Landau free energies for $n = 4$ and the subgroups of O(4)

Louis Michel
Institut des Hautes Etudes Scientifiques, 35, route de Chartres,
F 91440 Bures sur Yvette, France.

Jean-Claude Tolédano
Centre National d'Etudes des Télécommunications, 196, rue de Paris,
F 92220 Bagneux, France.

and Pierre Tolédano
Groupe de Physique Théorique, Université de Picardie, 33, rue St-Leu,
F 80000 Amiens, France

I. Introduction

The Landau theory of continuous phase transitions [1] defines a free energy which is a polynomial expansion of the order parameter components, truncated to the fourth degree, and invariant by the symmetry group of the more symmetric phase. At each temperature, the absolute minimum of this expansion provides the stable state of the system. Allowing for all possible values of the expansion's coefficients, one finds, in general, that several phases with different symmetries can be stable, some of which can be related to the more symmetric phase through a second order transition. Hence, knowing the Landau expansion permits a determination of a phase diagram for the system.

Though there is a great variety of physically distinct order parameters, the possible forms of their truncated expansions are very limited. For instance, two fourth degree expansions (one « isotropic » and one « cubic ») describe all the continuous transitions possessing a three-components order parameter. The reason underlying this simplicity is that the truncated expansion has an own symmetry which is usually much higher than that of the considered system.

Beyond the validity of the Landau theory, the form of the fourth degree expansion keeps an important role in the determination of the critical behaviour of the system in the vicinity of its transition point. This form is identical to that of the homogeneous part of the hamiltonian considered in the renormalization group theory of the critical behaviour. In this theory a set of critical exponents is associated to a stable fixed point belonging to an abstract space spanned by the coefficients of the former expansion. The location, and stability of these points are determined by recursion relations whose form is deduced from that of the Landau expansion. However, it has been shown [2] that for systems with short range forces, whenever the number of components of the order parameter is less than four, this precise form is unimportant : one finds a stable fixed point corresponding to the part of the free energy with $O(n)$ symmetry. For $n \geqslant 4$ the anisotropic terms of the expansion become relevant and the nature and stability of the fixed points are expected to depend closely on the form of the order parameter expansion. Examples examined up to now [3, 4] have shown two types of situations. Either an « anisotropic » fixed point exists giving rise to a new set of critical exponents, or, more frequently, all fixed points are unstable. The latter circumstance has been conjectured [4] to correspond to the occurrence of a first order transition.

At present the form of all possible fourth degree expansions is only known [5, 6] for $n \leqslant 3$. For higher dimensions an enumeration has recently been performed [7] in the restricted framework of the order parameters relative to continuous transitions between strictly crystalline phases (i.e. not modulated). For this case one finds 13 different expansions for $n = 4$, five expansions for $n = 6$, and three expansions for $n = 8$. This list does not exhaust all possibilities. In particular it does not contain the expansions arising from all the space-group irreducible representations but only from those complying with the restrictive « Lifschitz criterion » [7]. Besides, order parameters pertaining to other physical systems than structural transitions were not considered.

In the present paper, a complete enumeration of possible Landau expansions is derived for $n = 4$. As mentioned above, this is the smallest dimension expected to give non-trivial results for the critical behaviour. Besides, though the method used is, in principle, valid for any value of n, its application relies on the knowledge of a list of subgroups of $O(n)$, as well as on the form of the irreducible representations of $SO(n)$. Both were readily available for $n \leqslant 4$ and not for higher dimensions.

II. Landau polynomials and subgroups of $O(n)$

In the Landau theory, the order parameter spans a representation Γ of the high-symmetry phase which has the property to be irreducible on the real [8]. It can be reducible on the complex and it equals then the sum of two complex conjugate irreducible representations. Γ being orthogonal (real and unitary), its image G (i.e. the set of its distinct matrices) acting in the vector space V_n spanned by the n-components of the order parameter is isomorphous to a subgroup of the full orthogonal group $O(n)$. The matrices represent the transformation of the coordinates under the action of G, and their set coincides with the so-called « vector representation » [8] of this group. Like Γ, the image G is irreducible on the real. This condition puts a restriction on the subgroups of $O(n)$ which are likely to represent the symmetry of an irreducible order parameter. Such subgroups can be called « irreducible ». Thus, in three dimensions the only irreducible subgroups of $O(3)$ are the invariance groups of the regular tetrahedron (T, T_d), the cube (T_h, O, O_h) and the icosahedron (Y, Y_h) [Schoenflies notation].

The Landau free-energy F is a sum of homogeneous polynomials of successive degrees of the order parameter components, invariant by the matrices of Γ and consequently G-invariant. The pth degree contribution to F is a linear combination, with arbitrary coefficients, of G-invariant, linearly independent polynomials. The pth degree homogeneous polynomials of the coordinates generate a vector space of dimension C^p_{p+n-1}, denoted $[V^p_n]$ which spans the symmetrized pth power of V_n. In this space, the G-invariant polynomials generate a subspace E^G_p.

Because a given degree is singled out in the infinite expansion, it can happen that several groups $G \subset O(n)$ have in common the same space E^G_p. One of these groups, denoted G_p, will contain all the others. It is the largest subgroup of $O(n)$ leaving invariant every vector of E_p^G. We call it the centralizer of the set of vector spanning E_p^G. Physically, G_p is the symmetry group common to the entire set of degree-p contributions to a Landau expansion, based on the polynomials of E_p^G, and having arbitrary coefficients. On the other hand, it can happen that G_p is not the highest symmetry group of any specified vector $\boldsymbol{\alpha}$ of the E_p^G space. The invariance groups will be a family of groups G_p^α verifying $G_p \subseteq G_p^\alpha \subseteq O(n)$: G_p^α is the little group of $\boldsymbol{\alpha}$ in the usual sense [9] (*).

For any irreducible G there is no linear (degree one) invariant, and there is one degree-two invariant. In a real basis (x_i) of V_n, this invariant has the form $(\sum x_i^2)$. Its little group is $O(n)$. For higher degrees the number and form of the invariants spanning E_p^G depend on G_p.

When dealing with continuous transitions, the free energy is the sum of the quadratic term, with $O(n)$ symmetry and of a quartic term corresponding to E_4^G. As a consequence, *the enumeration of possible Landau expansions with arbitrary coefficients will consist in finding the various irreducible centralizers G_4, and their E_4^G spaces.* For each choice of the coefficients, the complete symmetry group of the expansion will be one of the G_4^α little groups. This investigation can be extended to expansions which contain a degree-three invariant. In a similar way, the enumeration of their possible forms will be related to the groups G_3 and G_3^α of degree-three polynomials.

An important point to notice is that these enumerations can be performed up to a conjugation in $O(n)$. Actually, each group G has an infinite number (except $SO(n)$) of conjugate groups. Their respective E_p^G have the same dimensions, and the Landau polynomials associated with them are physically equivalent as they are transformed from eachother by a mere change of reference frame in the V_n space of the order parameter components.

III. Determination of the centralizers and little groups of degree-p polynomials in $O(n)$

Let $C_p(G)$ be the dimension of E_p^G. Every vector of this space being G-invariant, $[V_n^p]$ spans a reducible representation of G_p which contains $C_p(G)$ times the trivial representation of this group. $C_p(G)$ is called the subduction number [10] of G in $[V_n^p]$. The procedure of determining the irreducible centralizers G_p is then the following [9, 12] :

i) Compute $C_p(G)$ for each irreducible subgroup of $O(n)$, up to a conjugation.

ii) If $C_p(G) > C_p(G')$ for any $G' \supset G$, then G is one of the G_p centralizers, since this establishes that any larger group than G has a subspace of invariant vectors smaller than E_p^G. Conversely, if $C_p(G) = C_p(G')$, G is obviously not the centralizer of E_p^G as this space is invariant by the larger group G'.

In order to determine the G_p^α little groups, we can notice that they possess the following properties :

j) $G' \supset G_p^\alpha \supseteq G_p$ and $C_p(G') < C_p(G_p^\alpha) < C_p(G_p)$. (The equality corresponds to $G_p = G_p^\alpha$.)

jj) $\bigcup_\alpha E_p^{G\alpha} = E_p^G$.

(*) G_p is the intersection of the G_p^α.

Condition j) means that G_p^α is the centralizer of a subspace $E_p^{G\alpha}$ of E_p^G and that it therefore belongs to the set of G_p groups first determined, up to a conjugation in $O(n)$. Condition jj) expresses, in conformity with the definition of G_p^α that any vector of E_p^G belongs to the invariance subspace of a higher symmetry group. As pointed out previously [11], this condition cannot be fulfilled if α takes a discrete number of values since only a continuous set of subspaces $E_p^{G\alpha}$ can fill a space E_p^G with a higher dimension. Hence, the search of the G_p^α will consist in considering an infinite set (*) of conjugate subgroups of a G'_p, having G_p as a common subgroup, and check condition jj).

Finally, the polynomials let invariant by the G_p must be determined in a given basis of V_n (or in given setting of these groups among conjugated subgroups). This can be done using well established techniques [5, 8, 9, 11]. Such a procedure has previously been applied to $O(2)$ and $O(3)$. The results are recalled on table I [6, 11, 12].

TABLE I

Groups G_p and G_p^α of degree-p polynomials ($p = 3, 4$) in $O(n)$ ($n = 2, 3$). $C_p(G_p)$ is the number of anisotropic invariants. When $G_p^\alpha \neq G_p$, each invariant by G_p has a symmetry group coinciding with a conjugate of G_p^α. In $O(2)$ one mirror plane is along x and the polar coordinates are used to express the invariants. In $O(3)$ the axes are those of the cube. The little group $O(n)$ and its invariants ρ^2 and ρ^4 which are common to all the Landau expansions must be added to the table.

n	p	G_p	G_p^α	$C_p(G_p)$	*Invariants*
2	3	C_3	C_{3v}	2	$\rho^3 \cos 3\theta ; \rho^3 \sin 3\theta$
2	3	C_{3v}	C_{3v}	1	$\rho^3 \cos 3\theta$
2	4	C_4	C_{4v}	2	$\rho^4 \cos 4\theta ; \rho^4 \sin 4\theta$
2	4	C_{4v}	C_{4v}	1	$\rho^4 \cos 4\theta$
3	3	T_d	T_d	1	xyz
3	4	O_h	O_h	1	$x^4 + y^4 + z^4$

IV. Irreducible centralizers and little groups in $O(4)$

Let us first apply the preceding method to the determination of the G_p and G_p^α groups in $SO(4)$. This preliminary step makes more convenient the deduction of the subduction numbers as their computation uses the irreducible representations of $SO(4)$ which have a simple form.

a) *Subgroups and irreducible representations of $SO(4)$*

$SO(4)$ is the homomorphic image of the product $SU(2) \times SU(2)$, while $SO(3) \times SO(3)$ is the homomorphic image of $SO(4)$. On the other hand, $SU(2)$ is isomorphic to the group of quaternions with unit moduli [13]. More precisely an element of $SO(4)$ can be constructed from a set of two 3-dimensional rotations. Let $(\boldsymbol{\theta}_1, \mathbf{u}_1/\theta_2, \mathbf{u}_2)$ be this set (axes $\mathbf{u}_i$, angles θ_i). Each rotation corresponds to two distincts elements of $SU(2)$ represented by the quaternions $\pm q_i = \pm \left(\sin \frac{\theta_i}{2} \cdot \mathbf{u}_i ; \cos \frac{\theta_i}{2} \right)$. The former

(*) The conjugation is performed by the elements of the normalizer [12] of E_p^G.

set then gives rise to two distinct elements of $SO(4)$ which are denoted $\pm S = \pm (q_1, q_2)$. If the current vector in V_4 is associated to the quaternion $(x, y, z ; t) = (\mathbf{U} ; t)$, the action of S is expressed by :

$$S(\mathbf{U} ; t) = q_1(\mathbf{U} ; t)\, q_2^{-1} \tag{1}$$

where the multiplication between quaternions is [13] :

$$(\mathbf{A} ; a)(\mathbf{B} ; b) = (a\mathbf{B} + b\mathbf{A} + \mathbf{A}_{\Lambda} \mathbf{B} ; ab - \mathbf{AB}) . \tag{2}$$

The form of the irreducible representations of $SO(4)$ as well as the construction and notation of its subgroups are based on the above homomorphisms. Thus, the irreducible representations have the form [9] :

$$(j \mid k) = D^j \oplus D^k$$

where D^j and D^k are Wigner's representations of $SU(2)$, with the restrictions that $2j$, $2k$, and $(j + k)$ are non-negative integers. In particular, the character of

$$S = (q_1, q_2)$$

in the irreducible space $(j \mid k)$ is :

$$\chi^{(j \mid k)}(S) = \chi^{(j)}(q_1)\, \chi^{(k)}(q_2) = \frac{\sin (2j + 1)\frac{\theta_1}{2} \sin (2k + 1)\frac{\theta_2}{2}}{\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}} . \tag{3}$$

On the other hand each subgroup of $SO(4)$ can be denoted [13].

$$G = (L/L_k ; R/R_k) \tag{4}$$

with L, L_k, R, R_k subgroups of $SU(2)$, L_k (resp. R_k) an invariant subgroup of L (resp. R), and the quotient group L/L_k isomorphous to R/R_k (*). Explicitely we can write :

$$L = \bigcup_{i=1}^{m} g_i L_k ; \qquad R = \bigcup_{i=1}^{m} g_i' R_k \tag{5}$$

then

$$G = \bigcup_{i=1}^{m} \bigcup_{l \in L_k} \bigcup_{r \in R_k} (g_i\, l ; g_i'\, r) \tag{6}$$

where g_i, g_i', l, r are quaternions of unit moduli (**). The definition and notation of the subgroups of $SU(2)$ is recalled on table II. A list, based on eq. (4) of the subgroups of $SO(4)$ is given by Du Val [13].

(*) Several isomorphisms can exist, some of which are unequivalent up to a conjugation.
(**) When $L = L_k$, $R = R_k$, G is the direct product $L \times R$.

TABLE II

Subgroups of $SU(2)$ up to a conjugation; notations are derived from those of the homomorphic [13] subgroups of $SO(3)$. Elements of $SU(2)$ are represented by unit quaternions $(\mathbf{U}, t)$. The chosen setting for C_m, D_m is : the m-fold axis along z and one twofold axis along x. In the other groups x, y, z are perpendicular to the faces of the cube. Y^+ is conjugated to Y. It has been distinguished in the table because both groups appear together in the following tables, $\tau = (1 + \sqrt{5})/2$.

Labelling	Order	Definition
C_m	m	$\left(\sin \frac{2\pi}{m} \mathbf{k} ; \cos \frac{2\pi}{m}\right) = (r_{\mathbf{m}})^2$
C_∞	∞	$(\sin \theta \mathbf{k} ; \cos \theta)$
D_m	$4\,m$	$[I \oplus (\mathbf{i} ; 0)]\, C_{2m} = (I \oplus x)\, C_{2m}$
D_∞	∞	$(I \oplus x)\, C_\infty$
T	24	$\left[I \oplus \frac{1}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k} ; I) \oplus \frac{1}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k} ; -I)\right] D_2$
O	48	$\left[I \oplus \frac{1}{\sqrt{2}}(\mathbf{i} ; I)\right] T$
Y	120	$\left[\bigoplus_{r=0}^{4} \left(\frac{\tau}{2}, \frac{1}{2}, 0, \frac{\tau^{-1}}{2}\right)^r\right] T$
Y^+	120	$\left[\bigoplus_{r=0}^{4} \left(-\frac{\tau^{-1}}{2}, \frac{1}{2}, 0, -\frac{\tau}{2}\right)\right] T$

b) *Irreducible centralizers and little groups in $SO(4)$*

The four dimensional space V_4 spans the irreducible representation $(1/2 \mid 1/2)$ of $SO(4)$. Its symmetrized 2nd, 3rd, and 4th powers respectively decompose into :

$$[V_4^2] = (0 \mid 0) + (1 \mid 1) \tag{7}$$

$$[V_4^3] = (1/2 \mid 1/2) + (3/2 \mid 3/2) \tag{8}$$

$$[V_4^4] = (0 \mid 0) + (1 \mid 1) + (2 \mid 2)\,. \tag{9}$$

For irreducible groups G, the representation $(1/2 \mid 1/2)$ remains irreducible on the real. It then follows from (8) that G-invariant degree-three polynomials necessarily belong to $(3/2 \mid 3/2)$. To determine their centralizers we can restrict the computation of the subduction number $C_3(G)$ to the $(3/2 \mid 3/2)$ subspace of $[V_4^3]$. On the other hand, we know that the only degree two invariant has $SO(4)$ symmetry and spans $(0 \mid 0)$. We deduce from (7) that no vector of $(1 \mid 1)$ is invariant by an irreducible G. In agreement with relation (9), the degree-4 invariants will comprise one polynomial spanning $(0 \mid 0)$ with $SO(4)$ symmetry, and possible anisotropic invariants belonging to the representation $(2 \mid 2)$. Consequently, the search of their centralizers can rely on the computation of the subduction numbers in $(2 \mid 2)$. These calculations consist in evaluating the number of times the former representations of $SO(4)$ contain the trivial representation of G, with the help of the characters $\chi^{(3/2 \mid 3/2)}$ and $\chi^{(2 \mid 2)}$ supplied by formula (3).

Among the subgroups of $SO(4)$ listed by Du Val, the selection of the irreducible ones can be based on the following remarks :

i) Groups of the form $(L/C_1, L/C_1)$ and $(L/C_2, L/C_2)$ are reducible as they represent rotations of $SO(3)$ thus preserving the fourth dimension [13].

ii) $(C_{mr}/C_m, C_{nr}/C_n)$ are Abelian groups, hence reducible.

iii) $(D_{mr}/C_m, D_{nr}/C_n)$ involve in addition to an Abelian group of the preceding type, a generator of the form (x, x), with $x = (1, 0, 0; 0)$, which is a real diagonal operator. These groups are also reducible.

iv) If in eq. (4) either L_k or R_k is a group D_n ($n \geqslant 2$), T, O, or Y, then G is irreducible. This derives from the irreducibility of $D^{1/2}$ with respect to these subgroups of $SU(2)$. Accordingly $(1/2 \mid 1/2) = D^{1/2} \oplus D^{1/2}$ is irreducible on the real (it is irreducible on the complex if both L_k and R_k coincide with the former groups).

Few subgroups of $SO(4)$ escape these criteria. Their reducibility has been checked directly by constructing their set of elements, and looking for an invariant subspace in $(1/2 \mid 1/2)$. Irreducible subgroups of $SO(4)$ are listed on tables III and IV as well as their subduction numbers in $(3/2 \mid 3/2)$ and $(2 \mid 2)$. The list of irreducible subgroups agrees with the partial one worked out by Brown *et al.* for the « crystallographic » subgroups of $SO(4)$ [14].

Table III contains the groups without a central rotation (i.e. the element $-I$). These groups are the only ones compatible with degree three polynomials since no such polynomial is preserved by $(-I)$. Likewise the centralizers of degree-4 polynomials belong necessarily to table IV which contains the groups with the central rotation. *The groups listed in these two tables represent the symmetries of all possible irreducible order parameters with four components.* Application to their subduction numbers of the criteria indicated in section III allows a straightforward determination

TABLE III

Irreducible subgroups of $SO(4)$ not containing $(-I)$ centralizers and little groups of third degree polynomials in $SO(4)$. $C_3(G)$ is the subduction number in (3/2, 3/2). The sign + in columns 3 and 4 means that G is a centralizer (resp. little group). * and ** are families of groups not contained in ref. [13] and whose definition is :

$$* \quad \bigoplus_{1}^{4} (C_4^1, x)^j \,[I + (x, C_4^1)]\, C_m \times C_n$$

$$** \quad [(x, C_{4m}^1) \oplus (C_{2n}^1, C_{2m}^1) \oplus (xC_{2n}^1, C_{4m}^3) \oplus I]\, C_m \times C_n\,.$$

The frame of reference adopted is the same, for the two $SO(3)$ rotation components as in table I.

Labelling of G		$C_3(G)$	G_p	G_p^α	Invariants
$\left(\dfrac{D_{4k+2}}{C_{2k+1}}; \dfrac{D_{4h+2}}{C_{2h+1}}\right)^*$	$h = k = 1$	1	+	+	I_1'
	$h \geqslant 1,\ k > 1$	0			
$\left(\dfrac{D_{2h+1}}{C_{2h+1}}; \dfrac{C_{8k+4}}{C_{2k+1}}\right)^{**}$	$h = k = 1$	2	+		$I_1' ; I_2'$
	$h \geqslant 1,\ k > 1$	0			
$\left(\dfrac{Y^+}{C_1}; \dfrac{Y}{C_1}\right)$		1	+	+	I_3'

TABLE IV

Irreducible subgroups of $SO(4)$ containing $(-I)$ and groups G_p, G_p^x of degree-4 polynomials. $C_4(G)$ is the subduction number in (2, 2). * is a group not contained in ref. [13] :

$$[I + (C_{4m}^1, x) + (x, C_{4n}^1) + (C_{4m}^1\, x, xC_{4n})]\; C_{2m} \times C_{2n}\,.$$

I_0 (little group $O(4)$ is invariant by all the group).

Labelling	Indices		$C_4(G)$	G_p	G_p^x	Invariants
$C_{2m} \times D_n$	m	n				
	1	2	10	+		$I_1\, I_2\, I_3\, I_5\, I_6\, I_7\, I_8\, I_9\, I_{10}\, I_{12}$
	1	> 2	5			
	2	2	6	+		$I_1\, I_2\, I_3\, I_5\, I_9\, I_{10}$
	2	> 2	3			
	> 2	2	2			
	> 2	> 2	1			
	1	∞	5	+		$I_1\, I_5\, I_9\, I_{12}\, I_{14}$
	2	∞	3	+		$I_1\, I_5\, I_9$
	∞	2	2			
	∞	> 2	1			
	∞	∞	1			
$(D_m/C_{2m}, C_{4n}/C_{2n})$	m	n				
	> 2	2	1			
	2	> 2	1			
	> 2	> 2	1			
$(D_2/D_1, C_8/C_4)$	—		4	+		$I_1\, I_5\, I_8\, I_{11}$
$(C_{4m}/C_{2m}, D_{2n}/D_n)$	m	n				
	1	2	5	+		$I_1\, I_5\, I_6\, I_8\, I_9$
	1	> 2	3			
	2	2	3	+		$I_1\, I_2\, I_{10}$
	2	> 2	1			
	> 2	1	2			
	> 2	2	1			
	> 2	> 2	1			
$(T/D_2, TD_2)$	—		2	+	+	$I_2\, I_3$
$(O/D_2, O/D_2)$	—		1	+	+	I_2
$(C_{6m}/C_{2m}, T/D_2)$	m					
	1		4	+	+	$I_2\, I_3\, I_6\, I_7$
	2		2	+		$(I_2 + 2\, I_6)\,(I_7 + 6\, I_3)$
	> 2		0			
$D_m \times D_n$	m	n				
	2	2	4	+	+	$I_1\, I_2\, I_3\, I_5$
	2	> 2	2			
	2	∞	2	+	+	$I_1\, I_5$
	> 2	> 2	1			
	> 2	∞	1			
	∞	∞	1	+	+	I_1
$(D_{2m}/D_m, D_{2n}/D_n)$	m	n				
	1	2	3	+	+	$I_1\, I_5\, I_8$
	1	> 2	2			
	2	2	2	+	+	$I_1\, I_2$
	2	> 2	1			
	> 2	> 2	1			

TABLE IV (*suite*)

Labelling	Indices		$C_4(G)$	G_p	G_p^{α}	Invariants
—	—		—	—	—	—
$(D_{2m}/D_m, D_n/C_{2n})$	m	n				
	2	> 2	1			
	> 2	1	3			
	> 2	2	2			
	> 2	> 2	1			
$(D_{2m}/C_{2m}, D_{2n}/C_{2n})^*$	m	n				
	2	2	2			
	2	> 2	1			
	> 2	> 2	1			
$(D_{3m}/C_{2m}, O/D_2)$	m					
	1		2	+	+	$I_2\ I_6$
	2		1	+	+	$(I_2 + 2\ I_6)$
	> 2		0			
$(Y^+/C_2, Y/C_2)$	—		1	+	+	I_4
$C_{2m} \times T\,; C_{2m} \times O\,; C_{2m} \times Y$ $D_m \times T\,; D_m \times O\,; D_m \times Y\,; T \times T$ $O \times O\,; Y \times Y\,; T \times Y\,; O \times Y\,; O \times T$ $(O/T, O/T)\,; (C_{4m}/C_{2m}, O/T)$ $(D_m/C_{2m}, O/T)\ (D_{2m}/D_m, O/T)$ $(D_{3m}/C_{2m}, O/T)$			0			

of the G_3 and G_4 centralizers. They are indicated in the tables together with the symbols of their invariant polynomials. The expression of the various polynomials in a specified basis of V_4 is shown in table V.

One more step is necessary to select the G_p^{α} groups among the G_p. To illustrate this selection, consider the example of the two symmetry related G_4 groups :

$$(C_8/C_4, D_4/D_2) \subset (D_4/D_2, D_4/D_2) \tag{10}$$

whose respective E_4^G spaces are spanned by the set of invariant polynomials (I_1, I_2, I_{10}) and (I_1, I_2). The setting of the smaller group leaves unspecified the direction of a twofold axis in the left-side D_2 component of the larger group. By rotating arbitrarily this axis in the (x, y) plane one generates a continuous set of conjugate groups which have $(C_8/C_4, D_4/D_2)$ as a common subgroup. It can be checked that, in this set, the group which has a twofold axis making an angle $\pi/4$ with x leaves the I_{10} polynomial invariant. Hence, like I_1 and I_2, I_{10} has a symmetry group conjugated to $(D_4/D_2, D_4/D_2)$. The smaller group in eq. (10) is not a little group G_p^{α}. This method has been applied to all the relevant G_3 and G_4 groups (Tables III and IV).

c) *Irreducible centralizers and little groups in $O(4)$*

The full rotation group $O(4)$ is generated from $SO(4)$ by :

$$O(4) = SO(4) + \mathcal{C} \times SO(4) \tag{11}$$

TABLE V

Invariant polynomials of degree-4 (I_j) and of degree-3 (I'_j). The frame of reference chosen for the two $SO(3)$ rotations constituting of an element of $SO(4)$ is the same as in table I.

$$I_0 = (x^2 + y^2 + z^2 + t^2)^2$$
$$I_1 = (x^2 + y^2)^2 + (z^2 + t^2)^2$$
$$I_2 = x^4 + y^4 + z^4 + t^4$$
$$I_3 = xyzt$$
$$I_4 = 5(x^4 + y^4 + z^4) + t^4 + (60/\sqrt{5})\, xyzt + 12\, t^2(x^2 + y^2 + z^2)$$
$$I_5 = (x^2 - y^2)(z^2 - t^2) + 4\, I_3$$
$$I_6 = xy(z^2 - t^2) - zt(x^2 - y^2) + xz(y^2 - t^2) + yt(x^2 - z^2) - xt(y^2 - z^2) + yz(x^2 - t^2)$$
$$I_7 = -xy(x^2 - y^2) + zt(z^2 - t^2) + xz(x^2 - z^2) + yt(y^2 - t^2) + xt(x^2 - t^2) - yz(y^2 - z^2)$$
$$I_8 = xt(3(y^2 - z^2) - (x^2 - t^2)) - yz(3(x^2 - t^2) - (y^2 - z^2))$$
$$I_9 = xy(z^2 - t^2) - zt(x^2 - y^2)$$
$$I_{10} = xy(x^2 - y^2) - zt(z^2 - t^2)$$
$$I_{11} = xz(3\, y^2 + 3\, t^2 - x^2 - z^2) + yt(3\, x^2 + 3\, z^2 - y^2 - t^2)$$
$$I_{12} = (xz + yt)(x^2 + y^2 - z^2 - t^2)$$
$$I_{13} = xt(3\, y^2 + 3\, z^2 - x^2 - t^2) - yz(3\, x^2 - 3\, t^2 + z^2 - y^2)$$
$$J_4 = I_8 + 2(I_6 + I_7 + I_{10} - I_9 - I_{12})$$
$$J_1 = (I_2 + 2\, I_6) \qquad J_2 = (I_7 + 6\, I_3) \qquad J_3 = (I_5 - 4\, I_3)$$
$$I'_1 = x(x^2 - 3\, y^2) + z(z^2 - 3\, t^2)$$
$$I'_2 = y(y^2 - 3\, x^2) + t(t^2 - 3\, z^2)$$
$$I'_3 = t^3 - t(x^2 + y^2 + z^2) + (10/\sqrt{5})\, xyz$$
$$I'_4 = x(z^2 - t^2) + t(x^2 - y^2) - 2\, yz(x + t)$$
$$I'_5 = x(x^2 - 3\, y^2) - t(t^2 - 3\, z^2)$$

where $\mathcal{C}$ is the « axial reflection » defined by $\mathcal{C}(\mathbf{U}\,; t) = (-\,\mathbf{U}\,; t)$. $O(4)$ contains « proper » subgroups $G \subset SO(4)$ and « improper » ones G^* which have the form :

$$G^* = G + S\mathcal{C} \times G \tag{12}$$

with G a proper subgroup of index 2 of G^* having the form $(L/L_k, L/L_k)$ and the element $S \in SO(4)$. Several G^* can have the same proper subgroup. If G is irreducible, so is G^*. However the converse is not true, and there is one family of irreducible G^*, indicated in ref. [13] which has reducible proper subgroups of index 2. It is denoted [13] (see Table II) :

$$(D_{nr/2}/C_n, D_{nr/2}/C_n)^*_{shk} = (I \oplus (xr^h_n, r^k_n))\,(D_{nr/2}/C_n, D_{nr/2}/C_n)_s\,. \tag{13}$$

The subduction number of G^* is equal to the number of invariant polynomials of G left invariant by the operation $S\mathcal{C}$. The determination of the centralizers in $O(4)$ can then be performed according to the following properties which are easily established :

i) G is an irreducible centralizer (resp. little group) in $SO(4)$ with $L \neq R$, or $L_k \neq R_k$, then G will remain a centralizer (resp. little group) in $O(4)$.

ii) G is an irreducible centralizer, subgroup of index two of one or several G^*. For such a group the invariant polynomials have been determined in the preceding section and the action of $S\mathcal{C}$ is readily determined for each G^*.

iii) G is an irreducible subgroup of index two of one or several G^*, not a centralizer in $SO(4)$. The application of the relation $C_p(G^*) \leqslant C_p(G)$ to the subduction numbers reproduced in tables III and IV show that no such G^* is a centralizer in $O(4)$.

iv) G^* belongs to the family (13). For this case the subduction numbers of the reducible centralizers G have been calculated and the effect of the $S\mathcal{C}$ operation examined in order to determine the subduction number of G^*.

The groups G_3, G_4, G_3^{α}, and G_4^{α} in $O(4)$ are presented on figure 1 and table VI which summarize the results of the investigation.

TABLE VI

Meaning of the symbols used in figure 1 for the centralizers of degree-3 and -4 polynomials. The notation used is that of Du Val [13] for the subgroups of $O(4)$. For the centralizers of degree-3 polynomials, the invariants of degree-3 and -4 are indicated. Column 3 contains Mozrzymas notation for the crystallographical groups [19]. The I_0 invariant is common to all the groups and is not listed.

Symbol (order)	Notation	Mozrzymas	Invariants
1	$(D \times D)^*$	—	I_1
2	$D_2 \times D$	—	$I_1\ I_5$
3	$C_4 \times D$	—	$I_1\ I_5\ I_9$
4	$C_2 \times D$	—	$I_1\ I_5\ I_9\ I_{12}\ J_4$
384	$(O/D_2, O/D_2)^*$	115.01	$I_2\ I_3$
240	$(Y^+/C_2, Y/C_2)^*$	112.01	I_4
192	$(T/D_2, T/D_2)^*$	109.01	$I_2\ I_3$
128	$(D_4/D_2, D_4/D_2)^*$	101.01	$I_1\ I_2$
120	$(Y^+/C_1, Y/C_1)^*$	100.02	$I'_4\ ;\ I_4$
96	$(D_6/C_4, O/D_2)$	—	J_1
72	$(D_6/C_3, D_6/C_3)^*$	90.01	$I'_1\ ;\ I_1$
64	$(D_2 \times D_2)^*$	82.01	$I_1\ I_2\ J_3$
48	$(D_3/C_2, O/D_2)$	77.2	$I_2\ I_6$
48′	$(C_{12}/C_4, T/D_2)$	—	$J_1\ J_2$
40	$(D_5/C_2, D_5/C_2)^*$	66.01	$I_1\ I_{13}$
36	$(D_3/C_3, D_3/C_3)^*$	65.01	$I'_5\ ;\ I_1$
32	$(D_2/D_1, D_4/D_2)$	57.1	$I_1\ I_5\ I_8$
32′	$(C_8/C_4, D_4/D_2)$	52.1	$I_1\ I_2\ I_{10}$
32″	$D_2 \times D_2$	56.1	$I_1\ I_2\ I_3\ I_5$
24	$(C_6/C_2, T/D_2)$	49.2	$I_2\ I_3\ I_6\ I_7$
20	$(D_5/C_1, D_5/C_1)^*$	36.01	$I'_4\ ;\ I_1\ I_{13}$
18	$(D_3/C_3, C_{12}/C_3)^*$	33.1	$I'_1\ I'_2\ ;\ I_1$
16	$(D_2/D_1, C_8/C_4)$	29.1	$I_1\ I_5\ I_8\ I_{11}$
16′	$(C_4/C_2, D_4/D_2)$	31.1	$I_1\ I_5\ I_6\ I_8\ I_9$
16″	$C_4 \times D_2$	26.1	$I_1\ I_2\ I_3\ I_5\ I_9\ I_{10}$
8	$C_2 \times D_2$	13.1	$I_1\ I_2\ I_3\ I_5\ I_6\ I_7\ I_8\ I_9\ I_{10}\ I_{12}$

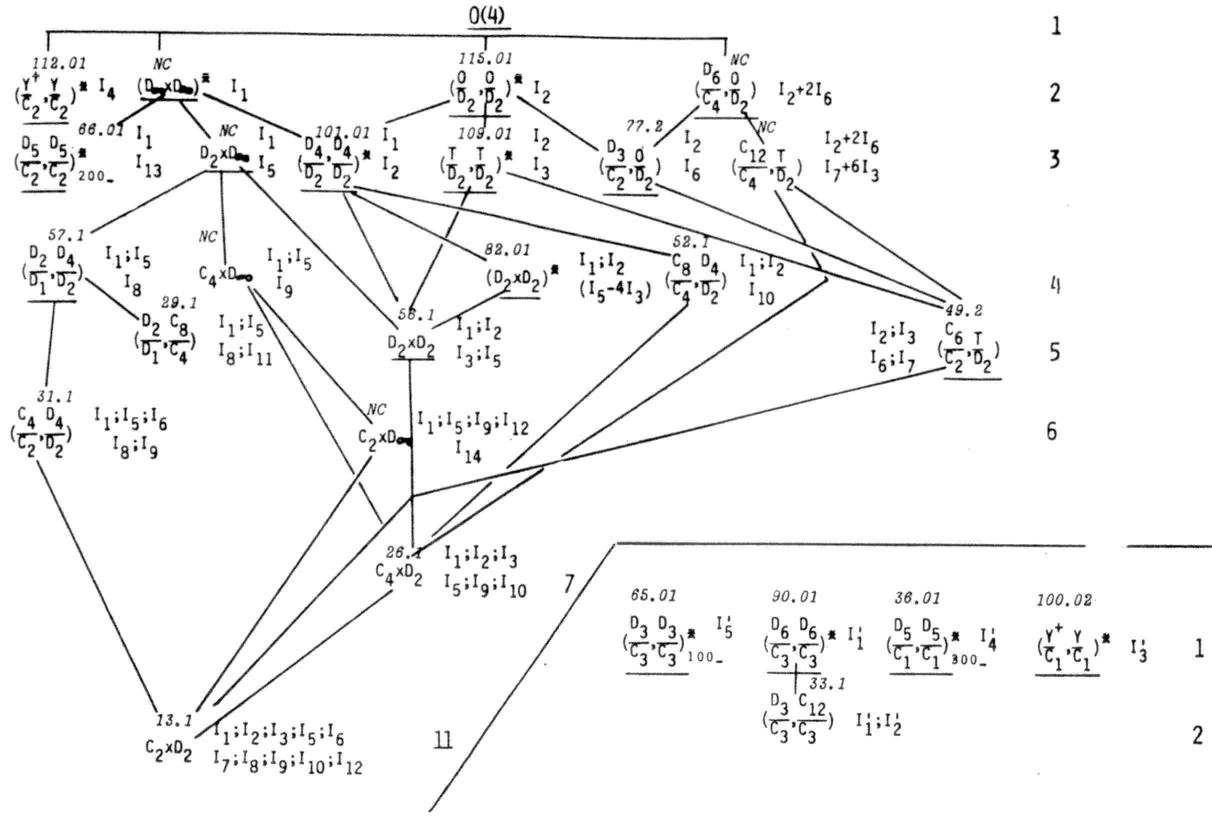

FIG. 1. — Partial order (by inclusion up to a conjugation) of the conjugation classes of irreducible subgroups of $O(4)$ which are centralizers of a vector space of 4-variables homogeneous polynomials of degree-3 and -4. Each subgroup is labelled by its order if it is finite and by a Γ symbol otherwise. For groups on the same horizontal line, the dimension of the vector space of invariant homogeneous polynomials is the same. It is indicated in the left column, for degree-3 polynomials and in the right one for degree-4 ones. The latter groups have no degree-3 invariants. All relevant information on the groups is contained in table VI. Little groups are singled out among centralizers by an underlining of their symbols.

V. Discussion and conclusion

Figure 1 shows that degree-4 homogeneous polynomials have 22 G_4 centralizers, including $O(4)$, while degree-3 terms have 5 G_3 centralizers. Thus there are 27 distinct types of Landau expansions (with or without degree-3 terms) corresponding to the four-components order parameters. Up to a conjugation the symmetries of these expansions are represented by 18 little groups, subgroups of $O(4)$. However it is the complete set of 27 expansions which have to be considered in the Landau theory. Actually, a variation of the expansion coefficients always exists, either arbitrarily imposed in order to plot the phase diagram of the system, or provoked by a change in temperature or pressure. Such variations have the effect of replacing the symmetry group G_p^α of the expansion by a conjugated group. Therefore the effective symmetry of the physical system is the common subgroup G_p of all these groups. In the case where G_p and G_p^α do not coincide, allowing for an infinitesimal variation of the expansion coefficients, we can keep the same symmetry group G_p^α and rotate instead the reference frame in the order parameter space. With respect to this frame, the stable state of the system which corresponds to an orbit [9] of directions specified with respect to the orientation of G_p^α, will also be rotated. As a consequence we can expect that for the considered G_p, part or all of the stable orbits form a stratum [9] containing a continuous family of orbits i.e. they will not be in directions which are isolated from a symmetry point of view. This property is realized, for instance in the groups $C_2 \times D_2$ and $C_4 \times D_2$ [7].

In the determination of the fixed points, one deals with a flow of the expansion coefficients, and the symmetry group common to all the stages of the renormalization should also be G_p and not G_p^α. Recently, it has been shown in an example of 2-component order parameter expansion for which $G_p \neq G_p^\alpha$, that there was a continuous line of fixed points [15]. This property seems to result from the same remark made above for the stable orbits.

The situation for $n = 4$ appears as much more complex than for lower values of n (Table I). Many of the listed expansions are likely to be realized by phase transitions in crystals. In ref. [7], 13 of the 22 expansions without cubic invariant were already found. They arise from the irreducible representations of the space groups compatible with standard crystalline transitions. Physical realizations have been observed in VO_2 [16] (expansion with little group $D_2 \times D_2^*$) and in NbO_2 ($|D_4/D_2, D_4/D_2|^*$) [3]. Other expansions are encountered in the study of modulated magnetic structures such as in DyC_2 ($D_\infty \times D_\infty^*$) [3] or structural transitions to an incommensurate phase such as in $Ba_2NaNb_5O_{15}$ ($D_\infty \times D_\infty^*$) [17] and in $BaMnF_4$ ($D_\infty \times D_2$) [18]. We notice that the latter transitions, which are also induced by space groups irreducible representations, can be associated to non-crystallographic little groups in 4-dimensions [14]. Actually, incommensurate transitions will always be related to continuous little groups (such as $D_\infty \times D_\infty^*$) : the translation matrices always form an infinite set dense in a continuous group.

Only 4 of the 21 anisotropic expansions without degree-3 terms have been examined up to now in the renormalization group theory. For $D_\infty \times D_\infty^*$ and $(D_4/D_2, D_4/D_2)^*$, an anisotropic fixed point was found, while for $D_\infty \times D_2$ and $(T/D_2, T/D_2)^*$ no stable fixed point exists in the approximation used [3, 18]. *Investigation* of the remaining expansions should provide an interesting test of the conjecture that most expansions have no stable fixed point for $n > 3$ [4].

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