The description of the symmetry of physical states and spontaneous symmetry breaking

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Pierre Curie was one of the first physicists to study the role of symmetry in physics. From the deep and important paper he published in 1894 in the *Journal de Physique Théorique et Appliquée*, 3e série, p. 393, I want to quote the two famous sentences :
« Lorsque certaines causes produisent certains effets, les éléments de symétrie des causes doivent se retrouver dans les effets produits. »
« Lorsque certains effets révèlent une certaine dissymétrie, cette dissymétrie doit se retrouver dans les causes qui lui ont donné naissance. »
In this talk we will study how symmetries are implemented in the physical formalism. Since these Curie's principles are well understood, we will be mainly interested by their violations. The natural occurrence of spontaneous symmetry breaking suggests that physical laws might have more symmetry than physical states. This is the leading thread of the present tentatives for the unification of fundamental interactions in physics.

The symmetries of a physical state, or of a physical problem, or of a physical theory form a group : the symmetry group G, and this group appears in physics through its action on the physical formalism.

Fifty years ago were published the fundamental books of Weyl and of Wigner on applications of group theory to quantum mechanics ; since, some knowledge of the theory of linear group representations has become necessary to nearly all physicists. However the most basic concepts concerning group actions are not introduced in these famous books and, in general, in the physics literature. Let us do it here rapidly.

For any mathematical object M (a set, a vector space, a manifold, a physical theory...) one can define its group of automorphisms Aut M. An action on M of the group G is given by a group homomorphism f :

$$G \xrightarrow{f} \text{Aut } M \qquad (1)$$

If the kernel of f, Ker $f \equiv f^{-1}(1)$, is not trivial, G does not act effectively, but only through its image :

$$f(G) \equiv \text{Im } f \approx G/\text{Ker } f \qquad (2)$$

isomorphic to the quotient group $G/\text{Ker } f$.

Given an element $m \in M$ one defines :

the little groupe $G_m = \{ g \in G, f(g) m = m \}$, the set of elements of G which leave m fixed,

the orbit $G(m) = \{ f(g) m, g \in G \}$, the set of transforms of m by G.

Points on the same orbits have conjugated little groups : $G_{g.m} = gG_m g^{-1}$, $g.m$ shorthand for $f(g) m$.

Conversely, if G_m and G_m' are conjugated in G, by definition m and m' belong to the same *stratum*. This concept of strata is very useful to physicists although it is very rarely used explicitly. For instance, for the action of the Lorentz group on Minkowski space, the strata correspond to the time like, space like, light like and null vectors. For the action of the orthogonal group $O(3)$ on the crystal lattices, the strata correspond to the seven crystallographic systems : cubic, hexagonal, trigonal, tetragonal, orthorhombic, monochimic, triclinic and for the action of a space group on our three dimensional space the strata are the Wyckoff positions.

The equivalence of two G-actions : $G \xrightarrow{f} \text{Aut } M$, $G \xrightarrow{f'} \text{Aut } M'$ is denoted $f \approx f'$; very naturally this is the case when there exists an isomorphism $M \xrightarrow{\theta} M'$ such that ($\forall$ reads « every »)

$$\forall g \in G, \qquad f(g') = \theta \circ f(g) \circ \theta^{-1} . \tag{3}$$

When M is a vector space (or a Hilbert space), $\text{Aut } M$ is the general linear group on M (or the unitary group on M) then (1) defines a linear (unitary) representation of G on M and (3) is the usual equivalence of linear representation well known to physicists. Two G-orbits which carry equivalent G-actions are, by definition, of the same type : then they have the same little groups. So the G-orbit types can be labelled by the conjugation classes of subgroups of G; this is also the case of the strata of a G-action since a stratum is the union of all orbits of a given type. We denote by $[H]$ the conjugation class of the subgroup $H < G$. A prototype of orbit of type $[H]$ is given by the action of G on the set $[G : H]$ of left cosets of $H : g.xH = gxH$. (When $H \triangledown G \equiv H$ invariant subgroup of G, then $[G : H]$ has moreover a group structure, the quotient group G/H.) Of course the orbit $[G : H]$ inherits some of the structure of G and M e.g. when M is a manifold (in particular a vector space) and H is a closed subgroup of the Lie group G, then $[G : H]$ is a submanifold of M.

Before going back to physics let us make a simple remark : let

$$K = \bigcap_{m \in G(m)} G_m \equiv \bigcap_{g \in G} gG_m g^{-1} . \tag{4}$$

Then K is the largest invariant subgroup of G contained in every G_m and it is the kernel, Ker f, of the action (1) on $[G : G_m]$.

When a physical problem has a symmetry group G, this group acts on the set $\mathcal{S}$ of solutions and each solution s is only invariant by its little group G_s. Although it is in general much easier to find the G-invariant solutions, one must not forget that others, with less symmetry, do often exist ! If the solutions depend on parameters « λ » (e.g. temperature, pressure, external fields), by varying λ the states of the system may describe a curve in $\mathcal{S}$: there is a symmetry change when this curve passes from one stratum to the other. This may be an increase of symmetry, thus obeying the Curie principle ; but if the transformation is reversible, the transition in the other way produces a « spontaneous symmetry breaking » which is a violation to Curie principles. Of course principles come first, and symmetry principles are useful in physics ; but exceptions to them are fascinating to study.

Landau theory of second order phase transition is an example of spontaneous

symmetry breaking well known to this audience. It predicts rather well the symmetry change in crystals; forgetting the Lifschitz criteria, it even applies to uncommensurate transitions; fluctuations can be added to this mean field theory and the application of the renormalization group techniques makes some interesting predictions. If G is the group of the highest symmetry phase, the subgroup $H < G$, symmetry of the lowest symmetry phase is a *little group of an irreducible* (*on the real*) *linear orthogonal representation of G*. This condition in general eliminates some H. Indeed assume that G and H are known (this is quite often the case). From the remark about eq. (4), we know that the kernel K of the irreducible orthogonal G-representation is the largest invariant subgroup of G contained in H (or any of its conjugate). Therefore the image of the representation is G/K; this requires that G/K has faithful real irreducible representations and this is ruled out for instance if G/K has a non cyclic center. Remark also that for a genuine crystal to crystal transition H contains a rank three translation group, so does K; hence the image G/K is finite. In that case there is a natural partial order (by inclusion up to a conjugation) on the set $\mathcal{K}$ of the conjugation classes of the subgroups of G/K. There is a well spread conjecture in the literature that the broken symmetry group H is such that $[H/K]$ is maximal in the set $\mathcal{K}' \subset \mathcal{K}$ of the conjugation classes of little groups of the action of G/K. This is not true and a counter example was recently given by the Toledano brothers [1]: it is obtained from the representation of $C_4^6 \equiv I\,4_1$ for the point N of the Brillouin zone.

It was obvious that this conjecture was not mathematically sound for the following reason: Landau theory usually approximates the thermodynamical potential to be minimized by a fourth degree polynomial. Due to this restriction, the Landau polynomial has in general an invariance group $\tilde{G}$ (subgroup of the orthogonal group $O(m)$, where m is the dimension of the representation) larger than the image group G/K given by the physics. Of course a mathematical conjecture can only be related to the exact invariance group $\tilde{G}$ of the Landau polynomial. If the degree of the polynomial were not limited — or if the representation were real reducible — it is easy to prove that any little group can be that of the minimum [2]. But for a fourth degree polynomial, I conjectured [2] that the little group of the minimum is maximal in the set $\tilde{\mathcal{K}}'$ of conjugation classes of the little groups of the action of $\tilde{G}$, but I have only been able to give some sufficient conditions on $\tilde{G}$ for the truth of this conjecture. This is an interesting problem to solve, and it applies also to the fourth degree Higgs polynomials (then $\tilde{G}$ is compact instead of finite) that we will meet later.

If we do not restrict ourselves to minima, but consider extrema of G invariant functions, then there is an interesting general theorem [3]:

Theorem. Given a smooth action of a compact G group on a real manifold M, every real valued G invariant function on M has extrema on each stratum corresponding to maximal little groups. Moreover all G invariant functions on M have in common some orbits of extrema; they are exactly those orbits which are isolated in their stratum.

These critical orbits are often found as the solutions of spontaneous symmetry breaking in nature. Then it is comforting for the author of a physical model to predict them, but it is wise to remember that this is not a specific prediction of the model: it is simply the verification of a general geometric theorem and any other function to be varied (for the same group action) would have also yielded this solution!

Looking for extrema of G invariant functions is only one mechanism predicting spontaneous symmetry breaking in physics; it is in fact a particular case of i) in the present known list of spontaneous symmetry breaking mechanisms :

  i) bifurcation theory,
  ii) thermodynamic limit,
  iii) renormalization in field theory,
  iv) decomposition of G invariant quantum states into pure states.

I refer to a review just published [4] for more references and some details. Here I just want to make some comments and introduce some open problems.

In bifurcation theory, the non linear integro-differential equations depend on some parameters : the nature and the symmetry of solutions change for some critical value of parameters. If these equations come from a variational principle we are in the particular case already studied. It is only recently that group action considerations have been introduced in bifurcation problems (see e.g. [4] for references).

Then, one proves again under very general conditions, that the symmetry G of the equations and boundary conditions can be spontaneously broken only into a subgroup $H < G$ which appears as a little group of an irreducible linear representation (*) of G when we exclude the exceptional case of accidental degeneracies (in that case, often, there is a larger symmetry $\tilde{G}$ involved).

As an example of bifurcation problem, I wish to recall one of the most famous and most studied : that of the equilibrium shape of rotating celestial bodies (planets, stars, galaxies, ...). Considering only the gravitational interaction and the rotation, it is convenient to introduce the dimensionless parameter

$$t = \frac{\text{rotational energy}}{|\text{ gravitational energy }|}. \tag{5}$$

From the virial theorem $0 \leqslant t \leqslant \frac{1}{2}$. Newton had already predicted that the Earth's shape is an axially symmetric oblate ellipsoid and he calculated its flatness. The surprise came when, in 1834, Jacobi showed, in the case of an uncompressible fluid, that for $t = 0.137$ ... there is a bifurcation. Above this critical value, the stable shape is an ellipsoid with three unequal principal axis. Poincaré found a new bifurcation which looses the center of symmetry (egg or convex pear shape) and which is followed by an enumerable infinity of others which all occur for $t < 0.141$. It is not known if there are bifurcations from the axially symmetric ellipsoid which do not destroy the axial symmetry and the static equilibrium in the rotating frame. However, by using group theory techniques, an enumerable set of bifurcation destroying the axial symmetry have been found recently [5]. I wish to attract your attention on a remarkable theorem due to Lichtenstein [6]. The symmetry through the plane orthogonal to the angular momentum cannot be spontaneously broken.

There is now a general method for finding symmetry breaking bifurcations (e.g. [5]). However Lichenstein's theorem seems ad hoc for the rotating celestial bodies and an open problem would be to generalize it to any bifurcation problem : *which symmetries cannot disappear ?* For instance there is now a very active study of the

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(*) When G is the usual rotation group SO(3) I could not find a complete and correct list of these little groups in the literature ! So I published it in [4].

shape of nuclei, because very high spin states ($j \lesssim 50\,h$) are produced, some three unequal axis and some pear shape nuclei have recently been observed, but I have no idea if a Lichenstein-like theorem is true for nuclei.

When one deals with a finite number of constituants, no sharp mathematical discontinuities occur in statistical mechanics although Nature with 10^{23} molecules or computer simulations with a few thousand show us vivid illustrations of these discontinuities. But the theorician has to go to the thermodynamic limit with an infinite number of constituants. This often implies spontaneous symmetry breaking; cf the historical example of Peierls [7] on Ising model. I refer to [8] for a recent review. For fifteen years spontaneous symmetry breaking has been computed by renormalization in quantum field theory. This technique has been applied for instance to Landau theory of phase transitions for taking account of the fluctuations. Not only it yields very interesting predictions on the critical exponents but some theorems on symmetry breaking have already been found. We will hear more on this subject from Professor Dzyaloshinskii in the next lecture. A lot of work on this subject is in progress and it would be interesting to have a review on this topic in a similar colloquium, let us say two years from now.

Finally, mechanism iv) is now well understood (see e.g. [9]). The measure which gives the decomposition of a G invariant state into extremal states must be finite, G-invariant and the action of G on its support must be either transitive (i.e. one orbit) or ergodic. For instance when G is the Euclidian group, the classification of transitive states falls into classes which correspond to known phases of matter [10] : crystal, nematics, cholesterics, smectics A and C, rod like structures (e.g. lyotrops). *The classification of ergodic states has not yet been made.* Obviously it will describe uncommensurate phases, but surely also other phases not well understood presently or perhaps to be discovered.

I wish now to make some general remarks on the physical description of a physical state with a symmetry group G. All its physical properties must be described by G-invariant functions (or perhaps distributions...). For the last twenty five years there has been some important progress in the mathematical theory of group invariants and physicists cannot afford to ignore them. For instance in the case of smooth G-actions with finite or compact image (and a finite number of strata in the later case), Mostow has shown [11] that these actions are equivalent to linear actions; then smooth G-invariant functions are just smooth functions of G-invariant polynomials [12]; and now the structure of the ring of G invariant polynomials is better known (for reviews readable by physicists see e.g. [2], [13], [14]).

Let me emphasize first that the invariants *depend only on the image* of the group action. They loose some knowledge of the structure of the symmetry group G, e.g. macroscopic effects in crystal do not depend on the space group, but only on its quotient by the translations (i.e. the point group). And the natural equivalence of action for physical problems might become weaker : we call it quasi-equivalence :

Definition. Two actions $G \xrightarrow{f} \text{Aut } M$ and $G' \xrightarrow{f'} \text{Aut } M$ are *quasi-equivalent*, $\sim$, if their images $f(G)$ and $f'(G')$ are conjugated in Aut M.

Not only the invariants do not distinguish between G and G' when $f \sim f'$, but then even do not distinguish between inequivalent G-actions which are quasi-equivalent.

The art of physics is to idealize and simplify the description of complex pheno-

mena in order to obtain general predictions. So physical considerations will often select a subset of the G-invariants where G is the image of the symmetry group action. This subset might no longer be characteristic of the group G and well possess a larger symmetry $\tilde{G}$. As I already pointed out for Landau theory, the physical predictions on symmetry breaking depend on $\tilde{G}$; and the simpler is the model, the more weight will have purely geometrical considerations (the dynamics is completely fixed by the model simplification). As in Landau theory, in many other physical models we have only to consider low degree, d, polynomials invariant by the finite or compact image G of an irreducible m dimensional orthogonal representation of the symmetry group. If these polynomials are defined up to a conjugation by $O(m)$, i.e. up to a choice of orthogonal coordinates, they will depend on few parameters and their exact symmetry groups $\tilde{G}_x$ are in a small selected subset of the irreducible subgroups of $O(m)$. For instance for $m = 3$, there is respectively for $d = 3$ and $d = 4$ a one and a two dimensional vector space of polynomials whose $\tilde{G}_x$ are :

$$m = 3, \qquad d = 3 \qquad \lambda xyz, \qquad\qquad\qquad\qquad \tilde{G} = T_d \qquad (6)$$

$$m = 3, \qquad d = 4 \qquad \lambda_1(x^2 + y^2 + z^2)^2 + \lambda_2(x^4 + y^4 + z^4), \qquad \tilde{G} = O_h. \qquad (6')$$

So the five other irreducible subgroups of $O(3)$: T, T_h, O, Y, Y_n are not symmetry groups of 3 variable polynomials with degree $\leqslant 4$. For $m = 4$ the classification is presented at this conference [15] : the dimension of the polynomial vector space are 5 for $d = 3$ and 14 for $d = 4$; and only 18 conjugate classes of symmetry group $\tilde{G}_x$ appear.

Let us give another example from crystallography. There are 153 equivalence classes of irreducible unitary representation of the 32 point groups : they have only 13 inequivalent images. These representations can be considered as a particular case of the unitary irreducible representations of the little space group G_k for the high symmetry wave vector k of the Brillouin zone. There are 59 orbits ($=$ stars) of such k's for the 14 Bravais lattices and more than three thousand irreducible linear representations (which have been all tabulated in the sixties and seventies) of the 219 isomorphic classes of G_k. These few thousand representations play a great role for instance in the study of lattice vibrations. They have only 36 inequivalent images [16] !

I strongly believe that this concept of quasi-equivalence will become more and more important for the study of symmetry in physics.

There is an important lesson to be learned from the occurrence of spontaneous breaking of symmetry. The laws of physics might have much more symmetry than that which appears in physical phenomena. Dream for a while that the electrons of a crystal are good physicists : they have well understood the triple periodicity of their world and they even know its space group. But will they be clever enough to find that the law of physics of their world is invariant by rotations and translations (i.e. by the Euclidian group). Physicists studying the fondamental interactions are confronted with a similar problem concerning the internal symmetry of fundamental particles. They have been tremendously successful and physics has had quite a revolution in the seventies. For leptons (electrons and their neutrinos, μ and v_μ, τ discovered in 1975 and v_τ), keep Dirac and Maxwell equations but give to the potential vector A_μ four degrees of freedom ($\rightarrow A_\mu^\alpha$, $\alpha = 0, 1, 2, 3$) such that the gauge group, instead of depending on a phase ($U(1)$), depends on an element

of the four dimensional group $U(2)$ (isomorphic to the group of 2×2 unitary matrices). This does require the addition of a quadratic term in A_μ^α in Maxwell equations and you obtain the unified theory of weak and electromagnetic interactions. Such unification, comparable to the Maxwell unification of Optics and Electrodynamics, is mainly due to the successive works of S. Glashow, A. Salam and J. C. Ward, S. Weinberg, G. t'Hooft (three of them Nobel prizes 1979). The four components of A_μ correspond to the massless photon γ and to three predicted particles Z^0, $W^\pm$ of atomic mass about 100 and 90 ! High energy physicists are very eager to have bigger accelerators for producing them, such as the 30 km of circumference LEP near Geneva.

Hadrons are composite particles; they are made of quarks. But the same set of equations, with $U(3)$ as gauge group gives chromodynamics. This $U(3)$ symmetry is exact. The nine components A_μ describe the photon and the eight massless colored gluons. This is now the fundamental theory of strong interactions. We do hope that it will explain quark confinement and that it contains the usual nuclear forces as secondary effects (just as Van der Vaals forces between atoms are secondary effects of the atome structure which is governed by the electromagnetic interaction).

Of course physicists work hard presently to unify all these interactions. The smallest candidate is $G = SU(5)$ but it does contain a 3-value degree of freedom which escapes classification. Exceptional Lie groups have also been considered for G. In any case if the equations of physics are G-invariant, this symmetry is spontaneously (and grossly) broken in our world. In the actual theories this is obtained by the old trick : add to the Lagrangian density a G-invariant degree 4 polynomial made from a new field, the Higgs field. Of course its absolute minimum is $\neq 0$; the degree 4 restriction is imposed by renormalizability. The trick is quite elaborate; the corresponding Goldstone bosons do not exist : they are incorporated in the A_μ^α components which do not belong to the preserved symmetry $H < G$ and these A_μ^α fields become massive (the lowest ones are the Z^0, W^+, W^-; all others have even a much bigger mass). Although these massive particles have not yet been observed, they are strongly expected by the physicist community. We feel less sure of the existence of Higgs field quanta; in any case their properties are more difficult to predict and they look too much as « deus ex machina ».

We do hope to reach a full unification, including Einstein's equations of general relativity. Probably this will be done by superseding even the notion of symmetry group. Physicists have already invented the beautiful new concept of supersymmetry, for which Fermions and Bosons can be transformed into each other. Much more is to be done ! I firmly believe in the sentence of introduction of a Dirac paper « Quantized singularities in the Electromagnetic field », written very nearly fifty years ago [17] « The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced..., a mathematics that continually shifts its foundations and get more abstract ». Let me just end by recalling that at the end of this introduction, Dirac predicted the existence of anti-electrons (discovered the year after) and anti-protons (discovered twenty-four years later). This was also the first step for introducing C, the charge conjugation symmetry; C has been the first known element of the « internal symmetry » group G of unified fundamental interactions, and C is spontaneously broken as it was found experimentally in January 1957 [18, 19].

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