and g_1 are in the same left coset of H, i.e. $g_1H = g_2H$. The correspondence $g.m \rightsquigarrow gH$ is therefore a map $E \xrightarrow{\theta} [G:H]$;

it is equivariant: $g_1\theta(g_2.m) = g_1g_2H = \theta(g_1g_2.m)$;

it is one-to-one: $\theta(g_1m) = \theta(g_2m) \Leftrightarrow g_1H = g_2H \Leftrightarrow g_2^{-1}g_1 \in H \Leftrightarrow g_2^{-1}g_1m = m$, i.e. $g_1m = g_2m$;

it is onto: every gH is the image of a $g.m$.

So the G-homogeneous space E is isomorphic to the "standard" G-homogeneous space $[G:H]$ with G-action by translation. It is easy to see that $[G:H]$ and $[G:gHg^{-1}]$ are isomorphic G-homogeneous spaces. Hence we have proved the

Theorem:

There is a natural bijective map between the classes of isomorphic G-homogeneous spaces and the classes of conjugate subgroups of G.

A class of isomorphic G-homogeneous spaces is also called a *type* of G-orbits.

2.2. Partial Order on the Set $\mathcal{O}_G$ of Types of G-Orbits

We denote by (H) the class of subgroups of G conjugate to H.

With the inclusion $(H_1 \subset H_2)$, the subgroups of a group form a lattice. The classes of conjugate subgroups also form a lattice. This lattice structure can also be applied to the set $\mathcal{O}_G$, but it is done in the usual language by reversing the order: the larger a subgroup H of G, the smaller is the orbit $[G:H]$; indeed, for finite groups (Card H) . (Card $[G:H]$) = Card G.

In an action of G on E, a stratum is the union of all orbits of the same type. So the stratum space (see Section 1.8) $\sigma_G(E)$ can be identified with a subset of $\mathcal{O}_G$. One can therefore speak of maximal or minimal strata. For example, if there are fixed points, they form the unique minimal stratum.

2.3. Morphisms of Homogeneous Spaces. Imprimitivity

$$E_1 \xrightarrow{\theta} E_2$$

is an equivariant map from E_1 to E_2. θ is always surjective (= onto). If it is not injective (= one-to-one) $(H_1) \leqq (H_2)$, e.g.

$$\theta^{-1}(H_2) = [H_2 : H_1^{\sharp}] \subset [G : H_1].$$

If H_2 is a maximal proper subgroup of G (i.e. $H_2 \subset H \subset G$ implies for the subgroup H either $H = H_2$ or $H = G$), the action of G on E_2 is said to be *imprimitive*. If (H_1) is strictly smaller than (H_2), H_2 a maximal subgroup, the inverse images $\theta^{-1}(m_2)$ of elements of E_2 are called imprimitivity classes of E_1.

Example: The connected Lorentz group action on a light cone (without summit). The generatrices are imprimitivity classes. The little groups are isomorphic to $E(2)$, the 2-dimensional Euclidean group. They are not maximal in Lorentz, the maximal group $TR(2)$ is isomorphic to $E(2)$ and dilations

$$\left(TR_2 = \left\{ \text{triangular matrices} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \text{in } SL(2, \mathbb{C}) \right\} \right).$$

The set of points at infinity for the light cone (or any—real or imaginary—mass hyperboloid = asymptotia ...) can be identified with the orbit $[\mathscr{L} : TR(2)]$; it is homeomorphic to the sphere S_2.

2.4. Remark on the Isomorphy of G-Spaces and G-Orbits

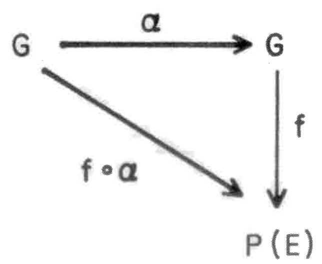

Diagram 4

Let $\alpha \in \operatorname{Aut} G$. If f of Diagram 4 defines an action of G on E, $f \circ \alpha$ defines another action. Are the two actions equivalent? The answer is yes if α is an inner automorphism of G; let $a \in G$ be an element which realizes α, i.e.

$$\forall g \in G \qquad aga^{-1} = \alpha(g). \tag{7}$$

Then $f(a)$ is an equivariant map on E which establishes the equivalence of the two actions. Indeed, for any $g \in G$, using (7) and the fact that f is a group homorphism:

$$f(\alpha(g)) \circ f(a) = f(aga^{-1}) \circ f(a) = f(a) \circ f(g) \tag{8}$$

If α is not an inner automorphism of G, it may happen that the two G-actions are inequivalent although they produce on E the same orbits and the same strata. But some orbits may be nonisomorphic G-homogeneous spaces. Indeed, consider the case when E is one orbit, isomorphic to $[G:H]$ for the action f. For the action $f \circ \alpha$, it is isomorphic to $[G:\alpha^{-1}(H)]$. If $\alpha^{-1}(H)$ and H are not conjugate in G, these two homogeneous spaces are nonisomorphic. Indeed, there is no $\varphi \in \mathscr{P}(E)$ such that

$$\forall g \in G \qquad f(\alpha(g)) \circ \varphi = \varphi \circ f(g). \tag{9}$$

Let $m \in E$ such that $G_m = H$ for the action f. Its little group for the action $f \circ \alpha$ is $\alpha^{-1}(H)$. If there is a φ satisfying (9), apply both sides of this equation to m for all $h \in H$; we obtain:

$$\alpha f(a(h))\, \varphi(m) = \varphi(m),$$

so that H is also the little group of $\varphi(m)$. Since points on the same orbit have conjugate little groups, H has to be conjugate to $\alpha^{-1}(H)$, the little group of m; this is absurd.

2.5. Physical Application

Generalization of the McGlinn Theorem. McGlinn (*Phys. Rev. Lett.*, **12**, 467 (1964)) proved that if $G = S.P$ (i.e. $\forall g \in G$ can be decomposed uniquely into a product $g = sp$, $s \in S$, $p \in P$) where P is the Poincaré group ($P = T.L$) and S a semisimple Lie group, and if $\forall s \in S, \forall l \in L, sl = ls$, then $G = S \times P$, the direct product. Michel (*Phys. Rev.*, **137B**, 405 (1965)) extended this result to the case when S can also be an arbitrary simple group or direct product of simple groups, under the weaker hypothesis: there is a $p_0 \in P$, $p_0 \notin T$ such that $\forall s \in S, s^{-1}p_0 s \in P$. As we saw in Section 2.1, G acts on $[G:P]$, the set of G-left cosets of P, by the action: $sP \overset{f(g)}{\rightsquigarrow} gsP$. The hypothesis $p_0 sP = sP$ shows that p_0 acts trivially, so that $p_0 \in K = \operatorname{Ker} f \subset G$. And $K \cap P$ is an invariant subgroup of P which contains p_0. It must be P itself, so that $P \subset K$; i.e. $\forall s \in S, \forall p \in P, s^{-1}ps \in P$ so P is an invariant subgroup of G, and there is a homomorphism $S \xrightarrow{\varphi} \operatorname{Aut} P/P = R$. Since S is a product of simple groups (or a semisimple Lie group), φ is trivial and G is isomorphic to the direct product $S \times P$. Hint: $G = S' \times P$, $s' = sq(s^{-1})$ where $sps^{-1} = q(s)\,pq(s^{-1})$, and $q: E \xrightarrow{q} \text{Inner Aut } P = P$.

3. CONTINUOUS ACTION OF A TOPOLOGICAL GROUP G ON A TOPOLOGICAL SPACE X

3.1. Introductory Remarks

If E has a mathematical structure, it has a group of automorphisms Aut E for this structure. The elements of Aut E have various names, e.g.:

E	Name of elements of Aut E
set	permutation = bijective map
ordered set	order preserving map
topological space	homeomorphism
metric space	isometry
vector space	invertible linear operator
complex Hilbert space	unitary operator
real Hilbert space	orthogonal operator
differentiable manifold	diffeomorphism

and so on.

One considers only the actions of G given by the group homomorphisms $G \xrightarrow{f} \text{Aut } E$.

Example: If E is a vector space, f is a linear representation of G. One should distinguish the different structures on the same set, for example:

If E is the space–time *vector space*, i.e. the space of energy momenta, the vector zero must be invariant and the group which preserves the Minkowski scalar product is the Lorentz group. If E is the affine space–time (i.e. of x and t coordinate), then translation belongs also to Aut E, which is the full Poincaré group. If the space–time E is defined just as a set with the partial order relation due to causality:

$$x < y \Leftrightarrow y - x \in V^+,$$

the inside of the future light cone, it is then quite remarkable that (without linear and even topological structure for E) Aut E is the orthochronous Poincaré group with the dilations added (E. C. Zeeman, *J. Math. Phys.*, **5**, 490 (1964)).

This point of view is too general for our purpose, because we wish also to take into account a richer structure on the group G than that of the group law. For instance, G can be a topological group; then we do not want to consider as possible actions of G on the topological space X all homomorphisms of G into the group of homeomorphisms of X. We shall use the alternative way (eqns. (2) and (2′)) of defining the action of G and require, for example, that the map Φ preserve the structure on G and X.

3.2. Definition and General Remarks

A continuous action of a topological group G on a topological space X is defined by a continuous map

$$G \times X \xrightarrow{\Phi} X$$

which satisfies (2′).

If G is a topological group, the homogeneous space $[G:H]$ is a topological space with the quotient topology: i.e. the open sets of $[G:H]$ are all the subsets S of $[G:H]$ such that $\rho^{-1}(S)$ is an open set of G, where ρ is the canonical map $G \xrightarrow{\rho} [G:H]$. By definition, the topology on $[G:H]$ is the finest one such that ρ be continuous.

Consider the continuous action $G \times X \longrightarrow X$. On any orbit $G.x (x \in X)$, one has to distinguish two topologies: That of the homogeneous space $[G:G_x]$ and that of topological subspace $G.x \subset X$ (the open sets of $G.x$ are then the intersections of $G.x$ and open sets of X).

If the canonical bijective map $[G:G_x] \overset{l}{\rightsquigarrow} G(x)$ is not continuous (i.e. if the topology of $[G:G_x]$ is not equal to or finer than that of $G.x$), then the G-action is not continuous.

The orbit space X/G is also a topological space with the quotient topology, defined by the canonical map $\pi : X \xrightarrow{\pi} X/G$.

Exercise

Let $K \subset H \subset G$, K an invariant subgroup of G. Then $[G:K]$ is also a topological group G/K and $[G:H]$ is homeomorphic to $[G/K:H/K]$.

3.3. Basic Facts on Continuous Group Action

$\Phi(g, .) = \varphi_g$ is a homeomorphism of X.

$\Phi(., x) = \psi_x$ is a continuous map from G to X.

Let L be a subset of G, S a subset of X, $LS = \{l.s, l \in L, s \in S\}$.

	G	X	L	S	LS
a)	topological	topological	arbitrary	open	open
b)	Hausdorff	Hausdorff	compact	compact	compact
c)	Hausdorff	Hausdorff	compact	closed	closed
d)	compact	Hausdorff	closed	closed	closed.

Note 1. Definition of X, Hausdorff: $\forall x, y \in X$, $\exists$ open neighborhood of x, V_x and open neighborhood V_y of y, $V_x \cap V_y = \emptyset$. Moreover, to be compact or to have compact subsets, X must be Hausdorff. Remark: a point of a Hausdorff space is a closed set.

Proofs

(a) Φ_e = homeomorphism $\to l.S$ open, $L.S = \bigcup_{l \in G}$ open sets = open set.

(b) $L.S$ is the continuous image of a compact set, product of two compact sets.

(c) Choose sets g'_α in L, s_α in S, $g_\alpha s_\alpha \to x \in \overline{LS}$ (closure of LS) L compact $\Leftrightarrow \exists$ a subset g_α in L, $g_\alpha \to l$, $\lim s_\alpha = \lim g_\alpha^{-1}(g_\alpha s_\alpha) = l^{-1}$, $x \in S$ since S closed, so $x \in LS$ and $\overline{LS} = LS$.

(d) L closed in G compact $\Rightarrow L$ compact $\Rightarrow$ (c).

Then for $L = G$, $S = \{x\}$ (one-point set, which is closed, $LS = G.x$, an orbit, so in case (d) the *orbits are closed*). They are also compact, since $G.x = \psi_x(G)$, the continuous image of a compact set.

In the general case (a), if S open, $G.S = \pi^{-1}(\pi(S))$ open, hence $\pi(S)$ open (by definition of the quotient topology), so that π is an open map. In case (d), S closed $\Rightarrow$ $\Rightarrow G.S = \pi^{-1}(\pi(S))$ closed, so that π is then also a closed map.

3.4. Particular Case when G Is Locally Compact and Compact and X Is Metrizable

A locally compact group G has a left-invariant Haar measure dg. If, furthermore, G is compact, then dg (which is also right-invariant) can be normalized by

$$\int_G dg = 1. \tag{10}$$

Let $\Delta(x, y)$ be a distance in X. It can be averaged by the Haar measure of the compact G into

$$\tilde{\Delta}(x, y) = \int_G \Delta(gx, gy)\, dg. \tag{11}$$

This is a G-invariant metric on X and it yields the same topology as Δ. The orbit space X/G is also metrizable, with the metric:

$$d(\pi(G.x), \pi(G.y)) = \operatorname*{Min}_{x' \in G.x;\ y' \in G.y} \tilde{\Delta}(x', y'). \tag{12}$$

The orbit space X/G is indeed a generalization of the meridian section of an $SO(2)$-invariant domain in our 3-dimensional space.

3.5. Physical Applications. Crystallography

Our space $\mathscr{E}$ as an affine space has for its automorphism group $IL(3, \mathbb{R})$, the inhomogeneous general linear group in 3 dimensions; it is the semidirect product of the translations T by $GL(3, \mathbb{R})$. We denote by $GL^+(3, \mathbb{R})$ the subgroup of index 2 of $GL(3, \mathbb{R})$ formed by the 3×3 real matrices of positive determinant; we denote by $IL^+(3, \mathbb{R})$ the semidirect product $T \wedge GL^+(3, \mathbb{R})$.

The Euclidean group $E(3)$ is a subgroup of $IL(3, \mathbb{R})$. A crystal is represented as a space lattice C of $\mathscr{E}$, and the crystal symmetry group is the subgroup $K \subset E(3)$ which transforms C into itself. An equivalent but more abstract definition of the crystallographic groups K is: a discrete subgroup of $E(3)$ such that the homogeneous space $[E(3):K]$ is compact. Two crystallographic groups K_1 and K_2 are considered as equivalent if they are isomorphic and their actions on $\mathscr{E}$ are equivalent. However, in order to distinguish between left and right, crystallographers have restricted the equivalence map θ of diagram 2 to be in $IL^+(3, \mathbb{R})$ and not in $IL(3, \mathbb{R})$. So there are 230 classes of equivalent crystallographic groups in 3 dimensions; among them 11 pairs are isomorphic: they are conjugate in $IL(3, \mathbb{R})$ but not in $IL^+(3, \mathbb{R})$. In 2 dimensions there are 17 classes of crystallographic groups. It was one of the famous Hilbert problems to know if the number of crystallographic classes is finite for any dimension n. The positive answer was given by Bieberbach in 1911.

4. SMOOTH ACTION OF A COMPACT LIE GROUP G ON A C^∞-MANIFOLD M

Smooth $= C^\infty =$ infinitely differentiable.

Then G and M are both (smooth) manifolds and the map defining the action $G \times M \xrightarrow{\Phi} M$, which satisfies (2′), is a morphism (= smooth map) of smooth manifolds.

4.1. General Results

Let $\Delta(x, y)$ be a Riemann metric on M. By averaging with the Haar measure of G one obtains a G-invariant Riemann metric and, as we have seen, M/G is metrizable. If in V_m, neighborhood of m, we choose geodesic coordinates, G_m transforms geodesics through m into geodesics through m, so G_m acts linearly in V_m.

However, there are stronger results:

(i) R. C. Palais (*Amer. J. Math.*, **92**, 748 (1970)): C^1 (= continuous action) $\Rightarrow C^\infty$ action when M is compact.

(ii) Myers and Steenrod (*Ann. Math.*, **40**, 400 (1939)): G acts isometrically $\Rightarrow G$ acts differentially.

For C^1-action: Mostow proved (*Ann. Math.*, **65**, 513 and 432 (1957)) *Theorem* 1: If M is compact, the number of strata is finite. *Theorem* 2: If the number of strata is finite, the G-action on M is equivariant, by an injective map, to a linear (orthogonal) representation of G on a finite dimensional vector space.

(iii) Palais (1961): When M is compact, the number $N(G, M)$ of nonisomorphic G-actions on M is at most countable, and there exists a k such that on the k-dimensional sphere S_k, $N(G, S_k)$ is infinite (countable).

(iv) Montgomery and Yang (1961) have proved for C^∞-action that there is a stratum which is open dense. More precisely, let $M_{r,c}$ be the set of $m \in M$ such that $\dim G_m = r$, number of connected components of $G_m = c$. Any connected component of $M_{r,c}$ is in one stratum only. The open dense stratum is $M_{r_{\max}, c_{\min}}$; it is the maximal stratum (i.e. the little group is *the* minimal one among all little groups which appear in the G-action on M; see Section 2.2). Let $t < r_{\max}$. Then

$$\bigcup_{r \leqq t,\, c} M_{r,c}$$

is closed and its dimension is $\leqq n - r_{\max} + t - 1 \leqq n - 2$.

References for (iii) and (iv) can be found in the review paper of D. Montgomery, "Compact Groups of Transformations", p. 43 of *Differential Analysis*, Bombay Colloquium (1964). A good reading for an introduction to the subject is Palais, "The Classification of G-Spaces", *Memoirs Amer. Math. Soc.*, No. 36 (1960).

(v) Consider the set $\mathscr{F}$ of real-valued, G-invariant smooth functions on M ($\forall g \in G, \forall m \in M, f(g.m) = f(m)$). We call an orbit $\mathscr{O}$ *critical* if for *all* $f \in \mathscr{F}$, $m \in \mathscr{O} \Rightarrow (df)_m = 0$. Then (L. Michel, *C. R. Acad. Sc., Paris*, **272**, 433 (1971)):

Theorem 1

An orbit is critical if and only if it is isolated in its stratum. (I.e., any connected component of this orbit is a connected component of its stratum). If a stratum has a finite number of orbits, it is closed and all its orbits are critical.

The importance of this theorem for physical problems blending an invariance by G and a variational principle is obvious. This theorem is a generalization of the intuitive result: A C^1 real even function $f(x)$ of the real numbers $\mathbb{R}$, $f(x) = f(-x)$, satisfies $df/dx|_{x=0} = 0$.

We shall introduce some essential tools for the proof of this theorem.

4.2. The Equivariant Retraction

Let $d(x, y)$ be the G-invariant Riemann metric on M. Let Q be a G-invariant submanifold of M. There exists a neighborhood V_Q such that

$$x \in V_Q, \qquad \operatorname*{Inf}_{y \in Q} d(x, y)$$

is well defined and unique. We denote by $r_Q(x)$ the point of Q for which this minimum is reached: this point is the foot of the geodesic passing through x and normal to Q. Since it is defined in terms of the metric and G acts isometrically, the *retraction* $V_Q \xrightarrow{r_Q} Q$ is equivariant, i.e.

$$\forall g \in G \qquad \Phi_g \circ r_Q = r_Q \circ \Phi_g. \tag{13}$$

Applying this equation to x, we see for $g \in G_x$ that

$$g \in G_x \Rightarrow g \in G_{r_Q(x)}. \tag{14}$$

So, if Q is the orbit $G.m$,

$$x \in V_{G.m} \Rightarrow G_x \leqq G_m \tag{15}$$

that is, for any $m \in M$, there is a neighborhood V_m such that for any $x \in V_m$, the little group G_x is equal or smaller (i.e. $G_x \subset G_m$ up to a conjugation). The submanifold $r_Q^{-1}(m)$ is called the (local) slice at m. We denote it by $N(m)$. In geodesic coordinates it is a linear manifold. As a particular case of (15),

$$x \in N(m) \Rightarrow G_x \subset G_m. \tag{16}$$

The set of points of $N(m)$ such that $G_m = G_x$ will be denoted by $F(m)$:

$$F(m) = N(m)^{G_m}. \tag{17}$$

Equation (16) implies:

$$S(m) \cap N(m) = F(m). \tag{18}$$

If $F(m) = m$, this is true also for all $m' \in G.m$ and we shall say that the orbit $G(m)$ is isolated in its stratum $S(m)$.

4.3. The Local Action of G_m

As we have seen, in a geodesic coordinate system G_m acts linearly on a neighborhood V_m of m, by an orthogonal (= real unitary) representation. So it leaves invariant a Euclidean scalar product (corresponding to the invariant metric) and the representation space decomposes into the direct sum of orthogonal subspaces.

$$V_m = \underset{\substack{\text{tangent plane to}\\ \text{the orbit}}}{T_m(G.m)} \oplus \underset{\text{slice}}{N(m)} \tag{19}$$

$$= \underbrace{T_m(G.m) \oplus F(m)}_{\substack{\text{tangent plane to}\\ \text{the stratum}}} \oplus K(m) \tag{19'}$$

($N(m)$, linear manifold is identical to $T_m(G_m)^\perp \subset T_m(M)$ in the tangent plane of M at m).

The linear representation of G_m on $T_m(G.m)$ depends only on G (It is the restriction of the adjoint representation of G to G_m, for the subspace of the Lie algebra $\perp$ to $\mathscr{G}_m$ for the Cartan–Killing metric).

The representation on $F(m)$ is trivial.

The representation on $K(m)$ does not contain the trivial representation.

For $m' \in S(m)$, $G_{m'} \sim G_m$ and the representation is equivalent. So (19) and (19') depends only on the stratum (at least $S(m)$ is connected).

4.4. Theorem on Critical Orbits of G-Invariant Functions

We now sketch the proof of Theorem 1. Let $f \in \mathscr{F}$, i.e. f is a G-invariant real-valued smooth function on M. Its differential at m, df_m, is in the cotangent plane to M at m. With the Euclidean local metric at m we use the dual notion, the gradient of f at m. The G-invariance of f implies that

$$(\operatorname{grad} f)_m \in T_m(G.m)^{\perp} = N(m) \tag{20}$$

and it has to be invariant under G_m, so that

$$(\operatorname{grad} f)_m \in F(m) \subseteq N(m). \tag{21}$$

Hence if $G.m$ is isolated in its stratum,

$$\forall f \in \mathscr{F}, \qquad \forall\, m \in G(m), \qquad (\operatorname{grad} f)_m = 0.$$

This proves the "if" of Theorem 1.

Conversely, if $F(m)$ has points other than m, i.e. $\dim F(m) > 0$ (since $F(m)$ is a linear manifold!), then one can explicitly build in V_m an invariant smooth function on M with compact support on $F(m)$ and a nonvanishing gradient (which is in $F(m)$ at m).

4.5. G-Invariant Gradient Vector Fields

Note that for any $f \in \mathscr{F}$, $(\operatorname{grad} f)_m \in T_m(S(m))$, the tangent plane to the stratum of m.

Therefore f and $f|_{S(m)}$, its restriction to the stratum $S(m)$, have the same gradient. Consider a minimal stratum (in the order defined in Section 2.2). Such a stratum is closed. If M *is compact*, a closed stratum $S(m)$ is compact so it has a finite number of (closed, compact) connected components. Let $S_0(m)$ be that of m. Either $S_0(m) \subset G.m$, and the orbit is isolated in its stratum, or the points of $S_0(m)$ belong to an infinite number of orbits. Then on the compact $S_0(m)$, for any $f \in \mathscr{F}$, $f|_{S_0(m)}$ reaches its maximum and its minimum on orbits on which its gradient vanishes. Since f has same gradient, we have (Michel, *loc. cit.*)*

Theorem 2

If $S(m)$ is compact and has an infinite number of orbits, every $f \in \mathscr{F}$ has at least $\operatorname{grad} f = 0$ on two different orbits of $S(m)$.

* As an application of this to Example 1.9(f), every $O(3)$-invariant smooth function on the 3-particle phase space Ω has at least two extrema, on the boundary of Ω. This is the case for smooth functions f defined on the Dalitz plot. Then $f_0\pi$ is defined on Ω.

However, these orbits generally depend on f. One can give more results on extrema of $f \in \mathscr{F}$ with an equivariant Morse theory (see A. G. Wassermann, *Topology*, **8**, 127 (1969)).

4.6. G-Invariant Vector Fields

It is easy to generalize to any G-invariant vector fields what we have done for gradient fields.

Decompose the vector field into two $\perp$ components, one normal to $T_m(G.m)$, hence in $F(m)$, the other in $T_m(G.m)$.

If $\dim F(m) = 0$, and if the representation of G_m on $T_m(G(m))$ does not contain the trivial one, then all G-invariant vector fields vanish on $G(m)$. If $T_m(G(m))$ contains a trivial representation of G_m, but the Euler characteristic of $G(m)$ is $\neq 0$, then each G-invariant vector field has to vanish on some orbit of $G(m)$.

4.7. Physical Applications

The directions of breaking of the hadronic internal symmetry by the electromagnetic, semileptonic and nonleptonic weak, and CP-violating interactions are on four critical orbits of the adjoint action of $(SU(3) \times SU(3)) \wedge (I, C, P, CP)$ on the unit sphere S_{15}. In this action there are twelve strata, and five critical orbits (each one forms a stratum). For lack of time we just refer to: L. Michel and L. A. Radicati, *Ann. Phys.*, **66**, 758 (1971). Let us end these two lectures by a question: what is the use of the fifth critical orbit?