THE PROBLEM OF GROUP EXTENSIONS OF THE POINCARÉ GROUP AND SU(6) SYMMETRY

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When I received an invitation to this conference, I was asked to speak on group extensions of the Poincaré group. In order to give a precise account on this subject, I wrote in advance the first three parts of this paper. At the same time I was trying to understand relativistic SU(6) symmetry and I intended to make this subject a substantial part of my oral lecture (Part IV here). Since I spoke on the last day, I have certainly been very much influenced by the free discussions and the inspiring atmosphere of this conference.

I. Simple Concepts in Group Theory

In order to present a self-contained paper, let us recall some basic facts of group theory.

I. 1 Known concepts.

We suppose that the reader knows what are a group G (we denote its group law multiplicatively), a subgroup H of G, a left coset x H, a right coset H x. We denote by $(G : H)_L$ or $(G : H)_R$ the set of left or right cosets of the subgroup H of G. If for every x (denoted $\forall x$) $\in$ G, xH = Hx, then H is called an invariant subgroup of G and one has only one set (G : H). It becomes naturally a group (group law x H.y H = x y H) called the quotient group of G by its invariant subgroup H and denoted G/H.

I. 2 Group homomorphisms.

A group homomorphism of the group G into G' is a mapping f of G in G' which preserves the group law, i.e., $\forall$ x, y $\in$ G, f(x)f(y) = f(xy) $\in$ G'. We shall denote such homomorphism by $G \xrightarrow{f} G'$. If the mapping f is one-to-one onto G', then the homomorphism is an isomorphism denoted $G \sim G'$. We denote by Ker f, the kernel of f that is the set $\{x \in G,\ f(x) = 1 \in G'\}$. Ker f is an invariant subgroup of G. We denote by Im f, the image of f, the set of values of

f. It is a subgroup of G', and we have the isomorphism:

$$G/\mathrm{Ker}\ f \sim \mathrm{Im}\ f \quad . \qquad (1)$$

We call the homomorphism $G \xrightarrow{f} G'$ and injection if Ker f = 1; then G is isomorphic to a subgroup of G'; we call it a projection if Im f = G' and we denote it $G \xrightarrow{f} G' \rightarrow 1$.

An isomorphism of G onto G is called an automorphism; the set of automorphisms of G forms a group denoted by Aut G.

As a mapping, homomorphisms can be composed. We denote by f o g the composed homomorphism, first g and then f.

Commutative diagram: it is a set of groups and homomorphisms between them such that all possible compositions of mappings which define a homorphism between two given groups of the diagram define the same homomorphism. (See examples below, e.g. Fig. 1).

1. 3 Group G acting on a set E.

Given a set **E** (finite or infinite), we denote by $\mathcal{P}(E)$ the group of permutations of its elements. We say that a group G acts on **E** when we consider a homomorphism:

$$G \xrightarrow{f} \mathcal{P}(E) \quad . \qquad (2)$$

The set **E** is called a homogeneous space of G if for every pair $\alpha, \beta \in E$ there exists ($\exists$) $x \in G$ such that f(x) transforms α into β. We also say that G acts transitively on **E**. In the general case of action of G in **E**, this set is a union of disjoint subsets which are the homogeneous space of G and are called the orbits of G in **E**. In that case, the set of $x \in G$, such that f(x) leaves $\alpha \in E$ fixed, form a subgroup of G called stabilizer of α or (by us physicists) little group of α ; we denote it by G_α. Let $\beta \in E$ and $y \in G$ such that f(y) transform α into β. Every element of the coset $y\, G_\alpha$ has the same action on α and one sees that there is a one-to-one correspondence between the elements of **E** and the cosets in G of the stabilizer G_α. The stabilizer of β is $G_\beta = y\, G_\alpha\, y^{-1}$, the subgroup G_α conjugated by y (or any $\in y\, G_\alpha^\beta$).

When, in (2), Ker f = 1, we say that G acts effectively on **E** ; if G acts ineffectively on **E**, then

$$\mathrm{Ker}\ f = \bigcap_{\alpha \in E} G_\alpha \qquad (3)$$

It can be proved that it is the largest invariant subgroup of G contained in the stabilizer G_α.

Examples:

1) If E is G itself and a is transformed by x into x a. Then G acts effectively and transitively in its set of elements. This action is said to be by left translations on G.

2) If E is $[G : H]_L$ and $x \in G$ transform a H into x a H (example 1 is a particular case of 2 for H = 1).

It can be proved that every homogeneous space of G is equivalent to an example 2, i.e., it is characterized by a subgroup H up to a conjugation in G (H is the stabilizer of an element).

I. 4 Group G operating on the group A.

If G acts on the elements of the group A by group automorphisms (i.e., $G \xrightarrow{f} \mathcal{P}(A)$ with Im $f \subset$ Aut $A \subset \mathcal{P}(A)$), then we shall say that G operates on A.

Examples: When A = G and f(x) transforms a into x a x^{-1}. then Ker f is called center of G (denoted $\mathcal{C}(G)$) and Im f is called the group of inner automorphisms of G (denoted Int G). Then in that case (1) is written:

$$G / \mathcal{C}(G) = \text{Int } G \tag{4}$$

One proves that Int G is the invariant subgroup of Aut G, and the quotient Aut G/Int G = Out G is called the group of outer automorphisms of G.

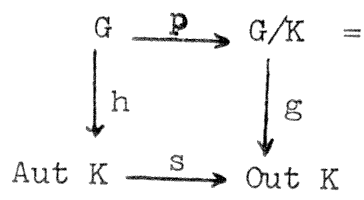
I. 5 A useful theorem.

Operations of G on itself by inner automorphisms $G \xrightarrow{f}$ Int G leave stable any invariant subgroup K, so that they induce automorphisms on K.

$$\begin{array}{ccc} G & \xrightarrow{f} & \text{Int } G \\ & {\scriptstyle h}\searrow \quad \swarrow{\scriptstyle 1} & \\ & \text{Aut } K & \end{array} \qquad h = 1 \circ f \tag{5}$$

so G operates on K (by $G \xrightarrow{h}$ Aut K).

This enables us to define a mapping g from Q = G/K on Out K = Aut K/Int K. Indeed, let $q \in Q$ and $x \in G$ such that (See Fig. 1):



Commutative diagram 1

Fig. 1 : Ker s = Int K, Ker p = K

$g \circ p = s \circ h$

$p(x) = q$; we write $h(x) = y$ and $s(y) = z$. We define $g(q) = z$. This correspondence g is a mapping; indeed, let us make another choice x' instead of x such that $p(x') = q$. Then $x' = \xi x$ with $p(\xi) = 1$, i.e. $\xi \in K$. Thus $h(\xi) \in \text{Int } K = \text{Ker } s$, $s \circ h(\xi) = 1$

and

$$z' = s \circ h(x') = s \circ h(\xi) . s \circ h(x) = s \circ h(x) = z = g(q) \tag{6}$$

Moreover, g is a group-homomorphism: indeed

$$g(q_1)g(q_2) = h \circ s(x_1) . h \circ s(x_2) = h \circ s(x_1x_2)$$

with

$$p(x_1) = q_1, \quad p(x_2) = q \quad .$$

Hence

$$p(x_1x_2) = q_1q_2 \quad ,$$

so

$$h \circ s(x_1x_2) = g(q_1q_2) \quad .$$

<u>To summarize</u>, if K is invariant subgroup of G, we have established the diagram of homomorphisms of Fig. 1 with:

$$g \circ p = s \circ h \quad , \tag{7}$$

i.e., it is a commutative diagram and we shall assert:

Theorem 1: K invariant subgroup of G ⇒ commutative diagram 1. If K is abelian, Aut = Out K, so s is the identity isomorphism.

I. 6 Extension problem.

If Q is a homomorphic image of G, we also say that G is an extension of Q : $G \xrightarrow{p} Q \to 1$, with kernel Ker p. A classical mathematical problem is "given Q and K, find all extensions G of Q with kernel K". This problem can be decomposed into a set of problems, one for each g $\in$ Hom (Q, Out K), the set of homomorphisms of Q into Out K. For some g the problem may have no solutions. There exist mathematical criteria on g for existence of a solution. One has to decide when two solutions for a given triplet Q, K, $Q \xrightarrow{g}$ Out K are to be considered as equivalent. The natural mathematical definition is the following: two extensions E_1, E_2, solutions for the triplet Q, K, g, are equivalent if there exists an isomorphism f such that the diagram (Fig.2) is commutative:

So two equivalent extensions are isomorphic but the converse might not be true.
(See below, IV. 1).

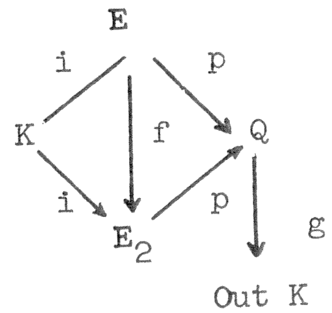


Diagram 2
i = injection, p = projection
Figure 2

For more information we refer the reader to the mathematical literature[1] or to some earlier lectures for physicists[2]. If g is the trivial homomorphism (Im g = 1), there are always solutions called "central extensions".

Particular extensions are the semi-direct products (one at most for each g); by definition Q is isomorphic to a subgroup of G that the corresponding injection i satisfies:

$$p \circ i = \text{Identity on } Q, \text{ where } Q = p(G)$$

If G is both a central extension and a semi-direct product, it is then a direct product.

II. Relativistic Invariance

II. 1 The Poincaré group P

Here we consider only the connected Poincaré group P. The translation group T is its only proper invariant subgroup. The quotient $P/T \sim L$ is the (homogeneous, connected) Lorentz group. Indeed P is the semi-direct product:

$$P = T \times L$$

To how many subgroups of P is L isomorphic? Wigner[3] has shown that all these subgroups are conjugated from each other in P. His proof is even purely algebraic, and does not use the topology of P, (see equations (38) and (39) of Ref. 3).

If $a \in T$, $A \in L$, we can denote the group law of P by

$$(a, A)(b, B) = (a + Ab, AB) \qquad (8)$$

II. 2 The universal covering $\bar{P}$ of P.

The group SL(2, C) is the universal covering of L. We denote it by $\bar{L}$. It has a two-element center Z_2 and $\bar{L}/Z_2 = L$. By composition of homomorphisms $\bar{L} \to L \to$ Aut T, $\bar{L}$ operates on T and $\bar{P}$ is the corresponding semi-direct product. The group $\bar{P}$ is the universal covering of P, i.e., it is simply connected and has the same Lie algebra ; it is an extension $\bar{P} \xrightarrow{r} P$ of P. This also means, (see, e.g. Ref. 2, chap. V) that given any central extension E with a continuous homomorphism $p : E \xrightarrow{p} P$ and with Ker p abelian, there exists a unique homomorphism $q : \bar{P} \xrightarrow{q} E$ such that the diagram

$$\begin{array}{ccccc} E & \xrightarrow{p} & P & \to & 1 \\ & {\scriptstyle q}\nwarrow & \nearrow{\scriptstyle r} & & \\ & & \bar{P} & & \end{array}$$

is commutative

Whether this property is still true if no continuity is imposed on P is not known.

II. 3 The action of P on the space-time $\mathcal{E}$.

The group P acts effectively and transitively on space-time. The stabilizer (= little group) of one point is L. The orbits of L on $\mathcal{E}$ are well-known to physicists; they are the origin 0, the two halves of the lightcone (without the origin), the sheets of the hyperboloid corresponding to constant Minkowski length of vectors of $\mathcal{E}$ with origin 0.

Through P, the group $\bar{P}$ also acts on $\mathcal{E}$.

II. 4 The group Aut P.

The group L has only one class of continuous outer automorphisms, that of space inversions. So for continuous automorphisms, $Out_c L = Z_2$, the two element group.

When one forgets the topology of L, one can consider the group Out L. This group is known (see for instance[4] § 7. 1) and is abhorrently large ($\sim \mathcal{P}$ (E), where the set E has the power of the continuum). The situation is similar for the translation group.

Space inversions also induce on P a class of continuous outer automorphisms. With the notation of (8) the correspondence,

$$(a, A) \dashrightarrow (-a, A) \quad , \tag{9}$$

is an automorphism of P called space-time inversion. Let α be a positive real number ; the correspondence:

$$(a, A) \dashrightarrow (\alpha a, A) \tag{10}$$

is called a dilatation. Hence, the multiplication group R^x of the field of real numbers operates on the Poincaré group P by (9) and (10). It is the direct product:

$$R^x = Z_2 \otimes R \quad . \tag{11}$$

An opinion poll among physicists shows that they all know that:

$$Out_c \; P = Z_2 \otimes Z_2 \otimes R \tag{12}$$

(space inversion, time inversion, dilatation) and that Aut P is the semi-direct product:

$$Aut \; P = P \times Out \; P \tag{13}$$

We have omitted the index c for continuity because it can be proved[5] that every automorphism of the Poincaré group is continuous and that Aut P is given by (13) and (12). This is related to the remarkable theorem of Zeeman[6] .

Theorem Z: The permutation group of points in space-time which preserves, or completely reverses, the partial order relation corresponding to causality is Aut P.

This theorem can also be formulated:

Theorem Z': The permutation group of points in space-time which transforms light rays into light rays is Aut P.

We shall now show that this remarkable theorem imposes strong conditions on the invariance group of a relativistic theory.

II. 5 The invariance group G of a relativistic theory.

With F. Lurçat, I have suggested elsewhere[7,8] that the invariance group of a physical theory is an extension of the Poincaré group. Indeed P, as automorphism group of the theory, is a quotient of the invariance group G of the formalism of the theory by the invariant subgroup K which does not act on space-time or on its dual, the energy-momentum vector space. This is a generalization of the classic Wigner analysis of ray representations of P for quantum mechanics[3]. Zeeman's theorem makes this argument even more powerful.

Every element of the invariance group G of relativistic theory acts or does not act (i.e., acts trivially) on space-time. By this action, either space-time is transformed into itself, although physicists, needing more freedom, invent the possibility of sending it into a larger manifold[9]. We will consider here the former alternative only: the group G permutes space-time points. In order not to destroy relativity, it transforms light rays into light rays. So, by the Zeeman theorem, there exists a mapping $G \dashrightarrow \text{Aut } P$. The day has not yet arrived in which someone dares to make this mapping not a group homomorphism. So I would say that today all physicists who do not escape from space-time agree on the existence of a homomorphism:

$$G \xrightarrow{\varphi'} \text{Aut } P \qquad (14)$$

for the invariance group of a special-relativistic theory.

Thanks to Lee and Yang, physicists learned that $\text{Im}\,\varphi'$ does not contain space inversions when the relativistic theory deals with weak interactions. Only a few relativistic theories are dilatation-invariant and the $K_2^0 \to 2\pi$ decay has recently revived doubts about time inversion invariance. By definition of a relativistic theory $\text{Im}\,\varphi' \supset P$. So, neglecting eventually inversions and dilatations, we are led to the conclusion: the group G of invariance of a relativistic theory is an extension:

$$G \xrightarrow{\varphi} P \longrightarrow 1 \qquad (15)$$

of the Poincaré group.

III. Mathematical Results on Extensions of P

III. 1 The extension problem.

We just give here the relevant known mathematical results and the pertinent references.

As we have seen in Part I, there is a general mathematical theory for the study of the following problem:

"Given the group K, Q and the homomorphism $Q \xrightarrow{g} \text{Out } K$, find all extensions E of Q with kernel K such that E operates on its invariant subgroup K according to diagram of Fig. 1".

In the case where K is the Poincaré group, for any pair Q, g, ($Q \xrightarrow{g} \text{Out } P$) there is one and only one solution, the semi-direct product: one solution, because Aut P is a semi-direct product, one only, because P has no center (ref. 1 for instance). It is to be remarked that no topological considerations are involved in that case.

However, the case of interest for us is when the Poincaré group is the quotient Q.

III. 2 Central extensions of the Poincaré group (i.e., g = 0).

The case of central extension is the only one to be considered for groups K such that there is one only homomorphism, $P \to \text{Out } K$, the trivial one. This is true when K is a finite group or a compact-simple Lie Group (ex : SU_3) even when its topology is neglected. If one considers only continuous g homomorphisms, this is then also true when K is a compact group or a semi-simple (finite-dimensional) Lie group.

Physically when K is generated by infinitesimal operators invariant under P (charge, isotopic spin, unitary spin and so on) one has of course to choose a central extension of P.

Then, there is always at least a solution, the direct product $K \otimes P$. I have shown[4] that there is a one-to-one correspondence between the central extensions of P by K and those of L by K. To obtain this result, again no topology is involved (another way to say it: it is a property of the abstract P and L groups). It can be shown that the only central extensions of P by K interesting for physicists are those of the form:

$$E_\alpha = \frac{K \otimes \bar{P}}{Z_2(\alpha)} \quad , \tag{16}$$

i.e., the quotient of the direct product $K \otimes \bar{P}$ by a two element group Z_2 generated by $(\alpha,\omega) \in K \otimes \bar{P}$ where $\alpha \in \mathcal{C}(K)$= Center of K and a square root of 1 ($\alpha^2 = 1$) and $\omega \in \bar{P}$ is the "2π rotation" (i.e., non unit element of the center of $\bar{P}$). If $\alpha = 1$, E_α is the direct product $K \otimes P$.

When the center of K has no divisible subgroup (A group is divisible if every element has at least an n^{th} root for every n) I have proven[4] that the only central extension of the abstract group P by the abstract K is of the form (16). (This is therefore true for every semi-simple Lie group K). For other groups K, it is not known if there are other extensions, but, if they exist, I have proven in Ref. 4 that they have properties so pathological (and related to the abhorent Aut L you have heard of previously) that physicists would not consider them.

III. 3 Noncentral extension of P.

Nothing as general can be said in that case as an application of the abstract group theory of extension. There is in the mathematics literature nothing written on extension theory for topological groups although there is a general theory for groups with Borel structure[10] (i.e., a structure weaker than a topology). But the problem is completely solved in the mathematical literature if we consider only Lie groups and Lie group homomorphisms. Indeed, in that case, we can pass to the Lie algebra. From the semi-simplicity of L, the last theorem of Ref. 11 tells us that the only solution in that case is the semi-direct product of the Lie Algebras[12].

However in physics we do use more general extensions of the Poincaré group. This is the case of the invariance group of electrodynamics where the kernel K of the extension $G \to P \to 1$ is the gauge group of the second kind. This group is not a Lie group, but a functional group, and P operates nontrivially on it. To my knowledge, the classification of this kind of extensions of P has not been made.

IV. Application To SU(6) Symmetry

IV. 1 Physical interpretation for the extensions of P

We have shown that relativistic invariance in elementary particle theory leads us to adopt as G, the invariance group of the formalism, an extension of the Poincaré group. Then the "mixing" of Poincaré invariance and the internal symmetry invariance K is "rather poor". It has, however, to be interpreted, and in 1961, with Lurçat[7], I proposed as an interpretation, the existence of the relation "b + ℓ + 2j is even" (b = baryonic charge, ℓ = sum of the leptonic charges, j = spin) for any physical state. The group K we considered was the direct product of U_1's (first kind gauge group for baryonic, electric, leptonic charges). It must be emphasized here that although in that case all extensions given by equation (16) are isomorphic, they are inequivalent, i.e., distinct for the mathematical theory of group extensions. The physical interpretation uses very naturally this distinction.

Indeed, physicists have not only to consider abstractly the invariance group G of a physical theory, but they have to label its elements, i.e., to decompose them in a product of "simple physical operations" such as space or time translation, pure Lorentz transformation, covering of a space rotation, isospin rotation, hypercharge gauge transformation, etc. The appendix shows how such a labeling leads very naturally to conditions on the mixing of internal symmetry and relativistic invariance.

IV. 2 The group of invariance of Wigner's supermultiplet theory

The physicists who introduced SU(6) symmetry[13,14,15] considered it as a generalization of Wigner's supermultiplet theory[16]. So let us first study the Galilean invariance of Wigner's theory.

In it, a nucleon wave function is a square integrable function $\psi(x, \sigma, \tau)$ of space-time, spin and isospin. So the one-particle states form the Hilbert space $\mathcal{H}^1$ which is a tensor product

$$\mathcal{H}^{(1)} = \mathcal{H}_x^{(1)} \otimes \mathcal{H}_\sigma^{(1)} \otimes \mathcal{H}_\tau^{(1)} \qquad (17)$$

of three Hilbert spaces ; the last two, $\mathcal{H}_\sigma$ and $\mathcal{H}_\tau$ have only two dimensions each. Physical n-particle states belong to $S \otimes \mathcal{H}^{(1)} = \mathcal{H}^n$ (where S is the projection on completely symmetric or completely antisymmetric tensors if the particle is a boson or a fermion) and the Hilbert space of state-vectors $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$ can also be written as

$$\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_\sigma \otimes \mathcal{H}_\tau \qquad (18)$$

When one can neglect spin and isospin dependence of nuclear forces, Wigner's supermultiplet theory is a good approximation in which the Hamiltonian operator is of the form

$$H = H_x \otimes (I_\sigma \otimes I_\tau) \qquad (19)$$

where I is the identity operator. We used the brackets in order to write simply $I \otimes A$ for any operator not acting on $\mathcal{H}_x$. Therefore it commutes with H. In order to preserve the hermitian metric, A must be unitary. The bold step made by Wigner was to pass from the invariance under the group $U_2 \otimes U_2$ (whose unitary representation $I \otimes [D_\sigma(U_2) \otimes D_\tau(U_2)]$ commutes with H) to invariance under the group $U_{2x2} = U_4$ the largest unitary group acting on $\mathcal{H}^{(1)}$, but not on space time or on $\mathcal{H}_x^{(1)}$. This internal symmetry group of invariance acts on $\mathcal{H}$ as $I \otimes D(U_4)$ and so commutes with H.

Let us study now the Galilean invariance of the theory. It appears through a central extension $\mathcal{G} : \mathcal{G} \xrightarrow{p} G \to 1$ of the Galilean group G (see for instance 17). Since G acts on space time, $\mathcal{G}$ does it also through G. But it also acts on spin, and therefore on $\mathcal{H}_\sigma$, through the homomorphic image SU_2 :

$$\mathcal{G} \xrightarrow{\varphi} SU_2 \to 1 \qquad (20)$$

where SU_2 is the covering of the three-dimensional rotation group.

Table 1 summarizes the actions of $\mathcal{G}$ and U_4 on $\mathcal{H}$. These actions do not commute. So what is the invariance group of the theory?

$$\mathcal{H} = \mathcal{H}_x \otimes (\mathcal{H}_\sigma \otimes \mathcal{H}_\tau)$$

action of $\mathcal{G}$: $D_x(\mathcal{G}) \otimes D_\sigma[\varphi(\mathcal{G})] \otimes I_\tau$

action of U_4 : $I_x \otimes D(U_4)$

action of $\mathcal{G}'$: $D_x(\mathcal{G}') \otimes I_\sigma \otimes I_\tau$

action of $\mathcal{G}' \otimes U_4$: $D_x(\mathcal{G}') \otimes D(U_4)$

TABLE 1

Let us denote by $\mathcal{G}'$ a group isomorphic to $\mathcal{G}$, acting as the Galilean group on space-time (therefore $\mathcal{G}'$ acts on $\mathcal{H}_x$), but acting trivially on spin and, of course, on isospin. Then one sees easily that the direct product $\mathcal{G}' \times U_4$ acting on $\mathcal{H}$ by the unitary representation $D_x(\mathcal{G}') \otimes D(U_4)$, is the invariance group of Wigner's supermultiplet theory. This group contains the physical extension $\mathcal{G}$ of the Galilean group as a subgroup : $\mathcal{G} \subset \mathcal{G}' \otimes U_4$, but $\mathcal{G}$ is placed in a skew way ; its elements are the form $[g, \varphi(g)]$ where $g \in \mathcal{G}'$ and $\varphi(g) \in SU_2 \subset U_4$. (Among elements of U_4, some are labeled "spin transformations" and form a sub-group SU_2, image of $\mathcal{G}$). (See figure 3). A supermultiplet corresponds to an irreducible representation $\mathcal{U}$ of $\mathcal{G}' \otimes U_4$; it becomes reducible when it is restricted to $\mathcal{G}$ and this gives the spin content of the supermultiplet.

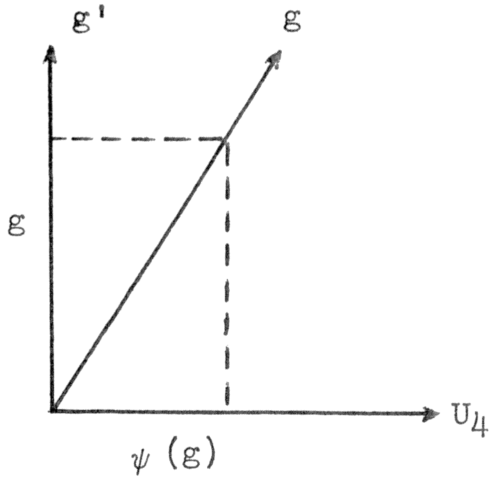


FIGURE 3

IV. 3 Possible group of invariance of a relativistic SU(6)-theory

The invariance of the SU(3)-theory is easy to describe. We can repeat for that case what we did in IV. 2. Since a one particle wave function $\psi(x, \mu, \rho)$ is a function of three kinds of variables: space-time variable x, spin or Lorentz indices μ, "unitary spin" indices ρ, we are led to consider the Hilbert space $\mathcal{H}$ of physical state vectors as a tensor product.

$$\mathcal{H} = \mathcal{H}_x \otimes \mathcal{E}_\mu \otimes \mathcal{H}_\rho \tag{21}$$

The invariance group is the direct product $\bar{P} \otimes SU(3)$ where $\bar{P}$ is the covering of the Poincaré group P. Through P, the group $\bar{P}$ acts on space-time, hence on $\mathcal{H}_x$. It also acts on μ-indices, hence on $\mathcal{H}_\mu$ through $\bar{L} = SL(2, C)$:

$$\bar{P} \xrightarrow{\theta} \bar{L} \longrightarrow 1 \quad , \tag{22}$$

the covering of the homogeneous Lorentz group. We summarize this in the first line of Table II.

We face now a difference with table I: the new fact, well-known, for instance, in Dirac theory, is that $D_\mu[SL(2,C)]$ is not a unitary representation and $\mathcal{E}_\mu$ is not a Hilbert space. Hence $D_x(\bar{P}) \otimes D_\mu [SL(2,C)]$ is not a unitary representation of the direct product $\bar{P} \otimes SL(2,C)$. However, the representation $D_x(\bar{P}) \otimes D_\mu[\theta(\bar{P})]$ is a unitary representation of $\bar{P}$. So with finite-value spin indices, in order to disconnect spin and space one has to pay the price of handling nonunitary representations.

In the "static" approximation, the Wigner step is to enlarge $SU(2) \otimes SU(3)$ to $SU(6)$. One can prove[18] that for $SL(2,C) \otimes SU(3)$ the minimum step is the enlargement to $SL(6,C)$. But as I learned at this Conference, some physicists are even bolder! Because of parity conservation one has to consider the orthochronous group $L^\uparrow$ whose fundamental finite representation[19] is real and four-dimensional. The corresponding Wigner step leads to a 4 x 3 = 12 dimensional linear group $U(6,6)$. This group is a noncompact real form of $U(12)$; it leaves invariant the pseudo-hermitian metric with 6 plus signs and 6 minus signs. It is a 144 parameter group. Its Lie algebra is that of the associative algebra on the real field generated by the tensor products of the 16 Dirac matrices and the 9 unitary 3 by 3 matrices. Of course $SU(6,6) \supset SL(6,C)$

$$\mathcal{H} = \mathcal{H}_x \otimes \mathcal{E}_\mu \otimes \mathcal{E}_\rho$$

$\bar{P} \otimes SU(3)$ acts as	: $D_x(\bar{P}) \otimes D_\mu[\theta(\bar{P})] \otimes D(SU_3)$
$\bar{P}' \otimes SL(2,C) \otimes SU(3)$	: $D_x(\bar{P}') \otimes D_\mu[SL(2,C)] \otimes D_\rho(SU_3)$
$\bar{P}' \otimes SL(6,C)$	: $D_x(\bar{P}') \otimes D[SL(6,C)]$
$\bar{P}' \otimes SU(6,6)$	: $D_x(\bar{P}') \otimes D[SU(6,6)]$

TABLE II

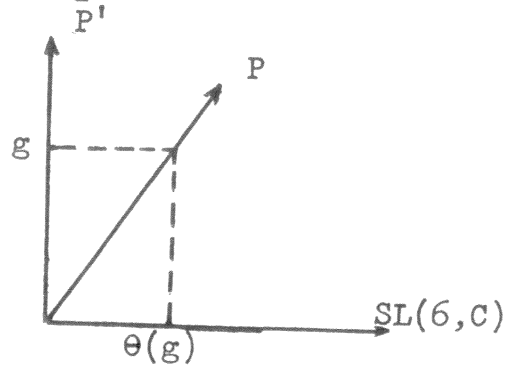


FIGURE 4

The two alternatives SL(6,C) (70 parameters) or SU(6,6) (143 parameters) are written in Table II. The covering of the physical Poincaré group $\bar{P}$ is placed as a subgroup of $\bar{P}' \times SL(6,C) \subset \bar{P}' \times SU(6,6)$ by the injection $\bar{P} \xrightarrow{i} \bar{P}' \times SL(6,6)$ with $i(g) = [g, \theta(g)]$.

Tensor products of irreducible representations of $\bar{P}'$ (unitary representations) and SL(6,C) (unitary infinite of finite nonunitary representations) are, for $\bar{P}' \otimes SL(6,C)$, irreducible representations whose restriction to $\bar{P} \otimes SU(3)$ is unitary and reducible. The reduction yields the spin and unitary spin content of the theory.

Since the translation group $T \subset \bar{P}$ is represented trivially in SL(6,C) ($T \subset \text{Ker}\ \theta$) this scheme cannot yield a mass formula by pure group-theoretical means.

Conclusion

In order to transform the invariance group of Wigner's non-relativistic supermultiplet theory without sending space-time into a larger manifold (36 dimensions at least for SU(6) theory[15]) one can adopt the scheme described above. In the burgeoning output of papers on relativistic SU(6) theory, I do not know yet if this scheme is widely accepted. In many papers it is difficult to guess the framework, generally very much hidden by the superstructures.

It is easy to write interaction terms of a Lagrangian, invariant under $\bar{P}' \otimes SL(6,C)$, but nobody has yet written a full Lagrangian with such a property. Indeed the gradient $\partial/\partial x^{\mu}$ is transformed by $\bar{P}'$ but cannot be transformed by SL(6,C); so for this group, it is to be considered as a spurion. It might be that the badly broken finite-parameter-Lie-group invariance is only a pale reflection of a deep invariance under an infinite parameter Lie group. In the recent years it has always come as a surprise that strong-coupling approximate symmetries are much better and more useful than one would have expected. Relativity does introduce a kinematical spin-orbit coupling (e.g. Thomas precession) which can be handled by pure group theoretical techniques. The recent boom of SU(6) theory seems to indicate that once these space-spin kinematical correlations have been (more or less awkwardly!)taken into account, the dynamics seems rather "spin-unitary spin" independent.

This situation is just the opposite of the revolutionary situation considered by F. Lurcat[20] where dynamics does mix intimately spin and space degrees of freedom.

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Appendix

On relations between internal symmetry and relativistic invariance

Let us study the invariance group G of a relativistic theory with an "internal symmetry" given by the invariance under a group S. Then (as we explained in IV. 1) every $g \in G$ has to be labeled with the elements of $\bar{P}$ and S, i.e., g is made (as a product) of elements of $\bar{P}$ and of S. (There exists a mathematical characterization of the abstract group G in terms of the abstract groups $\bar{P}$ and S). A natural hypothesis is that every $g \in G$ is a product:

$$g = ps \quad \text{with} \quad p \in \bar{P},\ s \in S. \tag{1}$$

That $\bar{P}$ and S have no common element implies (easy proof, given in Ref. 8) the uniqueness of the decomposition of (1). To summarize:

Hypothesis 1: $\bar{P}$ and S are subgroups of G with $\bar{P} \cap S = 1$ and every element $g \in G$ is a product as in equation (1). We use for this hypothesis the condensed notation: $G = P \cdot S$. This also implies $G = S \cdot \bar{P}$.

During the past year, under the name of McGlinn's theorem, many papers[21] have given long proofs of various versions of the following lemma:

Lemma G: If $G = \bar{P}.S$ and if $\forall \ell \in \bar{L} \subset \bar{P}$, $\forall s \in S$, $\ell s = s\ell$, then G is the semi-direct product $G = \bar{P} \times S$ (where $\bar{P}$ is kernel, S quotient).

$G = \bar{P}.S$ implies a natural one-to-one correspondence between $E = [G : \bar{P}]_L$ and S. By left translations G acts on the space E $G \xrightarrow{f} \mathcal{P}(E)$ (See I. 3, example 2) explicitly: $g \in G$ transforms $s\bar{P}$ into $gs\bar{P}$. The hypothesis $\ell s = s\ell$ implies $\ell s\bar{P} = s\bar{P}$ so $\bar{L} \subset G$ does not act in E, i.e., $\bar{L} \subset K = \text{Ker } f$. Hence $K \cap \bar{P}$ is an invariant subgroup of $\bar{P}$ which contains $\bar{L}$. Hence $\bar{P} \cap K = \bar{P} \subset K$ and $\forall p \in \bar{P}$, $ps \in s\bar{P}$ or $s^{-1}ps \in \bar{P}$. In other words, $\bar{P}$ is an invariant subgroup of $\bar{P}$; with $G = \bar{P}.S$, this proves $G = \bar{P} \times S$.

In fact, the proof uses only the weaker hypothesis:

Lemma $\bar{P}$: If $G = \bar{P}.S.$ and if there is one $p \in \bar{P}$ which is not the covering of a translation such that $\forall s \in S$, $s^{-1}ps \in \bar{P}$, then G is the semi-direct product $\bar{P} \times S$ (where $\bar{P}$ is kernel, S quotient).

By exchanging the role of $\bar{P}$ and S, one obtains the symmetric lemma:

Lemma S: If $G = \bar{P}.S$ with S a simple group (no proper invariant subgroup) and if there is one $s \in S$ such that for every $p \in \bar{P}$, $p^{-1} s p \in S$, then G is the semi-direct product $G = S \times \bar{P}$ (with P as quotient).

In III. 2, we have seen that if S is a simple Lie group, then G is the direct product $S \otimes \bar{P}$.

If one internal symmetry observable is relativistically invariant (e.g. hypercharge) any element $s \in S$ of the one parameter group it generates as infinitesimal operator satisfies the condition of the lemma S, which forbides "mixing" of internal symmetry and Poincaré invariance.

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DISCUSSION

Coleman - There is another objection that can be made to this last structure. The construction for finding representations which you gave is the sort of construction that one would go through if one were attempting to find the possible fields that are invariant under this group. However, there is, of course another way of applying representation theory to groups in physics, and that is to look for the infinite-dimensional unitary representations of the full inhomogeneous group, to look for the possible particles, as Wigner did for the Poincaré group in the late 30's. If one attempts to do that with this group, one finds, because the little group is noncompact, there are an infinite number of particles for any fixed momentum four-vector; every super-multiplet is infinitely degenerate. I think that this is very, very unsatisfactory from the viewpoint of physics, much more unsatisfactory than any of the handicaps you have mentioned.

Michel - Well, I know that, everybody here knows that, but when you have an objection, just apply it to a theory which you know works well. If you take a Dirac field without the Dirac equation - if it is not solution of the Dirac equation - the infinite representation is not unitary; it is unitary only if you restrict it to solutions of the Dirac equation. And, of course, here there is a difficulty, because the Dirac equation is not invariant under this direct product. We are really at the limit of symmetries which are formal, which exist not in the way of writing, but exist in nature. But you know that isotopic spin just happened this way. It was a formal symmetry which existed in writing. Now every physicist feels this symmetry.

Coleman - The example of the Dirac equation is an excellent argument against this sort of structure. One might argue that precisely because this sort of difficulty arises if you do not have a kinetic term, you must put the kinetic term in the Lagrangian. I think that an argument based on unitary representations of the group, an argument that does not go through field theory and does not resort to local causality or to Lagrangians is an insensitive argument. All it involves are the most fundamental principles of quantum mechanics and our basic ideas about how symmetry groups enter into quantum mechanical systems. In order to overthrow such arguments would seem to require a much more radical revision of our fundamental ideas than any that has been presented at this conference.

Sudarshan - Following up the train of thought that

Professor Michel outlined with regard to the "forms" that people have to fill in, I think when anybody writes a paper about a spin-dependent group they should fill in a blank space to say what they mean by "spin". In this argument you have presented here, for example, you have written that the spin in relativistic theory must be SL(2,C). This is a kind of spin but this is not the only kind of spin. The day before yesterday I briefly outlined some work of Mahanthappa and myself, Riazzuddin and Pandit, which used another definition of spin in relativistic theory. And, in fact, you can construct a particular model which happens to be invariant under the Poincaré group, that is, it furnishes the representations of the Poincaré group by unitary matrices in a Hilbert space, and it also furnishes the representation of a certain spin-dependent group. Both of these papers are correct but not as obviously precise as your paper, so they might actually be published!

Michel - I think it is useful to try to make everything precise and of course it is much more interesting to try to make precise the difficult things in physics, not just the things which are going well. There is some trouble here - I completely agree - but I do not have time to dwell on it. But you know that the way to test an objection is to test the same objection on a theory which works well. Of course, if you have for instance, a Klein-Gordon equation, which several people have mentioned during this conference, a part of our trouble is gone. But we need more in physics; we need also dynamics and we need some spin-orbit coupling. The Dirac equation is a very particular way to write the spin orbit coupling. You could have non-relativistic spin-orbit coupling. The Wigner super multiplet theory works because we, of course, by assumption, supppress any spin-orbit coupling. Therefore the Schrödinger equation has only the Laplacian which corresponds to the Klein-Gordon operator. The question rightly is very exciting because we know this difficulty. I think we have to insist on it, but we are really at the forefront when we are considering things which do not exist but which are still useful to consider. And we have to study the difficulty - that is the most exciting thing. But I think that if you accept the framework we have been obliged to think the last few years, you have really to consider direct products of groups. There is not other way of mixing things if you assume what everybody has assumed for the past few years.