 

c) If $D_1 \doteq D_2$, $S = 1$:

(3) $$X_A = \int d\mu(g)\, D(g) A D^{-1}(g) = \chi(A) I .$$

If one takes $A$ such that tr $A$ is well defined and $\neq 0$ (and this is possible *e.g.* when $A$ has finite rank), one obtains:

$$\operatorname{tr} X_A = \int d\mu(g) \operatorname{tr} A = \chi(A) d_D ,$$

where $d_D$ is the dimension of $D$.

Now if $\chi(A)$ would be 0, we would have had $X_A = 0$ hence tr $X_A$ = tr $A = 0$: which is contradictory. Hence $\chi(A) \neq 0$ and $d_D$ is the finite number $a/\chi$.

THEOREM 2. – Any irreducible representation of a compact group has finite dimension.

Application to Abelian groups: Take

$$A = D(g_0) , \qquad g_0 \in G ; \qquad \to X = D(g_0) = \chi I :$$

since the representation is irreducible, it must then have dimension 1.

*Orthogonalization relations.* – Since the $D(g)$ have finite dimension, they can be represented by matrices.

Let us choose

$$A = [c_{jk}] = j - \begin{pmatrix} 0 & \overset{k}{|} & 0 \\ \hline 0 & 1 & 0 \end{pmatrix}$$

Then

$$(X_A)_{il} = \int d\mu(g)\, D^{(r)}_{ij}(g)\, D^{(s)}_{kl}(g^{-1}) = \delta_{rs} \qquad \delta_{jk} \qquad \frac{1}{d_D} \qquad \delta_{il}$$

$$0 \text{ if } D^r \not\simeq D^s \quad \chi(A) = \operatorname{tr} A/d\, (I)_{il} \text{ for } r = s$$

or, since $D(g^{-1}) = D^{-1}(g) = D^*(g) = D_T(g)$:

(4) $$\int d\mu(g)\, D^{(r)*}_{ij}(g)\, D^{(s)}_{kl}(g) = \frac{1}{d_D} \delta_{rs}\delta_{jk}\delta_{il} .$$

Thus:

THEOREM 3. – The functions $D^{(r)}_{ij}(g)$ form an orthogonal set of functions in $L^2(G)$ (the set of square integrable functions on $G$).

One can prove that $D^{(r)}_{ij}(g)$ form a complete basis of this Hilbert space (Peter-Weyl theorem).

One can also prove that $L^{2}(g)$ is separable; therefore there is at most a denumerable set of inequivalent irreducible representations.

*Reality condition for irreducible representation.* – Let us now define

$$Y = \int \mathrm{d}\mu(g) D(g) A D^{-1}(g) .$$

If $D$ is a unitary representation, $\bar{D} = D^{-1T}$ is also a representation. By the same technique we prove $DY = Y\bar{D}$:

— If $D \not\sim \bar{D}$, $Y = 0$.

— If $D \sim \bar{D}$, $\exists S \to \bar{D} = S^{-1} D S = D^{-1T}$

$$Y = \int \mathrm{d}\mu(g) D(g) A D^{T}(g) = \alpha(A) S .$$

Now,

$D = \bar{S}^{-1} \bar{D} \bar{S}$, hence $\bar{S} S = \lambda I$ (Schur's lemma),

$D^{*-1} = D = S^{T} D^{-1T} S^{-1T}$, hence $S^{-1T} S = \lambda' I$ (Schur's lemma),

*i.e.* $S^{T} = CS \to C^{2} = 1 \to S^{T} = \pm S$ by convenient choice of a factor. The $S$ can be made unitary, hance $\bar{S}S = C$.

— If $D \sim$ to a real representation $\bar{S}$ is a multiple of the unit matrix and therefore $S^{T} = +S$, *i.e.* $C = 1$ (use the fact that $C$ does not depend onan equivalence). It is easy to prove that this necessary condition is sufficient. Let $\omega \neq$ all proper values of $\bar{S}$ and $|\omega| = 1$; then $D' = (\bar{S} - \omega) D (\bar{S} - \omega)^{-1} = \bar{D}$

— If $D \not\sim$ to a real representation $S^{T} = -S$.

Thus for any $A$

(5) $$\int \mathrm{d}\mu(g) D(g) A^{T} D^{T}(g) = \pm \int \mathrm{d}\mu(g) D(g) A D^{T}(g)$$

*Characters.* – The trace of a representation $D^{r}(g)$ is called its character $\chi^{(r)}(g) = \operatorname{tr} D^{(r)}(g)$. Since $\operatorname{tr} D = \operatorname{tr} SDS^{-1}$, the characters are defined on representations up to an equivalence. If two elements $g$, $h$ are conjugate ($\exists a \to g = aha^{-1}$), then $\chi^{r}(g) = \chi^{r}(g)$: the character is the same for all members of a conjugate class in the group. One has from (4) with $i = j$, $k = l$)

(6) $$\int \mathrm{d}\mu(g) \chi^{(r)}(g) \bar{\chi}^{(s)}(g) = \delta_{rs}$$

Furthermore, from (5) with $A = [e_{jk}]$ one gets:

$$Y_{i} = \int \mathrm{d}\mu(g) D_{ij}(g) D_{lk}(g) = \pm \int \mathrm{d}\mu(g) D_{ik}(g) D_{lj}(g) .$$

Taking the trace: ($j = l$, $i = k$, $\Sigma_{i,l}$) one has:

$$\int d\mu(g)\chi(g^2) = \pm\int d\mu(g)\chi^2(g) = \int d\mu(g)\chi(g)\,\overline{\chi}(g) = 1$$

(since $\overline{D} \sim D$ if $\Upsilon \neq 0$).

THEOREM 4.

$D \sim \overline{D}$ and $D \sim$ real rep. $\Leftrightarrow \int d\mu(g)\chi(g^2) = +1$

and $D$ [redacted] real rep. $\Leftrightarrow$ $d\mu(g)\chi(g^2) = -1$

$D$ [redacted] $\overline{D}$ $\Leftrightarrow$ $d\mu(g)\chi(g^2) = 0$

*Peter-Weyl Theorem.* – The original proof makes use of the spectral theory of Hermitian operators. We give a simpler proof due to STONE. It is valid only for groups with faithful representations.

THEOREM. – The functions $D^{(r)}_{ij}(g)$ of $g \in G$ form a complete basis in the space $L^2(G)$, *i.e.* if $\varphi(g)$ is orthogonal to $D^{(r)}_{ij}(g)$ for all $r$, $i$, $j$, it is the zero function. This basis is uniformly complete, *i.e.* all $\varepsilon > 0$.

$\exists$ a finite linear combination $f_\varepsilon$ s.t. $|f_\varepsilon(g) - \varphi(g)| < \varepsilon$ for any $g$. In what follows $D^{(r)}_{ij}$ is assumed to be real; for, if it is not, the representation formed by the direct sum $D^{(r)} \dotplus \overline{D}^{(r)}$ is equivalent to a real representation:

$$S\begin{pmatrix} D^{(r)} & \\ & \overline{D}^{(r)}\end{pmatrix}S^{-1} = \frac{1}{2}\begin{pmatrix} D+\overline{D} & i(\overline{D}-D) \\ -i(\overline{D}-D) & D+\overline{D}\end{pmatrix}$$

where

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} i & i \\ -1 & 1\end{pmatrix}.$$

Let $D$ be a real faithfull representation of $G$.
Consider the Kronecker products

$$\overset{n}{\otimes} D \qquad \text{for all } n$$

(note that the Kronecker product of two representations is itself a representation since $(A \otimes B)(A' \otimes B') = AA' \otimes BB'$).

We shall prove that the representation $(\overset{n}{\otimes} D)_{i_n j_n}$ form a complete set, so the same holds for the irreducible representations.

Suppose there exists $\varphi(g)$ orthogonal to all $(\overset{n}{\otimes} D)_{i_n j_n}$ and different from zero in some neighbourhood of $g_0$.

Construct

$$f(g, N) = A(N) \exp\left[-N \sum_{ij} \{D_{ij}(g) - D_{ij}(g_0)\}^2\right],$$

where $A(N)$ is chosen so that

$$\int \mathrm{d}\mu(g) f(g, N) = 1 .$$

It is well known that a function of the form of $f(g, N)$ can be expanded as a power series with an infinite radius of convergence. The terms in the series are evidently linear combinations of the $(\overset{n}{\otimes} D)_{i_n j_n}$

$$\lim_{N\to\infty} \int \mathrm{d}\mu(g) f(g, N) = 1$$

but $f(g, \infty) = 0$ for $g \neq g_0$ so that $f(g, \infty) = \delta(g - g_0)$.

If $\varphi(g)$ were orthogonal to all $(\overset{n}{\otimes} D)_{ij}$ then one would have

$$\lim_{N\to\infty} \int \mathrm{d}\mu(g)\, \bar{f}(g, N)\, \varphi(g) = 0 ,$$

but

$$\lim_{N\to\infty} \int \mathrm{d}\mu(g) \bar{f}\,(g, N)\, \varphi(g) = \varphi(g_0) ,$$

which by hypothesis is different from zero. Thus $(\overset{n}{\otimes} D)_{ij}$ form a complete set and can be decomposed into a direct sum of irreducible representations which is complete.

*This proof shows also that:*

1) The set of irreducible representations is at most denumerable.

2) Any faithful representation generates, by $\overset{n}{\otimes}$, all the irreducible representations.

*Example.* – The two-dimensional rotation group: if $\varphi$ is the angle of rotation $D^{(n)}(\varphi) = \exp[in\varphi]$ ($n$ integer)

$$\int_0^{2\pi} \exp[in\varphi] \exp[-im\varphi] \frac{\mathrm{d}\varphi}{2\pi} = \delta_{nm} .$$

We have therefore proved that for the Hilbert space of square integrable functions defined on the interval $(0, 2\pi)$, the functions $\exp[in\varphi]$ form an orthonormal complete set.

*Finite groups.* – Let $N_g$ be the number of elements of the group. The complex-valued functions defined on the group form a $N_g$ dimensional linear space (Exercise in GARDING's note). We have show that the $\sum_r a^2_{D^{(r)}} \Omega^{(r)}_{ij}$-functions are orthonormal. Since they form a complete set we can conclude

$$\sum_r d^2_{D^{(r)}} = N_g .$$

***The decomposition of a given representation of a compact group into a direct sum of irreducible representations.***

From Theorem 1, we can make it unitary.

First case: *The representation is finite dimensional.*

Then the accident mentioned at the bottom of page 7 will not occur. The representation is a direct sum of irreducible representations.

Let

$$U(g) = \bigoplus_r n_r D^{(r)}(g) , \tag{7}$$

where $n_r$ is number of times the irreducible representation $D^{(r)}(g)$ occurs in $U(g)$. Then

$$\chi(U(g)) = \sum_r n_r \chi^{(r)}(g)$$

and from (6)

$$n_r = \int d\mu(g) \overline{\chi}^{(r)}(g) \, \chi(U(g)) \quad .$$

If the representation is infinite one has to be more careful.

Let

$$E^{(r)}_{ij} = d_{D^{(r)}} \int d\mu(g) \, \overline{D}^{(r)}_{ij}(g) \, U(g) ;$$

one can prove

$$E^{(r)*}_{ij} = E^{(r)}_{ji} .$$

Let

$$E^{(r)} = \sum_i E^{(r)}_{ii} = d_{D^{(r)}} \int d\mu(g) \, \overline{\chi}^{(r)}(g) \, U(g) ;$$

calculation yields that $E^{(r)}$ are the projection operators in Hilbert space onto the subspace $\mathcal{H}^{(r)}$ on which act the $\oplus$ of representations equivalent to $D^{(r)}$ occurring in the direct sum of $U$.

The character [illegible] function only of the conjugation class $C$ and not of each group element. Let $N_c$ be the number of such classes.

The characters $\chi^{(r)} \in C$ span a vector space of at most $N_c$ dimensions. Since they are linearly independent (equation (6)) the number $N_D$ of irreducible inequivalent representations is $\leqslant N_c$.

THEOREM. – For finite groups $N_D = N_C$.

Consider the $N_G$ dimensional linear space of formal linear combinations of group elements. The group multiplication generates naturally a $N_G$ dimensional linear representation on this space, the so called regular representation $D_R$. It can be shown that $D_R$ contains all the irreducible representations, each one $d_D(r)$ times. We shall not prove it here.

## 4. – Structure of the Lorentz group.

*Notation.* – An event in space time is specified by the four numbers $(ct, \boldsymbol{r})$. We denote 3-dimensional vectors with an arrow above $\boldsymbol{r} = (x^i)$; $i = 1, 2, 3$. We denote 4-dimensional vectors with a bar below: $\mathbf{x} = (x^\mu)$; $\mu = 0, 1, 2, 3$. $(x^0 = ct)$.

The separation between two events, $\mathbf{x}$, $\mathbf{y}$ is given by:

$$(x - y)^2 - (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) = (x^0 - y^0)^2 - \sum_{i=1}^{3} (x^i - y^i)^2 .$$

We define the metric tensor: $G$

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x} G \mathbf{y} = x^0 y^0 - \sum_{i=1}^{3} (x^i y^i) = x^\mu g_{\mu\nu} y^\nu ; \quad G = (g_{\mu\nu}) = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

We define $g^{\mu\nu}$ as the elements of $G^{-1} = G$

$$g^{\mu\varrho} g_{\varrho\nu} = g^\mu_\nu = \delta^\mu_\nu \qquad \text{(The summation convention is understood).}$$

We define the covariant components of a vector

$$x_\mu = g_{\mu\nu} x^\nu .$$

*Definition of Lorentz group.* – The group $\mathcal{L}$ of all linear transformations $\mathbf{x} \to \mathbf{x}'$ such that $(\mathbf{x}' - \mathbf{y}')^2 = (\mathbf{x} - \mathbf{y})^2$. Unless otherwise specified, we shall limit ourselves to the real transformations.

So, we write:

$$x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu ,$$

$a = \{a^\mu\}$ is called the translation $\boldsymbol{r}\, T$;

$\Lambda = \{\Lambda^\mu_\nu\}$ is called a homogeneous transformation: $\boldsymbol{r}\, L$.

The most general Lorentz transformation is called inhomogeneous.

The inhomogeneous group will be noted by $\mathcal{L}$.

Given a homogeneous transformation $\Lambda$ and a translation $a$, one denotes the operation $\mathbf{x} \to \mathbf{x}' = \{x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}\}$ by: $\{a, \Lambda\}$, being understood that $\Lambda$ is performed first.

Multiplication law:

$$\{a, \Lambda\}\{b, M\} = \{a + \Lambda b, \Lambda M\}.$$

Neutral element (unity): $\{0, 1\}$.

Inverse:

$$\{a, \Lambda\}^{-1} = \{-\Lambda^{-1} a, \Lambda^{-1}\}$$

$$\{a, \Lambda\} = \{a, 1\}\{0, \Lambda\} \qquad \text{i.e.} \quad \mathcal{L} = T \cdot L.$$

Both $T$ and $L$ are subgroups of $\mathcal{L}$.

*Invariant subgroups.* – $H$ is an invariant subgroup of $\mathcal{L}$ is for all $\{a, \Lambda\} \in \mathcal{L}$, $\{b, M\} \in H$

$$\{a, \Lambda\}\{b, M\}\{a, \Lambda\}^{-1} = \{a + \Lambda b - \Lambda M \Lambda^{-1} a, \Lambda M \Lambda^{-1}\} \in H.$$

One can easily see that $T$ is an invariant subgroup.

The cosets of $\mathcal{L}$ modulo $T$ are:

$$\{a, \Lambda\}\{b, 1\} = \{a + \Lambda b, \Lambda\}.$$

The quotient group $\mathcal{L}/T$ is isomorphic with $L$.

$\mathcal{L} = T \cdot L$ but $\mathcal{L} \neq T \times L$ because, whereas $T$ is an invariant subgroup, $L$ is not. $\mathcal{L}$ is sometimes called the *semi* direct product of $T$ and $L$. The multiplication law is easily remembered by writing in matrix form:

$$\{a, \Lambda\} \to \begin{vmatrix} 1 & 0 \\ a & \Lambda \end{vmatrix}.$$

(Another illustration of a reducible representation which is not completely reducible).

— Translation group. $T$.

*The homogeneous group.* $L$. *Its four pieces.* – Since $\mathcal{L}$ leaves invariant $(\mathbf{x} - \mathbf{y})^2$, it leaves invariant $\mathbf{x}^2$ (put $y = 0$), $\mathbf{y}^2$ (put $\mathbf{x} = 0$), and therefore $\mathbf{x} \cdot \mathbf{y}$:

$$\mathbf{x} \cdot \mathbf{y} = \Lambda \mathbf{x} \cdot \Lambda \mathbf{y}, \qquad \text{i.e.} \qquad x^{\varrho} g_{\varrho\sigma} y^{\sigma} = \Lambda^{\mu}{}_{\varrho} x^{\varrho} \Lambda^{\nu}{}_{\sigma} y^{\sigma} g_{\mu\nu};$$

as this must be true for all $\mathbf{x}$, $\mathbf{y}$, one has:

$$\Lambda^\mu{}_\varrho \Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\varrho\sigma} \quad \text{or} \quad G = \Lambda^T G \Lambda .$$

This is necessary and sufficient.

The invariance of $\mathbf{x}\cdot\mathbf{x} = \mathbf{x}^2$ under $\Lambda \in L$ allows us to distinguish between

time-like vectors $\mathbf{x}^2 > 0$,

light-like vectors $\mathbf{x}^2 = 0$,

space-like vectors $\mathbf{x}^2 < 0$.

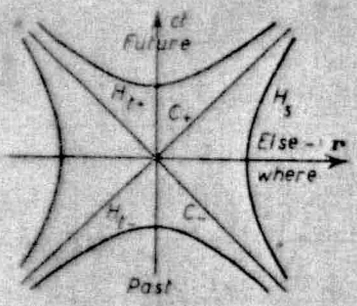


Fig. 3.

In the affine space (set of points $0+\mathbf{x}$), the sets of points such that $\mathbf{x}^2 = c > 0$ is the two piece hyperbolois $\mathcal{H}_{t+}$ $(x^0 > 0)$ and $\mathcal{H}_{t-}$ $(x^0 < 0)$. $x^2 = 0$ is the light cone $C$, $C_+$ $(x^0 > 0)$ and $C_-$ $(x^0 < 0)$; $x^2 = c < 0$ is the one piece hyperboloide $\mathcal{H}_s$.

Note if $\mathbf{x}'^2 = \mathbf{x}^2 = c < 0$ one can go continuously from

$$\mathbf{x} \to \mathbf{x}' .$$

If $c > 0$, it is possible only if $x'^0 x^0 > 0$.

*Connection of L.* – One has $\det G = \det \Lambda^T G \Lambda$, thus $\det \Lambda^T \Lambda = (\det \Lambda)^2 = 1$ so that $\det \Lambda = \pm 1$.

One cannot go by a homomorphism from unity ($\det 1 = 1$) to a $\Lambda$ for which $\det \Lambda = -1$; $L$ is therefore composed of two disconnected parts:

$L_+ = \{\Lambda, \det \Lambda = +1\}$ which is the invariant subgroup of proper Lorentz transformations,

$L_- = \{\Lambda, \det \Lambda = -1\}$ which is the set of improper Lorentz transformations (and is *not* a subgroup).

$L/L_+$ consists of two elements.

Now consider real Lorentz transformations: (the complex homogeneous group is denoted by $CL$).

One has:

$$g_{00} = \Lambda^\mu_0 \Lambda^\nu_0 g_{\mu\nu} = 1 = (\Lambda^0_0)^2 - \sum_i (\Lambda^i_0)^2 ,$$

thus

$$(\Lambda^0_0)^2 = I + \sum_i (\Lambda^i_0)^2 \geqslant 1 .$$

Thus:

$$\Lambda_0^0 \geqslant 1 \quad \{\Lambda, \Lambda_0^0 \geqslant 1\} = L^{\uparrow}: \quad \text{orthochronous group (invariant subgroup),}$$

or

$$\Lambda_0^0 \leqslant -1 \, \{\Lambda, \Lambda_0^0 \leqslant -1\} = L_{\downarrow}: \quad \text{antichronous transformations (not a subgroup).}$$

Tne justification of the terminology is the following:

$\Lambda_0^0 \geqslant 1 \Rightarrow$ the sign of the time component of a true like vector is not changed:

$$x^{0\prime} = \Lambda_{\mu}^0 x^{\mu} = \Lambda_0^0 x^0 + \Lambda_i^0 x^i ,$$

$$\Big(\sum_i \Lambda_i^0 x^i\Big)^2 \leqslant \Big(\sum_i \Lambda_i^0\Big)^2 \Big(\sum_i x^i\Big)^2 = [(\Lambda_0^{0\,2})^2 - 1] \sum_i (x^i)^2 \overset{*}{<} (\Lambda_0^0)^2 (x^0)^2 ,$$

(* for a time like vector: $\sum (x_i)^2 < (x^0)^2$.)

Thus $|\Lambda_i^0 x^0| > |\sum_i \Lambda_i^0 x^i|$, which proves the announced result (for space like vectors $\mathbf{x}^2 < 0$, the proof does not go through). Similarly if $\Lambda_0^0 \leqslant -1$ the time component of a time-like vector reverses sign under $\Lambda$.

Thus we have found in $L$ four disconnected-sets of transformations: $L_+^{\uparrow}$, $L_-^{\uparrow}$, $L_+^{\downarrow}$, $L_-^{\downarrow}$. Correspondingly $\mathcal{L}$ has four disconnected pieces.

*Decomposition of Lorentz transformations into plane reflections.*

*Def.* – The reflection $\Sigma_{\mathbf{n}}$ through a plane orthogonal to $\mathbf{n}$ is defined by

$$\mathbf{x}' = \mathbf{x} - 2\,\frac{\mathbf{n}\cdot\mathbf{x}}{n^2}\,\mathbf{n} .$$

Note that $\Sigma_{\mathbf{n}} = \Sigma_{-\mathbf{n}}$ and $\Sigma_{\mathbf{n}}\mathbf{n} = -\mathbf{n}$.

*Exercise* 1. – Show that $\Lambda\Sigma_{\mathbf{n}}\Lambda^{-1} = \Sigma_{\mathbf{n}'}$ with $\mathbf{n}' = \Lambda_{\mathbf{n}}$ ($\Sigma_{\mathbf{n}}$ is explicitely known, $\Lambda^{-1} = G\Lambda^T G$).

*Exercise* 2. – If $\mathbf{n}_i$ are linearly independent and $\Lambda = \Sigma_{n_1}\Sigma_{n_2}\cdot\Sigma_{n_k}$

$$\Lambda\mathbf{p} = \mathbf{p} \Leftrightarrow \mathbf{n}_i\cdot\mathbf{p} = 0 \quad \text{for all } \mathbf{n}_i \quad (\Leftarrow \text{evident}) ;$$

$\Lambda$ is of the form

$$\Lambda^{\mu}_{\nu} = g^{\mu}_{\nu} - \sum_{i,j} c_{ij} (n_i)^{\mu} (n_j)_{\nu} \qquad \text{with } c_{ii} = -2(n_i) .$$

and

$$(\Lambda p)^{\mu} = p^{\mu} + \sum_i \lambda_i n_i^{\mu} :$$

hence

$$\lambda_i = 0 \; ; \qquad \lambda_i = \sum_j e_{ij} \mathbf{n}_j \cdot \mathbf{p} \; ;$$

for $i = k \, n_k \cdot p = 0$ then $i = k-1 \Rightarrow n_{k-1} \cdot p = 0$, then.....

THEOREM. – Any $\Lambda \in L$ can be written as a product of at most 4 plane reflections. (This is a particular case of a stronger theorem: A «rotation» in $k$-dimensional space, *i.e.* a linear transformation leaving invariant the non-degenerate symmetric form

$$\sum_{ij=1} g_{ij} x^i y^j \qquad g_{ij} = g_{ji}$$

can be written as a product of at most $k$ plane reflections).

The proof goes by induction on $k$.

We have three cases.

1) $\exists \mathbf{x}$ such $\Lambda \mathbf{x} = \mathbf{x}$ and $\mathbf{x}^2 \neq 0$. Then the $k-1$ dimensional space $\mathcal{E}_{k-1} + x$ is left invariant by $\Lambda$: $\Lambda \mathcal{E}_{k-1} = \mathcal{E}_{k-1}$ and $\Lambda$ is at most the product of $k-1$ symmetries;

2) no invariant $\mathbf{x}$ but $\exists \mathbf{y}$ such that $(\Lambda \mathbf{y} - \mathbf{y})^2 \neq 0$. The symmetry $\Sigma_{\mathbf{a}}$ (where $\mathbf{a} = \Lambda \mathbf{y} - \mathbf{y}$) exchanges $\Lambda \mathbf{y}$ and $\mathbf{y}$. Hence $\Sigma_{\mathbf{a}} \Lambda$ leaves $\mathbf{y}$ invariant, *i.e.* belongs to first case;

3) for all $\mathbf{x}$, $\mathbf{a} = \Lambda \mathbf{x} - \mathbf{x}$ is a light vector,

$$\mathbf{a}^2 = 0 \, .$$

Part of this proof is more difficult in the general case, but in the case of real Lorentz group in 4 dimension it is very easy to prove that $\Lambda$ is then the identity.

*The symmetries and the 4 pieces.*

$$(\Sigma_{\mathbf{n}})^\mu_\nu = g^\mu_\nu - 2 \frac{n^\mu n^\nu}{\mathbf{n}^2} \qquad \Sigma^2_{\mathbf{n}} = 1 \qquad \operatorname{Tr} \Sigma_{\mathbf{n}} = 2 \; ; \qquad \det \Sigma_{\mathbf{n}} = -1$$

$$(\Sigma_{\mathbf{n}})^0_0 = 1 - 2 \frac{n^0 n_0}{\mathbf{n}^2} \quad i.e. \quad (\Sigma)^0_0 - 1 = 2 \frac{(n^0)^2}{\mathbf{n}^2} , \quad \text{the sign of which is } -\mathbf{n}^2 :$$

Hence $\Sigma_n \in L_-$; if $\mathbf{n}$ time like $\Sigma_n \in L^{\downarrow}_-$,
$\mathbf{n}$ space like $\Sigma_n \in L^{\uparrow}_-$.

Hence, elements of $L^{\uparrow}$ = product of even (2, 4) number of $\Sigma_{\mathbf{n}}$ with even (0, 2, 4) number of time-like $\mathbf{n}$

$L_+^\downarrow$ = product of even number of $\Sigma_{\mathbf{n}}$ with odd (1, 3) number of time-like $\mathbf{n}$

$L_-^\uparrow$ = product of odd (1, 3) number of $\Sigma_{\mathbf{n}}$ with even number of time-like $\mathbf{n}$

$L_-^\downarrow$ = product of odd number of $\Sigma_{\mathbf{n}}$ with odd number of time-like $\mathbf{n}$

$L_+^\uparrow$ *is connected*:

$$\forall \Lambda \in L_+^\downarrow\,, \qquad \Lambda = \Sigma_{\mathbf{n}_1}\Sigma_{\mathbf{n}_2}\Sigma_{\mathbf{n}_3}\Sigma_{\mathbf{n}_4}\,.$$

Since the sign of $\mathbf{n}$ is arbitrary, for time like vector $\mathbf{n}$ choose the time component $> 0$. Now it is possible to vary continuously the time like vector to a fixed time like $\mathbf{t}$; the space like, to a fixed $\mathbf{s}$; since $(\Sigma_{\mathbf{n}})^2 = 1$, one has varied continuously $\Lambda \to I$ (the identity), except in the case where the situation was for instance $\Lambda = \Sigma_s\Sigma_t\Sigma_s\Sigma_t = \Sigma_{t'}\Sigma_t$ where $\mathbf{s}' = \Sigma_s\mathbf{t}$ (from Exercise 2, p. 000), then when $t' \to t$, $\Lambda \to I$.

*Transitivity of* $L_+^\uparrow$ *on* $\mathscr{H}_{t+}$, $\mathscr{H}_{t-}$, $C_+$, $C_-$, $\mathscr{H}_s$.
Notation defined on pages 11-12.

*Remark.* – If $p'^2 = p^2$ and $(p' + p)^2 \neq 0$, $\Sigma_{p'+p}(-\mathbf{p}) = \mathbf{p}'$.
Hence if

$$(p' - p)^2 \neq 0\,, \qquad \Sigma_{p'-p}\mathbf{p} = \mathbf{p}'\,.$$

We define

$$S_{p',p} = \Sigma_{p'+p}\Sigma_p\,.$$

Now we can easily solve the problem:
Given $p$ and $p'$, $(p'^2 = p^2)$ in the same $\mathscr{H}_{t+}$ (respectively $\mathscr{H}_{t-}$, ...), find $S \in L_+^\uparrow$ such that $p' = Sp$.

Answer:

1) for $\mathscr{H}_{t+}$ (respectively $\mathscr{H}_{t-}$); $S_{p',p}$ (for $\mathbf{p}+\mathbf{p}' \in \mathscr{H}_{t\pm}$), hence $S_{p',p} \in L_+^\uparrow$;

2) for $C_+$ (respectively $C_-$):

If $p'$, $p$ are linearly independent $(p' - p)^2 < 0$ let $\mathbf{s}$ such that $\mathbf{s}^2 < 0$, $n \cdot p = n \cdot p' = 0$; $\Sigma_{p'-p}\Sigma_s$ is a solution.

If $p' = \alpha p$, take $\mathbf{t}$ time like $(\Rightarrow \mathbf{t} \cdot \mathbf{p} \neq 0)$ call $p'' = \Sigma_t p$ then $\Sigma_{p'-p''}\Sigma_t$ is a solution;

3) for $\mathscr{H}_s$ $(p+p')^2 + (p - p')^2 = 4p^2 = 4p'^2 < 0$, hance at least one of $p + p'$ or $p - p'$ is space like.

If $p + p'$ space like $S_{p',p}$ is solution $p + p'$ time like $\Sigma_{p'-p}\Sigma_s$ is solution $(s^2 < 0,\ s \cdot p = s \cdot p' = 0)$.

*Little group of vector.* – WIGNER called little group of $p = L_p$, the set of $\Lambda$ such that $\Lambda p = p$.

*p time like* from Exercise 2, page 12, $L_p$ is generated by $\Sigma_{\mathbf{n}}$ with $\mathbf{n}\cdot\mathbf{p} = 0$, *i.e.* $\mathbf{n}$ space like. Hence $L_p$ isomorphic to $O_{(3)}$ the orthogonal group into 3 dimensions. Its connected part is $O^+_{(3)}$, *i.e.* the group of rotations into 3 dimensions.

It can also be seen this way; $L_p$ and $\Lambda_0 L_p \Lambda_0^{-1}$ are isomorphic groups for a fixed $\Lambda_0$. Choose $\Lambda_0$ such that $\Lambda_0^{-1}\mathbf{p}$ is on the time axis (if $p$ is the four momentum, $\Lambda_0^{-1}\mathbf{p}$ brings the particle at rest.

*p space like* similar argumentation yields Lorentz group on space with one time axis, 2 space axis.

*p light like* the space orthogonal to $\mathbf{p}$ contains $\mathbf{p}$: $p^2 = 0$. Let $n_1$, $n_2$ transverse vectors orthogonal to $\mathbf{p}$, *i.e.*

$$\mathbf{n}_1\cdot\mathbf{p} = \mathbf{n}_2\cdot\mathbf{p} = \mathbf{n}_1\cdot\mathbf{n}_2 = 0 \quad \text{and} \quad \mathbf{n}_i = (0, \boldsymbol{n}_i) \quad \text{hence} \quad \boldsymbol{n}_1\cdot\boldsymbol{p}_2 = \boldsymbol{n}_2\cdot\boldsymbol{p} = \boldsymbol{n}_1\cdot\boldsymbol{n}_2 = 0$$

we take the $n$ unitary

$$\mathbf{n}_1^2 = -\boldsymbol{n}_1^2 = \mathbf{n}_2^2 = -\boldsymbol{n}_2^2 = -1\,.$$

Call

$$\mathbf{n}' = \cos\frac{\theta}{2}\,\mathbf{n}_1 + \sin\frac{\theta}{2}\,\mathbf{n}_2\,; \qquad \mathbf{n}'\cdot\mathbf{p} = 0\,, \quad \mathbf{n}'^2 = -1\,.$$

The most general unit vector orthogonal to $\mathbf{p}$ is of the form ($\alpha$ real arbitrary):

$$\mathbf{n}' + \mathbf{p}\,, \quad \text{indeed} \quad (\mathbf{n}' + \alpha\mathbf{p})\cdot\mathbf{p} = 0 \qquad (\mathbf{n}' + \alpha\mathbf{p})^2 = -1\,.$$

The elements of the little group of $\mathbf{p}$ are therefore the product of at most 3 arbitrary symmetrios of the type $\Sigma_{\mathbf{n}'+\alpha\mathbf{p}}$. (Exercise 2, page 12).

Connected little group. Its elements are the product of 2 symmeties

$$\Sigma_{n'+\alpha' p}\Sigma_{n'+\alpha' p}\,.$$

Let us call $R(\theta) = \Sigma_{n'}\Sigma_{n}$; it is a rotation around $\boldsymbol{p}$ of angle $\theta$; so is

$$\Sigma_{n''}\Sigma_{n'} = R(\theta'' - \theta')$$

and call

$$T'(\alpha') = \Sigma_{n'+\alpha' p}\Sigma_{n'}\,, \qquad \text{example} \quad T(\alpha_1)\,T_2(\alpha_2)\,,$$

note $\Sigma_{n'}\Sigma_{\mathbf{n}'-\alpha' p} = \Sigma_{n'+\alpha' p}\Sigma_{n'}$ (see pages 12-13).

Any element of the connected little group of $p$ can be written:

$$\Sigma_{n''+\alpha''p}\Sigma_{n'-\alpha'p} = \Sigma_{n''+\alpha''p}\Sigma^2_{n''}\Sigma^2_{n'}\Sigma_{n'-\alpha'p = T''(\alpha'')R(\theta''-\theta')T'(\alpha)'}$$

We can generate the whole group from

$$R(\theta)\,T_1(\alpha_1)\,T_2(\alpha_2)\;;$$

then $\theta$, $\alpha_1$, $\alpha_2$ are the three parameters of the group.

From $\Lambda\Sigma_n\Lambda^{-1} = \Sigma_{\Lambda n}$ that this group is just isomorphic to the 2 dimensional connected euclidean group, *i.e.* the group of translation $\boldsymbol{t} = \alpha_1\boldsymbol{t}_1 + \alpha_2\boldsymbol{t}_2$ and rotation around the origin of angle $\theta$. Indeed the group law is similar to that of the inhomogeneous Lorentz group and can be represented by the matrice

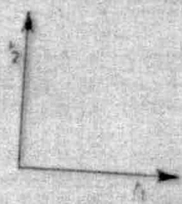

Fig. 4.

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_1 & \cos\theta & -\operatorname{sen}\theta \\ \alpha_2 & \operatorname{sen}\theta & \cos\theta \end{pmatrix}$$

The isomorphism can be extended to the symmetries. $\Sigma_{n1}$ corresponds to the symmetry defined by $t_1$, *i.e.* through $t_2$ ($\perp t_1$).

*Summary.* – Little group of $p$:

$p$ time like: isomorphic to 3-dimensional rotation group,

$p$ light like: isomorphic to 2-dimensional euclidean group,

$p$ space like: isomorphic to 3-dimensional Lorentz group.

***The covering group $U_2$ of $O_3^{(+)}$ (rotations in 3 dimensions).*** – We use the 3 Pauli matrices $\tau = \{\tau_i\}$ $i = 1, 2, 3$ such that $\tau_i^* = \tau_i$;

(30)
$$\tau_i\tau_j + \tau_{ji} = 2\delta_{ij}\,.$$

To each 3-vector $\boldsymbol{x}$, we associate the matrix $x = \sum_i x_i\tau_i = \boldsymbol{x}\cdot\boldsymbol{\tau}$. Note that

(31)
$$\boldsymbol{x}\cdot\boldsymbol{y} = \tfrac{1}{2}(xy + yx)\;; \qquad \boldsymbol{x}^2 = x^2$$

if $\boldsymbol{x}' = \Sigma_n\,\boldsymbol{x}$ we have

(32)
$$x' = x - (nx + xn)n^{-2}n = -\,nxn^{-1}\,.$$

Let $r_{21}=n_2n_1$; it corresponds to the rotation

$$R_{21}=\Sigma_{n_2}\Sigma_{n_1} \qquad \text{if} \qquad y=R_{21}x$$

one has

$$y=r_{21}xr_{21}^{-1} \qquad \text{with } r_{21}=n_2n_1 . \tag{33}$$

From this it is easy to conclude that the matrices $r$ multiply as do the rotations, *i.e.*

$$R\to r\,,\qquad R'\to r'\,,\qquad R''=RR'\to r'r=r''\,.$$

This correspondence is however up to a sign, because $\Sigma_n$ and $\Sigma_{-n}$ are the same symmetry; hence both $\pm r$ correspond to $R$.

If the rotation is defined by $\boldsymbol{n}$, $\omega$, axis and angle of rotation $r(\boldsymbol{n},\omega)$ can easily be computed (use $ab=a\cdot b+i(a\Lambda b)\cdot\boldsymbol{\tau}$). We obtain taking $\boldsymbol{n}^2=1$

$$r(\boldsymbol{n},\omega)=\pm\left(\cos\frac{\omega}{2}-in\sin\frac{\omega}{2}\right)=\pm\exp-\left[i\frac{\omega}{2}n\right]=\pm u(\omega,\boldsymbol{n})\,,$$

*Remarks*:

*a*) The matrices $u(\omega,\boldsymbol{n})=\exp[-(i\omega/2)n]$ for all values of $\boldsymbol{n}$ (with $n^2=1$) and $\omega$ generates a group, $U_2$ the unitary unimodular group.

Ideed

$$\omega_1,\,n_x,\,n_y,\,n_z \text{ real}\Rightarrow\left(\frac{\omega}{2}n\right)^*=\frac{\omega}{2}n\Rightarrow \text{u unitary}$$

$$\operatorname{tr} n=0\Rightarrow\det u=1\,.$$

*b*) Conversely a unitary 2 by 2 matrix $u$ can be written $u=\exp[-ih]$ where $h$ is a 2 by 2 hermitian matrix and any $h$ can be written $(\omega/2)n$.

*c*) What we have found therefore is not a true representation of the rotation group $O_3^{(+)}$ but a representation *up to a sign only.*

We also prove $U_2\xrightarrow{f}O_3^{(+)}$, where the homomorphism $f$ is a two to one correspondence. Kernel of $f$ is $u=1$ and $u=-1$ (which form the two element group $Z_2$). $\pm u(\omega,\boldsymbol{n})\to r(\omega,\boldsymbol{n})$.

*d*) Note that $u(2\pi,\boldsymbol{n})=-1$ whatever $\boldsymbol{n}$.

*Definition of Poincar, group,* $\pi_1$. – For a topological group consider continuous mapping $s_1$ of the circle $S_1$ (= sphere in 1 dimension; we can generalize to $S_n$) into the grop manifold and such that this mapping contains the identity. (By translation on the group any other fixed point can be chosen). If a given $s_1$ called $s_1'$ can be, *by continuous transformation*, transformed into $s_1''$ (another $s_1$), then $s_1'$ and $s_1''$ are said « homotopic ».

It is easy to see that « homotopy » is an equivalence relation among the $s_1$. We can define the « product » of two $s_1$, when $S_1$ is oriented. This $\Rightarrow$ a composition law for the equivalence class, which gives a group structure. (the identity class is that of the $s_1$ which, by continuous deformation can be reduced to a point). This group is called $\pi_1$ or Poincaré group.

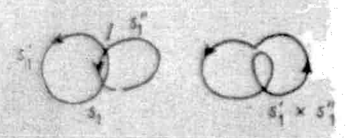


Fig. 5.

*Simply connected group.* – It is a group whose Poincaré group has only one element. All closed paths in the group are homotopic. Example, $U_2$ is simply connected.

The most general matrix of $U_2$ is of the form

$$\begin{vmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{vmatrix}$$

with $\alpha\overline{\alpha}+\beta\overline{\beta}=1$. If $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$, the condition is $\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2=1$. This is the equation of $S_3$, the 3 dimensional sphere. Consider a closed path in it, passing through the point 1, 0, 0, 0 (unit of the group). By continuous deformation this path can be made « plane » (*i.e.* contained in a 3 dimensional plane) and this plane can be moved continuously to the tangent plane in 1, 0, 0, 0.

*Covering group $C$ of $G$.* – It can be shown (see C. J. Pontrjagin: *Topological Groups*, § 47, for the proof and the precise sufficient conditions) that for any Lie group $G$, there exists a *unique* simplyu connected group $C$, locally isomorphic to $G$, homomorphic to $G$. Then, the kernel $\pi_1^*$ of the homomorphism is isomorphic to the Poincaré group of $G$. This unique $C$ is called the covering group of $G$. We have therefore proved that $U_2$ is the covering group of $O_3^+$. We also know the two homotopy class of paths of $O_3^+$. The identity class is that of closed paths which are image (by the homomorphism $U.\to O_3^+$) of closed path of $U$. They can be shrunk to a point since $U$ is simply connected. The closed paths in $O_3^+$ which are image of continuous path from 1 to $-1$ in $U$ cannot be shrunk to a point. A way to « see » the topology of $O_3^+$ is to plot each rotation $\boldsymbol{n}$, $\omega$ at the tip of the vector $\omega\boldsymbol{n}$ with $0<\omega<\pi$ and identify the points $\pm\omega\boldsymbol{n}$; closed paths which contain an odd number of such points $\pm\omega\boldsymbol{n}$ cannot be homotopic to zero.

*The covering group $C_2$ of $L_+^\uparrow$.* – What we have done for $O_3^+$ with the Pauli matrices can be done now for the Lorentz group $L$ and the Dirac matrices

(34) $$[i\gamma^\mu, i\gamma^\nu]_+ = 2g^{\mu\nu} \quad \text{when} \quad [A, B]_+ = AB + BA\,.$$

We write $i\gamma^\mu$ for the matrices, because it is possible to choose the $i\gamma^\mu$ real. Indeed the $4\gamma^\mu$ generate an algebra of 16 linearly independent matrices

$$\gamma^{[A]} = 1,\ \gamma^\mu,\ i\sigma^{\mu\nu} = \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]\,, \qquad \gamma^5\gamma^\mu\,, \qquad \gamma^5 = \gamma^0\gamma^1\lambda^2\gamma^3\,.$$

The 32 matrices $\pm\gamma^{[A]}$ form a group of elements $\gamma^{[A']}$. We can compute (see pages 555, 666)

$$(35)\quad \frac{1}{32}\sum_{A}\mathrm{Tr}\,(\gamma^{[A']})^2 = \frac{1}{16}\sum_{A}\mathrm{Tr}\,(\gamma^{[A]})^2 = 1\,.$$

We define the correspondence $\mathbf{x} \to x$.

Equations similar to (31), (32), (33) hold.

To summarize, $\Lambda \in L_+^\uparrow$ can be decomposed into (see page 14)

$$\Lambda = \Sigma_{n_4}\Sigma_{n_3}\Sigma_{n_2}\Sigma_{n_1} \qquad \text{with } |n_i^2| = 1.$$

The correspondence $\Lambda \to \pm n_4 n_3 n_2 n_1 = S(\Lambda)$ is a representation up to a sign of $L_+^\uparrow$.

This representation is reducible

$$i\gamma^5 S(\Lambda) = S(\Lambda) i\gamma^5\,.$$

Since $(i\gamma^5)^2 = 1$ and $\mathrm{tr}\,\gamma^5 = 0$, $\gamma^5$ can be written

$$\tau_3 \otimes 1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

and one finds easily that the real four dimensional representation $S(\Lambda)$ breaks into the direct sum of two charge conjugated two dimensional representations: *e.g.* take

$$\gamma^i = (\tau_1 \otimes \tau_i)\,; \qquad \gamma^0 = (i\tau_2 \otimes 1)\,.$$

Another method to prove it directly is:

We form the matrices $X$ for a given $\mathbf{x}$ according to

$$(36)\quad \mathbf{x}:\ \begin{cases} X = x^0\cdot 1 - \sum_i x^i\tau_i \\ X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x_1 - x^3 \end{pmatrix} \end{cases}$$

and notice that $\det X = \mathbf{x}^2$.

Corresponding to $\mathbf{x}' = \Lambda\mathbf{x}$ we write

$$X'^{\mu\nu} = \Lambda^{\mu\nu}{}_{\varrho\sigma} X^{\varrho\sigma}.$$

For a light-like vector $\mathbf{x}$ $(x^2 = 0)$ this has the form

$$\alpha^\mu \beta^\nu = \Lambda^{\mu\nu}{}_{\varrho\sigma} \xi^\varrho \eta^\sigma.$$

In this case one finds that it is possible to write

(37) $$X' = AXB^T \qquad \text{or} \qquad X' = AX^T B^T,$$

where the requirement $\det X' = \det X$ imposes the condition $\det A \cdot \det B = 1$ which leaves a great choice of possible $A$, $B$. If we suppose that $\det A = \det B = 1$ we still have an ambiguity in sign: we can choose either $A$ and $B$ or $-A$ and $-B$.

*The real homogeneous connected Lorentz group.* – For this group $\mathbf{x}$ is real and $X$ hermitian

(39) $$X^* = X, \qquad X'^* = X'.$$

So that

$$(B^T)^* X A^* = AXB^T$$

or equivalently

$$XA^*(B^T)^{-1} = (B^T)^{*-1} AX$$

or

$$XA^*(B^T)^{-1} = [A^*(B^T)^{-1}]^* X.$$

Writing

$$F = A^*(B^T)^{-1}$$

we have

$$XF = F^* X;$$

this holds for all hermitian matrices, in particular for $X = 1$ so that

$$F = F^*.$$

By Schur's Lemma $\exists$ a scalar $\lambda$ such that

$$F = \lambda I$$

so that

$$A^* = \lambda B^T :$$

but $\det A = \det B$ so that

$$A^* = \pm B^T .$$

For the group $L_+^\uparrow$:

$$X' = + AXA^* ; \tag{40}$$

for the set (not a group!) $L_+^\downarrow$: $X' = - AXA^*$.

Conversely, take an $A$ such that $\det A = 1$: the correspondence $X \to X' = AXA$ is a linear correspondence $\mathbf{x} \to \mathbf{x}'$ preserving $\mathbf{x}^2$.

Thus we have a homomorphism between the Unimodular group $C_2$, *i.e.* the group of all $2\times2$ matrices of determinant 1, and $L_+^\uparrow$:

$$\Lambda \in L_+^\uparrow \to \pm A \,\boldsymbol{n}\, G_2 .$$

*Examples.* – Rotation:

$$(\boldsymbol{n}, \omega), \quad A \text{ unitary } U = (U^*)^{-1} = \pm\left(\cos\frac{\omega}{2} - i\boldsymbol{n}\cdot\boldsymbol{\tau}\sin\frac{\omega}{2}\right). \tag{41}$$

« Pure » Lorentz transformation:

$$(\boldsymbol{l}, \beta). \quad A \text{ hermitian positive definite} = H = H^* = \pm\left(\mathrm{ch}\frac{\beta}{2} + \boldsymbol{l}\cdot\boldsymbol{\tau}\,\mathrm{sh}\frac{\beta}{2}\right) \tag{41'}$$

(a change in « velocity » $\beta$ ($\mathrm{th}\,\beta = v/c$) in the $\boldsymbol{l}$ direction).

Now, any matrix can be written as the product of a unitary matrix $U$ and a hermitian positive definite matrix $H$:

$$A = UH : \tag{42}$$

$AA^*$ is a positive definite hermitian matrix: $AA^* = H^2 \to \exists$ a unique positive definite square root of $H^2$: $H\cdot H^{-1}A\cdot A^*H^{-1} = (H^{-1}A)(H^{-1}A)^* = 1$; since $H^{-1}A)$ exists $H^{-1}A$ is unitary: $A = HU$.

Thus every Lorentz transformation has a unique decomposition into the product of a rotation and a pure Lorentz transformation.

We have also explicitly displayed the 6 parameters of the $L^\uparrow$. Note that if we consider « imaginary rotations » $\boldsymbol{n}$, $i\omega$ one just obtains the « pure » Lorentz transformation $\boldsymbol{l} = \boldsymbol{n}$, $\beta = \omega$ (see (41) and (41')).

This shows that $L_+^\uparrow$ is isomorphic to the complex rotation group $CO_3^{(+)}$ in the same way $C_2$ is the « complex group » of $U_2$ We leave to the reader to prove that $C_2$ is the covering group of $L_+^\uparrow$ with $C_2/Z_2 = L_+^\uparrow$.

*Complex Lorentz group $CL$.* – If we do not add condition (39) to the conditions (37) and (38) we obtain a representation up to a sign of the complex Lorentz group $CL$ (= complex connected orthogonal group $CO_4^+$), *i.e.* the group, for which $\mathbf{x}^2$ is invariant for all $\mathbf{x}$ with complex co-ordinates. Any $\in CL$ is represented, up to a sign, by a couple of unimodular $2\times 2$ matrices $A$ and $B$.

Hence the direct product $C_2\times C_2$ is a representation up to a sign of $CL$, *i.e.*

$$CL = C_2\times C_2/Z_2\,. \tag{43}$$

*Lie algebra of the inhomogeneous Lorentz group.* – As we saw, a general element of the group may be written

$$A = \begin{pmatrix}1 & 0\\ a & \Lambda\end{pmatrix} \equiv A(\dots\alpha^i\dots)\,,$$

where $\alpha^i$ are the parameters labelling the elements and for the identity

$$I = A(0, 0, \dots)\,.$$

We construct the infinitesimal operators in the neighbourhood of the identity defined by (see Prof. RACAH's lecture):

$$D_i = \left(\frac{\partial A}{\partial\alpha^i}\right)_{\alpha^i=0}. \tag{44}$$

For a one parameter abelian group (if it exists) we have

$$A(\alpha_1^i + \alpha_2^i) = A(\alpha_1^i)A(\alpha_2^i)\,,$$

where $a_1^i$, $\alpha_2^i$ are two values of the same parameter

We solve this functional equation by differentiating with respect to $\alpha_1^i$ holding $\alpha_2^i$ fixed, and then putting $\alpha_1^i = 0$:

$$A_1^{i\prime}(\alpha_2^i) = D_i A(\alpha_2^i)\,.$$

Thus we have

$$A(\alpha^i) = \exp[\alpha^i D_i]\,.$$

We remark that since the exponential of a matrix is well defined there is no difficulty when the operators are matrices; it should be remembered that if

$$[D_1, D_2] \neq 0\,, \qquad \exp[D_1 + D_2] \neq \exp[D_1]\cdot\exp[D_2]\,.$$

*Translation group.* – For an infinitesimal translation $A(a, 1) \simeq 1 + \mathcal{D}_\mu a^\mu$, where $a^\mu$ are the parameters and $\mathcal{D}_\mu$ are the four corresponding infinitesimal operator. For a finite translation we obtain

$$U(a, 1) = \exp[\mathcal{D}_\mu a^\mu] = \exp[iP_\mu a^\mu] , \tag{45}$$

where we have introduced

$$P_\mu = -\mathcal{D}_\mu .$$

Physicists introduce $P$, because when $U$ is unitary, $P^\mu$ are hermitian and are observables. Indeed they correspond to energy and momentum. The « transformation properties » of $P^\mu$ and $L^{\wedge}$ can be obtained by the use of ( ):

$$U(0, \Lambda)U(a, 1)U(0, \Lambda)^{-1} = U(\Lambda a, 1) \tag{47}$$

and (46); one obtains

$$U(0, F)P^\mu U(0, \Lambda)^{-1} = \Lambda^\mu_{\ \nu} P^\nu . \tag{7}$$

Furthermore the $\mathcal{D}^\mu$ commute hence

$$[P^\mu, P^\nu] = 0 . \tag{8}$$

Homogeneous group

$$\Lambda^T G \Lambda = G .$$

The treatment is very similar to that of the orthogonal rotation group. r $\Lambda \in O_n$, $\Lambda^T \Lambda = 1$.

Differentiation with respect to a parameter yields (by putting the parameter equal to zero)

$$\Lambda'^T|_{x=0} 1 + 1 \Lambda'|_{x=0} = 0 ,$$

$$D^T + D = 0 .$$

For the homogeneous Lorentz group we have form ( )

$$\mathcal{M}^T G + G\mathcal{M} = 0 \tag{}$$

putting

$$D = G\mathcal{M} . \tag{}$$

We have $D^t + D = 0$, *i.e.* $D$ is skewsymmetric. It is $4\times4$ so there are six parameters.

We define the matrices $e_{\alpha\beta}$ as before

(51) $$(e_{\alpha\beta})_{\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu}$$

that is

(52) $$e_{\alpha\beta}e_{\varrho\sigma} = e_{\alpha\sigma}e_{\beta\varrho}\,.$$

These $e_{\alpha\beta}$ provide us with a convenient representation of the $D_{\mu\nu}$ (infinitesimal operator of the rotation in the 2-plane $\mu-\nu$)

$$D_{\mu\nu} = e_{\mu\nu} - e_{\nu\mu}\,.$$

We have

$$\begin{aligned} D_{\mu\nu}D_{\varrho\sigma} &= (e_{\mu\nu} - e_{\nu\mu})(e_{\varrho\sigma} - e_{\sigma\varrho}) \\ &= e_{\mu\nu}e_{\varrho\sigma} + e_{\nu\mu}e_{\sigma\varrho} - e_{\nu\mu}e_{\varrho\sigma} - e_{\mu\nu}e_{\sigma\varrho} \\ &= e_{\mu\sigma}\delta_{\nu\varrho} + e_{\nu\varrho}\delta_{\mu\sigma} - e_{\nu\sigma}\delta_{\mu\varrho} - e_{\mu\varrho}\delta_{\mu\sigma} \end{aligned}$$

hence

(53) $$[D_{\mu\nu}, D_{\varrho\sigma}] = D_{\mu\sigma}\delta_{\nu\varrho} + D_{\nu\varrho}\delta_{\mu\sigma} - D_{\nu\sigma}\delta_{\mu\varrho} - D_{\mu\varrho}\delta_{\nu\sigma}\,.$$

Introducing

(54) $$M = i\mathcal{M} = i\mathcal{G}D\,,$$

(55) $$[M_{\mu\nu}, M_{\varrho\sigma}] = i[g_{\nu\sigma}M_{\mu\varrho} + g_{\mu\varrho}M_{\nu\sigma} - g_{\nu\varrho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\varrho}]\,.$$

Since the homogeneous Lorentz group is semi-simple we may find the Casimir operators in a straight forward way (see Prof. Racah lecture).

So far we have only in (55) the Lie algebra of $L_+^\uparrow$. That is, given any two elements, we know their commutator, not their products.

*Enveloping algebra of a Lie algebra.* – Example $L$ and $\mathcal{L}$.

This consists of all possible formal polynomials formed from the elements of the Lie algebra.

*Centre of the enveloping algebra.* – The centre $C$ of the enveloping algebra $E$ is the set of all elements $e\in E$ which commute with every other element:

$$C = \{c\in E\,|\,V f\in E\cdot fc = cf\}\,.$$

*Centre of the enveloping algebra of the* ***inhomogeneous group.*** – It is convenient to define the 3 dimensional « pseudovector »

(56) $$\boldsymbol{J} = (J^1, J^2, J^3) = (M^{23}, M^{31}, M^{12})$$

and the 3-vector

(56′) $$\boldsymbol{N} = (N^1, N^2, N^3) = (M^{01}, M^{02}, M^{03}) .$$

We have from (55)

(57) $$[J^i, J^j] = i\varepsilon^{ijk} J^k ,$$

which physicists write symbolically

(57′) $$\boldsymbol{J} \wedge \boldsymbol{J} = i\boldsymbol{J} .$$

We also obtain

(58) $$\boldsymbol{N} \wedge \boldsymbol{N} = -i\boldsymbol{J} ,$$

(58′) $$[J_i, N_j] = i\varepsilon_{ijk} N_k = [N_i, J_j] .$$

As is well known

$$[\boldsymbol{J}^2, \boldsymbol{J}] = 0 ,$$

but

$$[\boldsymbol{J}^2, \boldsymbol{N}] \neq 0 , \qquad [\boldsymbol{N}^2, \boldsymbol{N}] \neq 0 .$$

For the homogeneous group the invariants (*i.e.* elements of the center) are

(59) $$\boldsymbol{J}^2 - \boldsymbol{N}^2 = \tfrac{1}{2} M^{\mu\nu} M_{\mu\nu} ; \qquad \boldsymbol{J} \cdot \boldsymbol{N} = \tfrac{1}{2} \varepsilon^{\lambda\mu\nu\varrho} M_{\lambda\mu} M_{\nu\varrho} = \det M_{\mu\nu}$$

They do not, however, commute with the $P$'s and so do not belong to the Centre for the inhomogeneous group.

For the inhomogeneous group we have to include the elements $P_\mu$ of the translation group. We find

(60) $$[P_\lambda, M_{\mu\nu}] = i(g_{\lambda\mu} P_\nu - g_{\lambda\nu} P_\mu)$$

so that using $[AB, C] = A[B, C] + [A, C]B$ we find that: $P^2 = P^\mu P_\mu$ commutes withe $M_{\mu\nu}$.

It also commutes with the $P$, so that it is one element of the centre. The other invariant was found by Pauli: let

$$W_\lambda = \tfrac{1}{2} \varepsilon^{\lambda\mu\nu\delta} P_\mu M_{\nu\varrho}$$

then

$$P_\lambda W^\lambda = 0$$

and

$$[P_\mu, W_\lambda] = 0 .$$

We compute

$$[W_\lambda, M_{\mu\nu}] = i(g_{\lambda\mu} W_\nu - g_{\lambda\nu} W_\mu) ,$$

$$[W_\lambda, W_\mu] = i\varepsilon_{\lambda\mu\nu\varrho} P^\nu W^\varrho ,$$

(note that $\varepsilon^{\lambda\mu\nu\vartheta} = -\varepsilon_{\lambda\mu\nu\varrho}$ since $\det g = -1$; $\varepsilon^{0123} = 1$).

We see that

$$W^2 = W_\mu W^\mu$$

is also an element of the centre of the envelopping algebra.

*Summary.* – Invariants:

for $\mathcal{P}$ $P^2$, $W^2$,

for $L$: $J^2 - N^2$, $\boldsymbol{J}\cdot\boldsymbol{N}$.