COURSE ON
MATHEMATICAL PROBLEMS OF QUANTUM THEORY
OF PARTICLES AND FIELDS

(VARENNA 21 July - 9 August 1958)

C O N T E N T S

L E C T U R E S

(cont' d)

(cont.'d)

REPRODUCED PAPERS

VARENNA July 21 - August 9, 1958

PROGRAM

Cycle 1) Functional analysis

L. Gårding and J.L.Lions (14 hours). -

Abstract spaces and the function spaces of physics. Distributions. Lorentz invariant distributions. Hilbert space and spectral theory. The method of orthogonal projections for elliptic operators. The classical Hamiltonians. Fourier transform. Laplace transform. Nuclear theorem of Schwartz.

Cycle 2) Group theory and other formalisms

L. Michel (5 hours). -

Généralités sur les groupes abstraits. Sous-groupes distingués, quotient, homomorphismes de groupe; produit de groupe. Généralités sur représentations linéaires (- ou semi-linéaires - ou projectives), irréducibilité, réducibilité Lemme de Schur. Groupes finis.- Algèbre du groupe. Algèbre des classes. Détermination de toutes les représentations irréductibles. Produit tensoriel de représentations équivalentes et groupe symétrique. Critère pour la réalité d'une représentation. Groupe de Lie. - Groupes topologiques, groupes de Lie finis, groupe linéaire et ses sous-groupes. Mesure de Haar, intégration sur les groupes. Les groupes compacts et leurs représentations. Transformations infinitésimales. Algèbre de Lie, algèbre enveloppante, dérivation, représentation des algèbres de Lie, opérateur de Casimir, invariants.

L. Michel and A. S. Wightman (5 hours). -

Definition of homogeneous and inhomogeneous Lorentz groups (real and complex groups). Topology of group manifold; realization of covering group of homogeneous group by 2x2 unimodular matrices. Infinitesimal operators; their commutation relations; structure constants of the Lie algebra. Relativistic invariant representations up to a factor of relativity group. Physical equivalence and the unitary equivalence of representations; irreducible representations and elementary systems. Representations up to a factor and representations of the covering group. Which transformations unitary and which antiunitary? Physical meaning of the infinitesimal operator (preliminary discussion). Invariants of the representations: homogeneous and inhomogeneous group. Reduction of representations of the inhomogeneous group; the translation sub-group; the little groups. Irreducible representations of the little groups of time like and space like momenta. Adjunction of inversions. The representations of the little groups of space like and null momenta. The finite dimensional representations of the homogeneous Lorentz group.

E. R. Caianiello (4 hours-partially related also to cycle 4)).-

Algebra of products of field operators. Perturbative expansions. Regularization and renormalization.

Group Theory.

L. MICHEL
University of Lille

. – Generalities on abstracts groups.

Group. – A set G has the algebraic structure of a group if there exists a napping of $G \times G$ onto G which is associative;

There exists one and only one identity element:

— Each element has an inverse.

Two kinds of notations are used for groups:

a) Multiplicative:

The product of two elements $x \cdot y \in G$ is denoted by $x \cdot y$; one has $Z = x \cdot y$, $\in G$. Associativity: $(x \cdot y)Z = x(yz)$.

($\exists$ means «there exists».) For all $x \in G$, $\exists 1 \in G \to 1 \cdot x = x \cdot 1 = x$,

For all $x \in G$, $\exists x^{-1} \in G \to x^{-1}x = xx^{-1} = 1$.

b) Additive:

The composition of two elements $x, y \in G$ is denoted by $x + y$; the identity lement is denoted by 0.

One has:

$$(x + y) + Z = x + (y + Z)\,,$$

$$0 + x = x + 0 = x\,,$$

$$\exists(-x) \to x + (-x) = (-x) + x = 0\,.$$

Examples. – The set of real number excluding 0 forms a group for the mul-iplication.

The set of *all* real numbers forms an group for the addition.

Abelian group. – When the operation of the group is commutative, the roup is called *abelian* or commutative; the addictive notation is often used n this case.

Notation. – If X is a sub set of G one writes $X \subset H$

$$XY = \{x, y;\ x \in X,\ y \in Y\}$$

$$X^{-1} = \{x^{-1},\ x \in G\}$$

$$xX = \{xy,\ y \in X\}$$

$$aG = \{ay,\ y \in G\} = G$$

(for all $y \in G\ \exists\ x,\ ay = x;\ x = a^{-1}y$).

Left (right) translation. – Multiplying all elements of the group on the left (right) by one fixed element is called performing a *left translation*.

This is a one-to one mapping of the group onto itself.

Subgroup. – If 1) H is a subset of G,

2) H is a group,

then H is said to be a subgroup of G.

A necessary and sufficient condition is:

$$\text{if } x \in H,\quad y \in H, \qquad xy^{-1} \in H \quad (\text{or } x^{-1}y \in H).$$

Examples. – G and 1 are subgroups of G.

Definition: Equivalence relations: A relation $\sim$ is said to be an equivalence relation

1) It is reflexive: $x \sim x$.

2) It is symmetric: $x \sim y \Rightarrow y \sim x$.

3) It is transitive: $x \sim y,\ y \sim Z \Rightarrow x \sim Z$.

Equivalence class. – Given an equivalence relation (R) between elements of a set (E), the elements equivalent to an element $x \in E$ form the « equivalence class » $\bar{x}$.

The intersection of any two equivalent classes is empty. The reunion of all classes is E.

Application: Left cosets modulo H; invariant subgroups-factor groups. The relation $x^{-1}y \in H$ is an equivalence relation between x and y.

— it is reflexive: $x^{-1}x = 1 \in H$;

— it is symmetric: $x^{-1}y \in H \Rightarrow y^{-1}x = (x^{-1}y)^{-1} \in H$;

— it is transitive: $x^{-1}y \in H,\ y^{-1}Z \in H \Rightarrow x^{-1}Z = (x^{-1}y)(y^{-1}Z) \in H$.

The equivalence class $\dot{x} = \{y, x^{-1}y \in H\}$ is: $\dot{x} = x \cdot H$, since $y \in xH$; it is called the left coset of x modulo H.

Remarks. – If the group is finite, the product of the number of equivalence classes by the number of elements of H is equal to the number of elements of G.

The equivalence relation $x \sim y$ $(x^{-1}y \in H)$ is « compatible » with the left translations of the group: $x \sim y \to ax \sim ay$ since $(ax)^{-1}(ay) = x^{-1}y \in H$; but it is not in general compatible with the right translations, *i.e.* $xa \nsim ya$.

Compatibility of the equivalence relations also with the right translations requires that left and right cosets be the same, *i.e.*: $xH = Hx$ for all $x \in G$.

Indeed if $x^{-1}y \in H$, $xa \sim ya$ for all a in G, then $(xa)^{-1}(ya) \in H$; so that $a^{-1}(x^{-1})a \in H$.

If $xH = Hx$, H is called an « invariant subgroup.

There exists a group structure on the set of cosets $\dot{x}$ modulo H (invariant).

Element:	$\dot{1} = 1H = H$,
product:	$\dot{x}\dot{y} = xH\,yH = xyHH = xyH$,
inverse:	$xH\,yH = H$ when $y \sim x^{-1}$,
(xHyH)	($\sout{xH\,xH}$ $= xy\,HH = xyH$).

This group is called the quotient group $Q = G/H$.

Examples. – G = inhomogeneous Lorentz group.

H = group of translations.

$Q = G/H$ is isomorphic to the homogeneous Lorentz group.

if H_1 and H_2 are invariant subgroups of G_1, $H_1 \cap H_2$ is invariant.

G and I are invariant subgroups.

A group whose only invariant subgroups are itself and unity is called « simple ».

Homomorphism. Isomorphisms. – Consider two groups G, G' and a/f on G such that $D(F) = G$, $R(F) \subset G'$ (or $R(f)$ = image f.) If f, preserves the group property:

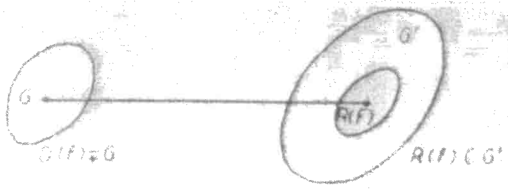

Fig. – 1.

$xy = Z \to f(x)\,f(y) = f(Z)$ it is called a homomorphism.

A homomorphism of G into G is called an endomorphism.
A homomorphism of G one to one onto G' is called an isomorphism.
An isomorphism of G onto itself is called an automorphism.
The image of a subgroup of G is a subgroup of G'.
The image of an invariant subgroup of G is an invariant subgroup of G'.
Therefore the unit element of G' is the image of an invariant subgroup K of G, which is called the kernel of the homomorphism:

$$K \text{ such that } f(K) = 1 .$$

The factor group G/K is isomorphic with the image of G.

Example. – If $a \in G$, $x \in G$, the isomorphism $x \to y = axa^{-1}$ is called an inner automorphism; it defines an equivalence relation: $y \sim x$ is called « conjugation ».

The set of all automorphism is a group $\mathcal{A}$.
The set of inner automorphisms is a group A.

Exercise. – Show that A is invariant subgroup of $\mathcal{A}$.

Product of two groups. – Given two groups G_1, G_2 one defines:

$$G = G_1 \times G_2 = \{(x_1, x_2);\ x_1 \in G_1,\ x_2 \in G_2\} .$$

The composition law is given by:

$$(x_1, x_2)\cdot(y_1, y_2) = (x_1 y_1, x_2 y_2) .$$

This definition can be extended with some caution to that of the product of an infinite number of groups.

Exercise. – If H_1 is an invariant subgroup of G_1

H_2 is an invariant subgroup of G_2,

prove that $(G_1/H_1)\times(G_2/H_2)$ is isomorphic with $(G_1 \times G_2)/(H_1 \times H_2)$.

Center Z of a group G. – It is the set of all x of H which commute with every element of G; $x \in Z$ if $xy = yx$ for all $y \in G$.

Exercise. – G/Z is isomorphic to the group of inner automorphism.

Group C of commutators of G. – All $Z = xyx^{-1}y^{-1}$ are called commutators of G.

Generally they do not form a group, but the group they generate is called C, group of commutators.

Exercise. – C is invariant subgroup.

Exercise. – If H, invariant subgroup such that $C \subset H$, G/H is abelian.

Groups in quantum mechanics. – As will be explained later by WIGHTMAN, the « relativity group » will play a basic part. More specially the group of invariants of a given problem.

I.e., If the problem, is in dependent of time, permanent satic, of has a spherical symmetry, or cylindrical symmetry or has n indistinguishable particles; then the translation group in time, or rotation group in 3 or 2 dimensions, or the permutation group of n objects will intervene through their representations. Therefore the following part of these lectures is on generalities about representations.

They are also groups in Q.M. whose elements are functionals (ex. groups of canonical transformation).

We shall not dare to speak about them.

2. – Representations of a group: linear operators.

A representation of a group G is a homomorphism between G and the group of endomorphism of a linear space L. If only linear endomorphisms are used *i.e.* linear operators A acting on L, the representation is said linear. The function $A(g)$ on L preserves the multiplication law:

$$A(g_1)A(g_2) = A(g_1 g_2)\,.$$

It will be required moreover that this function is continuous. In quantum theory as will be shown later by Prof. WIGHTMAN, more general representation will be needed. The $A(g)$ may also be antilinear operators *i.e.*

$$x, y \in L\,, \qquad A(x+y) = Ax + Ay\,,$$

but

$$A\alpha x = \bar{\alpha} A x\,.$$

Moreover, the $A(g)$ may form only a continuous representation « up to a factor », *i.e.*

$$A(g_1)A(g_2) = \omega(g_1, g_2) A(g_1 g_2)\,,$$

where $\omega(g_1, g_2)$ is a continuous complex valued function.

Today we shall just study general properties of linear operators and linear representations.

Invariant subspace. – Let $M \subset L$ be a linear subspace; M is said to be invariant under G if for all $x \in M$, g G, $A(g)x$ M.

Irreducible representations. – A representation is irreducible if the only subspaces left invariant are 0 and L, *i.e.*, if there is no « true » invariant subspace.

Reducible representation. – A representation which is not irreducible is called reducible, *i.e.* there is at least one invariant subspace $M \subset L$. If the representation in M is not irreducible, one can find an invariant subspace $M_1 \in M$, etc.

If M has a finite number of dimensions one ultimately can find an invariant subspace M_n such that $0 \neq M_n \subset M_{n-1} \subset \ldots \subset M \subset L$ and where the representation is irreducible.

If M has an infinite number of dimensions this may not be possible. In order to say more, we wish to have representation on an L space with a « richer » structure (more axioms).

Representations in Hilbert space. – Definition of a Hilbert space H.

1) It is a linear space « L ».

2) There is a scalar product, a mapping $L \times L \xrightarrow{\text{(scalar product)}} C$ with the following properties:

$$x \in H,\ y \in H \to (x, y): \qquad (\alpha x, \beta y) = \bar{\alpha}\beta(x, y)$$

$$(x_1 + x_2, y) = (x_1, y) + (x_2, y)$$

$$(x, y_1 + y_2) = (x, y_1) + (x, y_2)$$

$$(x, y) = \overline{(x, y)}; \ (x, x) = \|x\|^2 \geq 0; \ (x, x) = 0 \Leftrightarrow x = 0$$

3) Completeness:

Given a sequence x_n such that for given $\varepsilon > 0$, $\exists$ integer $N(\varepsilon) \to \to \|x_n - x_m\|_{m,n>N} < \varepsilon$; then there is a limit $x_n \to x$ for $N \to \infty$.

4) If in addition there exists a denumerable basis it is called « separable ».

Direct sum of Hilbert space. – The direct sum of linear spaces has been defined (see GÅRDING's notes). Therefore one has to worry only about the other axioms. Let us construct a scalar product:

$$\text{if} \quad x = \oplus\, x_i\,, \qquad x_i \in H_i\,,$$

define:

$$(x, y) = \sum_i (x_i y_i)\,, \qquad x = \oplus_i\, x_i\,, \qquad y = \oplus_i\, y_i\,,$$

the elements $x \in H$ are those which satisfy $(x, x) < \infty$. Then $H = \bigoplus_i H_i$ can proved also to be complete and, if all H_i are separable, H is separable.

Direct sum of linear operators. – If $x = \bigoplus x_i h$ A_i operates on x_i, define $A = \bigoplus_i A_i$ through $Ax = \bigoplus_i A_i x_i$.

Reducible and irreducible representations on a Hilbert space. – Let $A(g)$ be a reducible representation of G in H; it leaves invariant a subspace $M \subset H$. (A linear combination of vectors is a linear subset; it is a linear subspace only if it is closed). Let $H = M \oplus M'$ (M' is not unique:); one may take for M' the subspace M' orthogonal to M.

The most general form of $A(g)$ leaving M invariant is:

$$A(g) = \left(\begin{array}{c|c} A_1(g) & B(g) \\ \hline 0 & A_2(g) \end{array}\right)$$

It is easy to see that $A_1(g)$ and $A_2(g)$ are representations of G but in general $A \neq A_1 \oplus A_2$, *i.e.* $B(g) \neq 0$ and M' is not invariant subspace. If the representation can be written as a direct sum of irreducible representations it is said to be completely reducible.

Exercise. – Prove that the A_2 forms a representation on the linear space quotient H/M.

Example. – The representation $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ of the additive group of real numbers $x + y = z$ is reducible but *not* completely reducible, as it cannot be diagonalized.

Unitary representations. – For those representations of groups, if M is invariant $M^{\perp}$ is invariant. Consider a group G, a Hilbert space H, M an invariant subspace of H ($x \in M$), $M^{\perp}$ the orthogonal subspace ($y \in M^{\perp}$); therefore all x, y: $(x \cdot y) = 0$.

Given a unitary representation: $(U(g)x, U(g)y(= (x, y) = 0$. For all

$$z \in M\,, \qquad \exists\, x \to z = U(g)x \qquad\qquad (\text{take } x = U^{-1}(g)z).$$

Thus all $U(g)y$ is $\perp M$; thus $M^{\perp}$ is invariant.

The property: if M is invariant subspace, then $M^{\perp}$ is also invariant can evidently be extended to:

a) the antiunitary representations, since they are characterized by the following property:

$$(U(g)x, U(g)y) = (y, x) = \overline{(x, y)}\;;$$

b) the representations up to a factor, characterized by the following property:

$$|(x, y)| = |(U(g)x,\ U(g)y)|;$$

c) the representations of only multiplicative sets of operators with the property: if U is an element of the representation, U^* is also an element of the representation.

Bounded operators:

— norm of a vector: $\|x\| = \sqrt{(x, x)}$;

— norm of an operator: $\|A\| = \sup \dfrac{\|Ax\|}{\|x\|}$ for all $x \in H$.

— an operator is bounded if it has finite norm: $\|Ax\| < C\|x\|$,

where C is a constant, for all $x \in H$.

THEOREM. – A homomorphism $H \xrightarrow{A} H'$ is bounded if and only if it is continuous.

Proof. – If A is bounded:

$$\exists\, C \rightarrow \|A(x-x')\| < C\|x-x'\|, \text{ then if } x \rightarrow x',\ \|A(x-x')\| \rightarrow 0.$$

If A is not bounded: $\exists$ a sequence x_n such that:

$$\|Ax_n\| > n \quad \text{or} \quad \left\|A\frac{x_n}{n}\right\| > 1\,; \quad \text{if} \quad n \rightarrow \infty,\ \frac{x}{n} \rightarrow 0\,;$$

therefore A is not continuous in $x = 0$.

Schur's Lemna. (Finite dimensional case). – Consider two linear irreducible representations $A(g)$, $B(g)$; if $\exists\, T$ of the same multiplicative set G of elements g, such that for all g.

$$TA(g) = B(g)T, \text{ then: either } T = 0, \text{ or } T \text{ is regular and } A = T^{-1}BT.$$

Proof. – A acts on the linear space E,

B acts on the linear space F,

T is a homomorphism $E \xrightarrow{T} F$,

Let $M_e \subset E$ be the kernel of T: $M_e \xrightarrow{T} 0\,F$.

(*i.e.* M_e is the set of all $x \in E$ such that $Tx = 0$.) It is evident that M_E is a linear subspace invariant in E. Moreover: $TAx = BTx = 0$ for $x \subset M_E$.

Therefore Ax M_E: M_E is invariant under A.

Since A is irreducible either

$$M_E = E, \qquad T = 0$$

or $M_E = 0$, which means it is an «injection» in F.

c) the image of T (= range of T) is also an invariant linear space. If $T \neq 0$, it must be F (sice B irreducible). Then T is a one to one mapping and the two representations are called linearly equivalent.

b) If A and B are the same irreducible representations, and $\exists\, T \to TA = AT$, T is a multiple of the unit matrix: $T = \lambda I$.

Proof:

$$\lambda IA = \lambda AI .$$

Then: $[T - \lambda I, A] = 0$.

Since A is irreducible, from *a*) either $T - \lambda I = 0$
or $T - \lambda I$ is regular.

Since a matrix has at least one eigenvalue, take it to be λ; then $(T - \lambda 1)^{-1}$ does not exist; thus $T - \lambda I = 0$.

Possible extension to the infinite dimensional case.

Several difficulties may appear.

Has T at least one eigenvalue; it can be proved so if T is bounded? Even in this latter case, the image of T might not be a linear subspace (if the linear subset is not closed).

We shall not prove the

1st part of Schur's Lemna. – A set of operators A on H such that $A \in \mathcal{A} \to A^* \in \mathcal{A}$, is irreducible if and only if every bounded linear operator commuting with every element of A is a constant multiple of the identity operator. The proof uses spectral analysis of normal operators in Hilbert space.

2nd part of Schur's Lemna. – Let $\mathcal{A}$ and $\mathcal{B}$ two irreducible isomorphic sets of operators defined on H_1 and H_2 and such that $A \in \mathcal{A} \to A^* \in \mathcal{A}$; $B \in \mathcal{B} \to B^* \in \mathcal{B}$. If T is a linear operator $H_1 \xrightarrow{T} H_2$, such that for every corresponding pair of A_i and B_i, $TA_i = B_iT$.

Then either $T = 0$ or T is a regular operator (which maps H_1 on H_2 by one to one correspondence).

Proof:

$$B_iT = TA_i, \qquad B_i^*T = T_i^* .$$

Then

$$T^*B_i^* = A_i^*T^* = TT^*B_i = TA^*T^* = B_i^*TT^* ,$$

hence TT^* commuting with all B_i is a multiple of the unit; one similarly proves that $T^*TA_i = A_iT^*T$, *i.e.* T^*T is a multiple of the unit.

We shall need a powerful theorem on bounded operators; let us prove it.

THEOREM OF RIESZ. – A linear functional in H: $H \xrightarrow{f} C$, is a homomorphism of H into G,

$$f(x+y) = f(x) + f(y) \qquad x, y \in H,$$

$$f(\alpha x) \quad = \alpha f(x) \qquad f \in C.$$

a) A linear functional is bounded if and only if $\exists\, a$ such that for all x, $f(x) = (a, x)$. If a exists it is unique.

b) A sesqui-linear form $(f\,x, y)$, (linear in y antilinear in x, *i.e.* $f(\sum_i \alpha_i x_i, \sum_i \beta_i y_i) = \sum_{ij} \bar{\alpha}_i \beta_j f(x_i y_j)$) is bounded if $f(x, y) \leqslant C\|x\| \cdot \|y\|$. If f is bounded there exists a bounded operator A such that $f(x, y) = (x, Ay)$.

Proof. – Let us fix y_0; let M be the set of all $x \to f(x, y_0) = 0$ and consider $M^\perp$; if $M^\perp = 0 \to f(x, y_0) = 0 = f(x, 0)$. Otherwise $\exists\, x_0 \in M^\perp \to f(x_0, y_0) \neq 0$.

We shall prove that $f(x, y_0)$ is of the form (x, y_0') when $y_0' = \alpha x_0$: It is true if $x \in M$

$$f(x, y_0) = 0 = (x, \alpha x_0)\,,$$

also if $x = \beta x_0$

$$f(\beta x_0, y_0) = \bar{\beta} f(x_0, y_0) = \bar{\beta}\|x_0\|^2 \quad \text{where} \quad \alpha = \frac{f(x_0 y_0)}{\|x_0\|^2}$$

$$= (x, \alpha x_0) \quad \text{since} \quad \bar{\beta} = \frac{(x x_0)}{\|x_0\|^2}\,.$$

Therefore if $x \in M$ or $x = \beta x_0 \to f(x, y_0) = (x, \alpha x_0)$.

For x arbitrary: $x = x - \beta x_0 + \beta x_0$.

$$x - \beta x_0 \in M \quad \text{if} \quad f(x - \beta x_0, y_0) = (f(x, y_0) - \bar{\beta} f(x_0, y_0) = 0\,,$$

but $f(x_0 y_0) \neq 0$, therefore it is always possible to choose β so that $x - \beta x_0 \in M$, and for any x one has $f(x, y_0) = (x, \alpha x_0)$.

Therefore, for all y_0, $\exists\, y' = \alpha x_0$ such that $f(x, y_0) = (x, y')$.

The correspondence y_0, y' is linear since f is linear: $y' = Ay_0$.

Thus $f(x, y) = (x, Ay)$.

If $f(x, y) \leqslant C\|x\| \cdot \|y\|$ it follows that $(x, Ay) \leqslant C\|x\| \cdot \|y\|$, *i.e.* A is bounded.

In particular, if y is fixed, $f(x, y)$ is a linear form in x which can be written (Ay, x) which proves part a of the theorem.

3. – Representations of compact groups.

Integration on a group. Consider a topological group.

$x^{-1} \to x$ is a continuous automorphism of the group.

$(x, y) \to x \cdot y$ is a continuous mapping $G \times G \to G$.

Thus, if a is a fixed element of G, $x \to ax$ is an homomorphism.

Therefore it is enough to investigate the topology of the group in a neighbourhood of the unit element.

If U is a neighbourhood of unity, there exists another neighbourhood V such that $V \cdot V \subset U$. Also U^{-1} is a neighbourhood.

For a locally compact topological space one can define integration of functions of the group with respect to a measure: given a function $f(x)$, $x \in G$ and a measure $d\mu(x)$ we write

$$I = \int f(x) d\mu(x) .$$

Since left and right translations are homomorphisms, one may look for a measure which is left *or* right invariant; such a measure exists and is called a Haar measure. For compact groups it is both left and right invariant. (See for instance, PONTRJAGIN: *Topological Groups*).

It can be shown that this measure is unique, up to a constant which can be chosen, so that the « volume » of the group is 1:

$$\int_G d\mu(x) = 1 .$$

(It is infinite for a non compact group).

Lie group. – $x \in G$ are characterized by n parameters $(\sigma, l, \tau = 1, \dots, \eta)$: $x(\xi^\sigma)$, $y(\eta^\sigma)$ and the parameters of the product $xy = z(\zeta^l)$ are analytic functions of $\zeta^l = \varphi^l(\xi^\sigma, \eta^\tau)$.

Locally the group is homomorphic to R^n (euclidian space). Then one can speak of « volume element » $dv(\xi^\sigma)$ in a point x. To get it in another point z, one will use the jacobian $|\partial\varphi^\tau/\partial\xi^\sigma|$. (Since translations are homomorphisms, this jacobian never vanishes).

Example. – 1) Additive group of reals = Group of translations in one dimension: measure $d\mu(x) = dx$, but

$$\int_{-\infty}^{+\infty} dx = \infty ,$$

If this group is divided by the additive subgroup of integers one gets a group isomorphic to the group of rotations φ around an axis:

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}\varphi}{2\pi}, \qquad \text{indeed} \qquad \int_G \frac{\mathrm{d}\varphi}{2\pi} = 1 .$$

2) **Rotation group in 3 dimensions.** Angle of rotation ω; the direction of the axis of rotation is given by θ, φ, $\mathrm{d}\mu(\omega, \theta, \varphi) = \frac{1}{2}\sin^2(\omega/2)\sin\theta\,\mathrm{d}\omega\,\mathrm{d}\theta\,\mathrm{d}\varphi$ (see for instance WIGNER's *Gruppentheorie und Quantenmechaniks*).

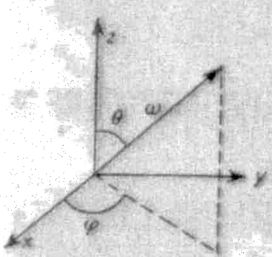

Fig. - 2.

Finite groupsare just a special case of compact groups. The topology is discrete. The invariant measure is: equal mass for every element. If N_G is the number of elements of the group, we take this mass $= 1/N_G$. Then the two following symbols

$$\boxed{\frac{1}{N_G}\sum = \quad \text{and} \quad \int_G \mathrm{d}\mu(g)}$$

are just corresponding for finite and infinite groups. Here we shall generally use only the more general sign

$$\int_G \mathrm{d}\mu(g) .$$

If $f(x)$ is a complex valued on the group, the translation invariance gives

$$\int_G f(x)\mathrm{d}\mu(x) = \int_G f(ax)\,\mathrm{d}\mu(x) = \int_G f(xa)\,\mathrm{d}\mu(x) .$$

This is also equal to

$$= \int_G f(x^{-1})\,\mathrm{d}\mu(x) .$$

Application of Riesz theorem. - For a locally compact group, $I = \int f(g)\,\mathrm{d}\mu(g)$ is a well defined expression if $f(g)$ is a good enough function (bounded, and vanishing outside a compact set). But what will be needed in the representation theory, is integration on operator-valued functions of the group.

Let us call $g \in G$. How to give meaning to such an expression as

$$\int_G \mathrm{d}\mu(g)\,\mathrm{f}(g) A(g),$$

where $A(g)$ is a bounded operator on H? If x, y are vectors in H $(x, A(g)y)$ is a bounded complex-valued function.

Then

$$I(x, y) = \int_G d\mu(g)\, f(g)\, (x, A(g)y)$$

is a sesquilinear form in x, y. From Riesz theorem $I(x, y) = (x, By)$. This bounded operator B is just the definition of

$$\int_G d\mu(g) f(g) A(g) .$$

Exercise. – With the help of Riesz theorem, prove:

$$\int d\mu(g) A(g) = \int d\mu(\ \ )\, A(gh) = \int d\mu(g) A(hg)$$

$$\left[\int d\mu(g)\, A(g)\right] B = \int d\mu(g)\, (A(g)B)$$

$$B\left[\int d\mu(g) A(g)\right] = \int d\mu(g)\, (B\, A(g))$$

$$\left[\int d\mu(g)\, \varphi(g)\, A(g)\right]^* = \int d\mu(g)\, \bar{\varphi}(g)\, A^*(g) .$$

THEOREM 1. – Every continuous linear regular representation of a compact group $g \to D(g)$ by bounded regular operators in a Hilbert space is equivalent to a unitary representation (regular; is to avoit the case $\begin{vmatrix} D & O \\ O & O \end{vmatrix}$.

Proof. – Consider

$$H = \int_G d\mu(g)\, D(g)\, D^*(g) .$$

Then

$$D(h)\, H\, D^*(h) = \int d\mu(g)\, D(h)\, D(g)\, D^*(g)\, (D^*(h = \int d\mu(g)\, D(hg)\, D^*(hg) = H .$$

$$(d\mu(h^{-1}g) = d\mu(g) .$$

Thus for all h

$$D(h)\, H\, D^*(h) = H \; ; \qquad \text{or} \qquad DH = HD^{*-1} .$$

Properties of H:

$$H^* = H \text{ since } \mathrm{d}\mu \text{ is real.}$$

$$\text{for all } x \ (x, DD^*x) \geqslant 0, \text{ hence } (x, Hx) \geqslant 0.$$

H is said to be positive semi definite. Therefore one can write

$$H = BB^* , \tag{1}$$

(H being hermitean $\geqslant 0$ in a finite dimensional space, it can be written as $H = \sum_i \alpha_i P_i$, where P_i are projection operators and α_i real $\geqslant 0$; then one may take $B = \sum_i \sqrt{\alpha_i} P_i$; here $\alpha_i \geqslant 0$ hence $B \bar{=} B^*$. In an infinite dimensional Hilbert space it can be written $H = \int \lambda \mathrm{d}E_\lambda$, with $\lambda \geqslant 0$; take $B = \int \sqrt{\lambda} \mathrm{d}E_\lambda$).

So,

$$DBB^* = BB^*D^{*-1}$$

or

$$B^{-1}DB = B^*D^{*-1}B^{*-1} - (B^{-2}DB)^{*-1} .$$

Let $U(g) = B^{-1}D(g)B$; this is a unitary representation of G equivalent to D. Therefore, from now on, it is sufficient to consider unitary representations.

Irreducible representations of compact groups. – Consider two irreducible representations $D_1(g)$, $D_2(g)$ and A, an arbitrary bounded operator. Let

$$X = \int_G \mathrm{d}\mu(g) D_1(g) A D_2^{-1}(g) . \tag{2}$$

One has: (proof similar to $DHD^* = H$) $D_1(h) X D_2^{-1}(k) = X$ for all h, *i.e.*, $D_1 X = X D_2$.

a) if D_1 is not equivalent ([redacted]) to D_2, $X = 0$ for any A;

b) if D_1 is equivalent ([redacted]) to D_2,

$$\exists S \text{ bounded} \to D_2 = S^{-2} D_1 S ;$$

hence

$$D_1 X = X S^{+1} D_1 S , \quad i.e. \quad D_1 X S^{-1} = X S^{-1} D_1 .$$

By Schur's lemma,

$$X S^{-1} = \alpha(A) I , \quad i.e. \quad X = \alpha(A) S ,$$

where $\alpha(A)$ is a scalar depending on A.