reference:
p. 162-174 in *Walifest-MRST15:*
New directions in the application of symmetry principles to elementary particle physics. edit. J. Schechter, World Scientific, (1994).
Unhappily the figure (page 12) could not appear in that book

Can we learn from other branches of Physics for the study of fundamental interaction symmetries?

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ABSTRACT

We first recall the basic concepts for symmetry and broken symmetry and their general application to crystals and liquid crystals. How that transposes to the cosmology of our Universe? We show the limitation of the use of Born von Karman finite groups as crystal symmetry groups. Some interesting physical phenomenon of topological nature are lost. We found in some space groups, quasi-invariant subgroups; this weaker equivalence has been very useful in mathematics. Physicists should be aware of it.

It is a great honor to have been invited at this Walifest. I thank the organizers to give me the opportunity and the deep joy to celebrate a well known physicist, founder of a school whose many members are gathered here, a man of great culture who has remarkably integrated Eastern and Western inheritance, a very respected scholar and a good writer. We are many here to share this dear friend: Kameshwar Wali.

Kamesh and I met in Madison in 1960 in the first (but not the last) Summer Institute organized by Bob Sachs. Since, not only we had many contacts as physicists but we became excellent friends (although we published six papers together in the last twenty years), and our two families share very good remembrances together. I learned much from Kamesh with his rich culture and his readiness to help and I am very pleased to have the occasion to thank him more formally today (not forgetting to thank also Kashi). This meeting of many friends, students and collaborators of Kamesh Wali is a marvellous way to celebrate him. Let me add that this meeting illustrates so well the universality of Scientific Culture that we all share. Indeed we were born in every part of the world and raised in so many different langages and backgrounds, but through Science we have so much in common!

I am not trying to minimize the difficulties that physicists have for understanding each others! I am too well aware of this handicap since most my research activity for the last fifteen years has been outside the domain of high energy physics in which most of you are working. But we all know how much we can learn from other fields of physics if ... we are able to communicate. Let me remind you how much the ideas of Goldstone, Nambu [1] ..., inspired by solid state physics, influenced our own views on symmetry. Mathematicians and physicists were aware of the fundamental role of gauge invariance: Weyl, Klein, Pauli, Yang and Mills, Sakurai, Gürsey,.... We could also learn from supraconductivity: a physical phenomenon where a gauge symmetry is broken.

It is time to outline the content of this lecture. Sure we can learn from other parts of physics, as well from their success than from their mistakes: I will illustrate both points, taking my examples mostly from condensed matter physics. In order to make the communication possible I will first recall the basic concepts to be used for the study of symmetry breaking. And the last part of the lecture will tell you of a small addition to these basic concepts, that some colleagues and I found lately, but it was discovered seventy years ago in other domains of science.

1. Basic concepts for symmetry and broken symmetry.

Symmetry groups intervene in physics through their action; such action might be not linear! Here are the simple and fundamental concepts for studying the action of a group G acting on a set M:
the *little group* G_m is the set of elements $g \in G$ leaving m fixed: $g.m = m$
the *orbit* $G.m \subset M$ is the set of transforms of m.
Here we will use the notation $M|G$ for the set of orbits (often called the orbit space because it may have nice geometric properties). The little groups of two points m and $m' = g.m$ of the same orbit are conjugate:
$G_{m'} = gG_mg^{-1}$ (an exercise is proposed in foot note [2]). The converse in generally not true. So one defines:

[1] I apologize to those I do not name.

[2] Which points of the orbit $G.m$ have exactly the same little group as m? Answer: the orbit $N_G(G_m).m$ where $N_G(H)$ denote the normalizer in G of its subgroup H, i.e. the largest subgroup of G which contains H as invariant subgroup.

a stratum is a set of all points of M with conjugated little groups.
So M is a disjoint union of strata; each stratum is a disjoint union of orbits with same conjugacy class of little group. That concept is fundamental in physics [3] and often $M||G$, the set of strata, is finite. For instance, the Lorentz group has an infinite set of conjugacy classes of subgroups, but four only appear as little groups in its action on the Minkowski space: the corresponding 4 strata are the time-like, space like, light like vectors and the null one.

The symmetry group of a physical problem acts on the set of its solutions; the symmetry group of each solution is its little group, so the different types of symmetry of the solutions correspond to the strata. I emphasize again that most often the nomber of strata is small; moreover the nature of the strata (or at least some of them) can often be predicted by the mathematical structure of the problem only [4]. That is the case of the famous and rather successful Landau model [3] which predicts [5] the symmetry change in second order phase transitions: the minima of the free energy potential form an orbit whose little groups are the symmetry groups of the spontaneously broken symmetry phase. For crystal to crystal transitions, the orbit has a finite number of minima. Which one is chosen? That is irrelevant: the choice is generally due to inhomogenities, or impurities or, in very good samples, fluctuations. The choice differs along the crystal; so it is composed of domains (called macles by mineralogists) whose symmetries correspond to the different conjugated little groups of the minima. For liquid crystals the orbit of minima is a manifold; in a perfect state the same minimum is chosen over the whole crystal. But imperfect states are more frequent and the chosen minimum varies along the crystal: that defines on the space occupied by the crystal a function valued in the manifold of minima. If the function is homotopic to the constant function, the state is simply a deformation of the perfect state; if the function belongs to another homotopy class, the state symmetry (and eventually its defects) can be classified

[3] Radicati and I introduced in [1] this name twenty five years ago, because this concept, although known to the mathematicians, had no name. One had to say "the union of orbits with same conjugacy class of little groups".

[4] That is the case for the extrema of smooth functions invariant by a compact group: there are few strata on which **every** such function has an extremum (for more details, see [2]). If you find their symmetry from a Lagrangian you have discovered, your result does not prove the value of your choice: any other Lagrangian would have given the same result!

[5] It did not explain the critical coefficients, so in the late seventies the renormalisation group technics were applied to it. How successfully? that might be a matter of opinion; but again, the symmetry predictions of the renormalisation group can often be obtained from the mathematical structure only: e.g. [4].

by topological invariants (in particular those of homotopy [6]).

2. Broken symmetry in our universe.

The concepts of broken symmetry are well known to most of you, perhaps expressed with a different vocabulary. The fundamental laws of physics are highly symmetric; outside the $U(3)$ gauge group of unified $Q(E+C)D$), many of the symmetries present in the big bang have disappeared with the cooling of our universe. And t'Hooft Polyakov monopoles are similarly the illustration of homotopy invariants obtained by spontaneous symmetry breaking. Unhappily they have not yet been observed and we know only a very low upper limit for their density in our universe. The exploration of our Universe is progressing rapidly as was progressing that of our Earth five hundred years ago. Is it not remarkable that in the XVI$^{\rm th}$ century, as soon as geographic maps were avalaible, F. Bacon asked if the similarity between the shapes of the coastlines of the eastern half of South America and the south-western half of Africa was purely spurious or had to be explained? Until the beginning of this century no one considered this remark as scientific. And during his life Wegener did not succeed to convince the scientists to accept his views on the drift of the continents. We now know when the primordial continent broke up: 200 millions years ago South America and Africa parted from each other.

A similar bold question was that of Dirac: does the value of the gravitational constant G depend on the history of our universe? For more than ten years some observers are measuring the variations of the distance between a point on the Moon and several on the Earth, with a precision better than 10^{-10}; that lead to a good upper limit for the change of value of G; the binary system of neutron stars discovered 18 years ago [6] and constantly observed since gives even a better upper limit [7]: $\dot{G}/G = (1.0 \pm 2.3)10^{-11}\text{year}^{-1}$. From the observations in space and in time we should be able to obtain as precise answer on the constancy of the value of the fine structure constant α. On the other hand it is a very natural question to wonder if the value of constants due to spontaneous breaking in the history of our Universe of the large symmetry which unified all the interactions are the same everywhere in space time. I believe that might be not the case for instance, for the "Cabibbo" directions of the weak current in the different families, but this statement is not very interesting as long as nobody is able to propose a physical test for answering such "natural" question. I hope to discuss with Kamesh, and many of you, these problems.

Another "natural" question has already been asked and a first tentative answer was given by Sakharov. How to explain the disymmetry between matter

[6] For many references in a general review, see [5].

and anti-matter in our Universe, since it is so natural to believe that the big bang had zero baryonic charge? To get his answer, in 1967, Sakharov used the observed fact of the large value of the ratio $\nu =$ (number of photons / number of baryons) $> 10^8$ in our Universe and CP violation discovered three years before [7]. He needed also proton decay that he was bold enough to predict! It is difficult to have a quantitative verification of these ideas. I am astonished that a very natural explanation, published more than ten years ago by Souriau and collaborators [25], seems to be unkown to the audience: the asymmetry between matter and antimatter might be purely local [8] and the total baryonic charge of the universe might still be zero. Indeed we have yet no proof of proton decay; so one could think that from a zero baryonic charge big bang the small number of surviving nucleons (compared to the number of photons) is due to a small spatial fluctuation of the density of baryonic charge: the simplest one is a dipole. Then, the universe expansion has separated matter and antimatter into two equal part of the Universe. Souriau *et al.* have found an observation supporting strongly their proposal. Indeed the farthest objects we can see in the Universe are the quasars. More than a thousand are observed. Plot their position in space time; to do it, you need a map of our Universe: the authors have chosen the one given by the Roberston Walker metric: the Universe has the topology $R \times S_3$ (it has O_4 symmetry). Their finding: quasars are distributed rather homogenuously up to the largest distance we see them, except for a regular gap slightly less than a billon light year wide and forming an equator of S_3. They give the position of this equator [9] (the nearest point is 8.10^9 lignt years away); beyond this equator, we see about 200 anti-quasars. Three quasars which are on the brim of this gap have very abnormal spectra. This gap should be an intense source of γ rays, decay products of the π^0 produced by nucleon anti-nucleon annihilation. The energy spectrum of these γ's (their average energy is around 130 Mev) is easy to predict: it depends continuously on the direction that we look at them. The observation of such extragalactic γ's is difficult because it is made after substraction of the large galactic background, but it has begun; so the predictions of the Souriau *et al.* model will be submitted to new tests.

3. Can we replace the symmetry group of a crystal by finite groups?

For the last twenty years, with the discovery of modulated crystals [8] and, more recently, of crystal with icosahedral symmetry [10] [9] (an impossible crystal

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[7] The conservation of CPT does not forbid a CP disymetry to appear in the evolution of a system which is not in thermodynamic equilibrium.

[8] Remember the macles!

[9] Every direction in space meet this equator whose topology is S_2.

[10] They are a very interesting and extreme case of modulated crystals.

symmetry as explained in all books [11]) our view on crystal structure have to be drastically modified; there is a great activity to make a new synthesis, not yet achieved. But periodical crystals, i.e. those crystals with a symmetry group of translation T isomorphic to Z^3, are still the more abundant and the more studied! Exactly one hundred years ago, the mathematician Schönflies and the mineralogist Fedorov, after correcting to each other small mistakes, agreed on the list of the 230 "space groups", a short name for symmetry groups of (ideal) periodic crystals. Such a space group G has the translation group $T \sim Z^3$ as invariant subgroup and the quotient $G/T = P$ is a finite subgroup (the point group of the crystal). Among the space groups, 73 of them as semi-direct products $G = T \rtimes P$. Most of the macroscopic properties of crystal depend only of the "geometrical class", i.e. the conjugacy class of P in the orthogonal group O_3. But there are 73 different possible actions of point groups on 3 dimensional lattices: they correpond to the 73 conjugacy classes of finite subgroups of Aut $Z^3 = GL_3(Z)$. These conjugacy classes are usually called "arithmetical class"; for each one, there is a space group which is a semi-direct product $G = T \rtimes P$ (see foot note [12] for more details) while the others are not.

The space groups of thousands of crystal have been determined. On the other hand solid state physics is studying many physical properties of crystals. But the majority of solid state physicists [13] study their symmetry properties assuming some periodic boundary conditions (usually called Born von Karman conditions); that procedure replaces the infinite space groups G by the finite group G_n (we use the notation $|M|$ for the number of elements of the finite set M):

$$n > 1, \ G_n = G/nT, \quad |G_n| = |P|n^3; \quad \text{notice that } G_0 = G, \ G_1 = P; \qquad (1)$$

where nT is the group generated by the nt with $t \in T$. If n is relatively prime to $|P|$, then G_n is a semi-direct product $Z_n^3 \rtimes P$ and it is the same for the different space groups of the arithmetic class. We explain in a foot note [14] a necessary condition

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[11] Since, 8-fold and 12-fold "forbidden" crystal symmetries (but no higher order) have also been observed.

[12] An arithmetic class defines an action of $P \xrightarrow{\phi} GL_3(Z)$, ϕ injective, on $T \sim Z^3$ and a group structure $H^2_\phi(P,T)$ (= second cohomology group) on the set of equivalence classes of extensions of T by P, i.e. of groups G which have T as invariant subgroup, P as quotient and with the action of the inner automorphisms of G on T corresponding to ϕ. Then the isomorphism classes of the space groups belonging to a chosen arithmetic class are given by the orbits of the normalizer $N_{GL_3(Z)}(\phi(P))$ in its natural action on the group $H^2_\phi(P,T)$.

[13] E. Wigner, and his student F. Seitz belong to the distinguished minority.

[14] Obviously, to obtain different G_n for the different G of the arithmetic class defined

to be satisfied by n in order that the different groups of the arithmetic class of G are represented by different Born von Karman groups [15] and we call "good n" any integer satifying this condition (they are all mutiple of the smallest good n).

Few years ago, when we made a classification of the symmetry of energy bands in crystal, H. Bacry, J. Zak and I thought that if a property is true for the infinite sequence of "good" G_n; it must be true for G. Since, *we have proven that it is not true.* Let us explain the problem.

More than thirty years ago, Burneika and Levinson [10], des Cloizeaux [11] showed that the set of states of an energy band forms the Hilbert space of a unitary representation of the space group G, induced from a representation of a little group appearing in the action of G on the Euclidean space (the strata of this action are called by crystallographers "Wyckoff positions"). Some were tabulated by Kovalev [12]. Of course band representations induced from equivalent representations of conjugate little groups (same stratum) *are equivalent.* However in the eighties, some equivalence between representations induced from different representations of a little group , or from little groups belonging to different strata, were found by Zak [13abc], and by Evarestov and Smirnov [14abc]. That could be well understood from the theory of induced representations as we explained in [15] where we gave a complete classification, up to equivalence, of the elementary band representations (those which cannot be decomposed into a direct sum of band representations [16]).

We found that in some crystallographic groups there might exist non conjugate finite subgroups H, H' which are isomorphic $H \xrightarrow{\iota} H'$, and the corresponding elements $h \in H$, $h' = \iota(h) \in H'$ are conjugate: in other words, for every $h \in H$, on can find an element of G, $g_h \in G$, such that $\iota(h) = g_h h g_h^{-1}$ but one cannot find a unique g, independent from h. We call such subgroups *quasi-conjugate.* Strictly speaking, conjugate subgroups are also quasi-conjugate; but in the following, except the appendix, we shall say simply "quasi-conjugate" as a short for "quasi-conjugate but non conjugate.

We listed in table 4 of [15] the 14 space groups containing a pair of conjugacy classes of finite subgroups which are quasi-conjugate (3 of them have two such pairs). We explain here the simplest example of these space groups; that is [17] $G = F222$.

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by ϕ, n must satisfies: $nH^2_\phi(P,T) = 0$, i.e. n must be divisible by the order of every element of $H^2_\phi(P,T) = 0$. (I have never seen this condition written in a book or a paper on solid state physics). The smallest number which satisfies this condition for all 73 arithmetic classes is 12.

[15] Such a "good choice" may introduce other unwanted features for mimicking the group G; e.g. G_n may have a non trivial center although G has not; see the example of $F222$ below.

[16] In [13b], Zak called them "band-irreducible representations" and gave a necessary condition: they must be induced from irreducible representations of largest little groups.

[17] More thatn sixty years ago, the crystallographers Hermann and Maughin have devised

Given 3 orthogonal vectors, generally of different norms λ_i:

$$e_i.e_j = \lambda_i \delta_{ij}; \quad \lambda_i > 0, \tag{2}$$

the translation group T is made of the vectors: $\sum_i n_i e_i$, with $\sum_i n_i$ *even*. The group $F222$ is the semidirect product $T \rtimes P$ with $P = \{I, R_1, R_2, R_3\}$ where the R_i are the rotations of order two around the three basis axes $\vec{e}_i$; they are represented by the matrices

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; R_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; R_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

So the quotient group P is isomorphic to Z_2^2. The group law of the semi-direct product $T \rtimes P$ is written:

$$a, b \in T, \; A, B \in P, \quad \langle a, A \rangle \langle b, B \rangle = \langle a + Ab, AB \rangle. \tag{4}$$

A matrix representation of this group is:

$$\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \varepsilon_1 & 0 & 0 & n_1 \\ 0 & \varepsilon_2 & 0 & n_2 \\ 0 & 0 & \varepsilon_3 & n_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon_i^2 = 1, \; \varepsilon_1 \varepsilon_2 \varepsilon_3 = 1, \; n_i \in Z, \; \sum_i n_i \in 2Z. \tag{5}$$

To this group presentation (the one given in ITC [16]), we prefer to choose, for elegance and simplicity, a **lattice basis**. We define:

$$u_1 = e_2 + e_3, \quad u_2 = e_3 + e_1, \quad u_3 = e_1 + e_2; \quad \text{so } T = \{\sum_i n_i u_i\}, \; n_i \in Z, \tag{6}$$

i.e. the vectors u_i generate the lattice T. In this basis the three elements of P different from I are represented by the matrices:

$$S_1 = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}. \tag{7}$$

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a remarkable notation for the 230 space groups; the label for each group, using 2 to 7 characters: letters, /, and digits (which can also be used as subscipt or carry a bar over them), contains enough information for reconstructing exactly the group law. They are used in the International Tables for Crystallography [16].

We easily verify:

$$S_i^2 = I, \quad S_i S_j = S_j S_i, \quad S_1 S_2 S_3 = I. \tag{8}$$

$$S_i u_i = -u_i \ \Leftrightarrow \ \langle 0, S_i\rangle\langle -u_i, I\rangle = \langle u_i, S_i\rangle \ \Rightarrow \langle u_i, S_i\rangle^2 = \langle 0, I\rangle \equiv I \in G \tag{9}$$

More generally:

$$\langle u_i, I\rangle\langle m\, u_i, S_i\rangle\langle u_i, I\rangle^{-1} = \langle (m+2)u_i, S_i\rangle \quad \langle u_i, S_i\rangle^2 = I \tag{10}$$

Also:

$$u = \sum_i u_i, \quad S_i u = u - 4u_i \Leftrightarrow \ \langle 0, S_i\rangle\langle u, I\rangle = \langle u - 4u_i, S_i\rangle. \tag{11}$$

With a little work, one shows that all elements $\neq I$ of $F222$ of finite order, are of order 2; they are of the form $\langle s_i, S_i\rangle$ with $S_i s_i = -s_i$; they are all conjugate to the elements $\langle m\, u_i, S_i\rangle$. The preceeding equations also show that those elements fall into six conjugacy classes which are labelled by the values 1,2,3 of the index i and the value of m modulo 2. All that implies that the maximal finite subgroups of $F222$ are isomorphic to $P \sim Z_2^2$.

The maximal finite subgroups of a crystallographic group are stabilizers of points: they leave fixed the barycenter of each of their orbits. Hence, a problem equivalent to the determination of the conjugacy classes of $F222$ subgroups isomorphic to P, is to determine their orbit on the Euclidean space. Let x a point of such an orbit; it satifies the three equations $s_i + S_i x = x \Leftrightarrow s_i = (I - S_i)x$. Let ξ_1, ξ_2, ξ_3 the coordinates of the point x. Since the translation s_i has integer coordinates, we solve these equations modulo 1 for the coordinates of x. The solutions are:

$$\mathrm{mod}\, 1: \quad \xi_1 \equiv \xi_2 \equiv \xi_3, \ \xi_i + \xi_1 + \xi_2 + \xi_3 \equiv 0 \ \Leftrightarrow \ 0 \leq \xi_1 = \xi_2 = \xi_3 < 1, \ 4\xi_i \equiv 0. \tag{12}$$

So there are four orbits, determined by their point in the fundamental domain [18] $0 \leq \xi_i < 1$:

$$a: \ 0,0,0; \quad c: \ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}; \quad b: \ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \quad d: \ \frac{3}{4}, \frac{3}{4}, \frac{3}{4}. \tag{13}$$

We write again that equation in the form:

$$x: \ (\xi_i = \frac{m}{4}); \quad \text{when } m = 0,1,2,3; \quad x = a, c, b, d. \tag{14}$$

The corresponding stabilizers are denoted by P_m:

$$P_m = \{\langle 0, I\rangle, \ \langle m\, u_i, S_i\rangle, i = 1,2,3\}. \tag{15}$$

[18] The notations a,b,c,d are those of the International Tables of Crystallography for "Wyckoff positions"; but they use the coordinates defined in Eq.(2),(3).

The set of points of the orbit $G.x$ is $\{x+t,\ t\in T\}$ and the corresponding little groups are conjugate of P_m by the translation t. So there are 4 conjugacy classes of $F222$ subgroups isomorphic to P: indeed Eq.(11) yields (use also that T is Abelian):

$$m,n\in Z,\quad P_m\langle nu,I\rangle=\langle nu,I\rangle P_{m-4n}. \tag{16}$$

That proves P_m *and* $P_{m'}$ *are conjugate.* $\Leftrightarrow\ m\equiv m' \bmod 4$
Moreover Eq.(15),(10) show
$m\equiv m' \bmod 2\ \Leftrightarrow P_m$ *and* $P_{m'}$ *are quasi-conjugate.*

For the Born von Karman groups $F222_n$, $n>1$, one shows easily that:
for n odd, all subgroups isomorphic to P are conjugate,
for $n\equiv 2 \bmod 4$, the subgroups $\sim P$ fall into 2 conjugacy classes labelled by $m \bmod 2$,
for $n\equiv 0 \bmod 4$, the subgroups $\sim P$ fall into 4 conjugacy classes labelled by $m \bmod 4$, each pair of conjugacy classes with same parity of m contains quasi conjugate subgroups.
We also notice the center of $F222_n$ is trivial when n is odd,
is $Z_2(\langle \frac{n}{2}\,u,I\rangle)$ when $n\equiv 2 \bmod 4$,
is $Z_4(\langle \frac{n}{4}\,u,I\rangle)$ when $n\equiv 0 \bmod 4$.

Representations of finite groups induced from equivalent representations of quasi-conjugate subgroups are equivalent (in the appendix we give an explicit expression for the characters of an induced representation). That is the case for those of $F222_n$ induced from equivalent representations of P_0 and P_2 or from P_1 and P_3. But we showed that it is not the case for the space group $F222$ [17]: the induced representations are infinite dimensional so we can no longer compare their characters; two representations induced from equivalent representations of quasi-conjugate subgroups (e.g. P_0 and P_2) have still the same content when decomposed into irreducible representations of [19] $F222$; however there is a topological obstruction for constructing a unitary intertwinning operator between them. So there are inequivalent [20] . That has a physical effect: one can measure a Berry like phase depending only on the symmetry of the band; we predicted [19] that they are different for the bands corresponding to such a pair of representations.

It is not so astonishing that when we replace our space time by a finite set of n^4 points, whatever the large value of n, we loose some physics specially those phenomena described by topological invariants. I think the still growing crowd of physicists engaged on computing gauge quantum field effects on finite lattices, might

[19] Their dimensions are 1,2,4.

[20] Bacry [18] obtained also that result from a detailed study of the dual of $F222$, i.e. the topological space canonically built on the set of equivalence classes of unitary irreducible representations; the explanation comes from its non seperate topology.

be interested by the limitations I just described of the use of Born von Karman approach on crystal symmetry. These limitations were quite unexpected by the large crowd of solid states physicists.

4. Quasi conjugate and almost conjugate finite subgroups of a group.

I was amazed by the existence of quasi-conjugate finite subgroups in 14 space groups. I asked myself many questions: e.g. what is the smallest number of elements of a finite group containing quasi-conjugate subgroups [21]? As usual my reaction was to search if other scientists had met quasi-conjugate subgroups. It took me sometimes to find out if many other examples were known. It will be easy to explain to you the infinite number of examples published by Gassmann [20] in 1926 (before Kamesh was born!): the quasi-conjugate subgroups are also isomorphic to Z_2^2 and are subgroups of the groups $\mathcal{S}_n$ of permutations of n objects for $n \geq 6$. Since you all read Dirac's book on quantum mechanics, you all know that permutation can be represented as a product of commuting cyclic permutations and those with the same lengths of cycles are conjugated. Consider the two subgroups of $\mathcal{S}_n, n \geq 6$:

$$H_1 = \{I, (12)(34), (13)(24), (14)(23)\}, \quad H_2 = \{I, (12)(34), (12)(56), (34)(56)\}. \tag{17}$$

Their elements $\neq I$ are all conjugate; since H_1 acts on 4 elements and H_2 acts on 6 elements, they cannot be conjugate in $\mathcal{S}_n$. So they are quasi-conjugate.

These examples were given in order to kill a Kronecker conjecture dating of 1880 about the fields of algebraic numbers: "isospectral fields are isomorphic"; I do not wish to explain here the meaning of this statement [22]. But I will explain a similar one: *Isospectral compact Riemann spaces might be non isometric.*

For any compact Riemannian space, one can define a Laplace operator acting on the functions defined on this space with the condition to vanish on the

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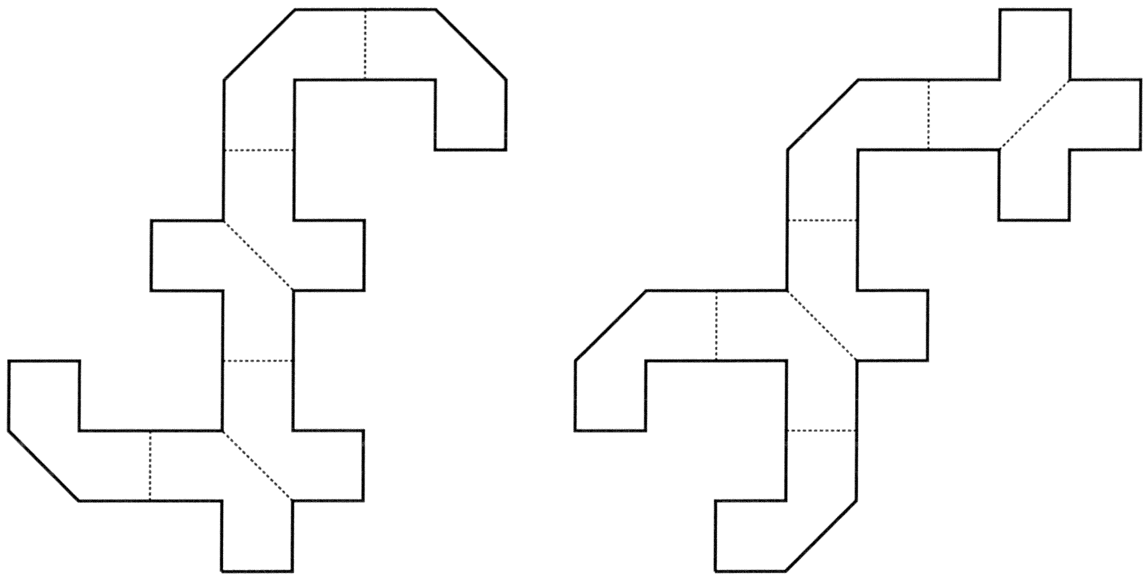
[21] The smallest I knew then was $|F222_4| = 256$; I now know that the answer is 32; I let to the reader the pleasure to browse by himself through the mathematical literature on the subject.

[22] Let us just say that a prime number in Z, the ring of integers, might be not prime in the ring of integers of the algebraic number field $\mathcal{F}$; the spectrum of $\mathcal{F}$ is defined by the decomposition (trivial or not), in its ring of integer, of every prime of Z. This seems to be far from physics, but... Mosseri and Sadoc have guessed a formula giving the number of atoms in the successive shells of an icosahedral packing. R. Moody and A. Weiss (to appear in J. Number Theory) have establish the rigourous expression. Its values differs from that conjectured one from 55th layer only! That is due that 11 can be factorized: $11 = (4 + \sqrt{5})(4 - \sqrt{5})$.



Two isopectral domains in the plane.
This figure is the reproduction of the one which appeared in the paper "One cannot hear the shape of a drum" by C. Gordon, D. Webb, S. Wolpert, Bull. Am. Math. Soc. 27 (1992) 134–137. However the original figure gave only the domains. In dashed lines we have indicated that the two domains are are from identical pieces. There are 7 for each domains; the construction relies on the existence of two quasi-conjugate subgroups in the 168 element simple group.
Unhappily this figure could not appear in the printed paper.

boundary when it exists [23] . Isometric Riemannian spaces are isospectral, i.e. their Laplacians have same spectrum. Is the converse true? In 1966 M. Kac was asking that question in a more picturesque manner "Can one hear the shape of a drum?". He seemed to have ignore that J. Milnor [21] had already given a negative answer: he had found a pair of isospectral but non isomorphic 16 dimensional Riemann manifolds. Similar examples were produced regularly from 1979.... In 1985 T. Sunada [22], by transposing the algebraic method of Gassmann to this problem, gave a general method for building such pair of Rimannian spaces. One has to find a Riemann manifold M with a finite group G acting on it by isometries with 0 or a finite number of fixed points. If G has a pair H_1, H_2 of quasi-conjugate subgroups acting freely [24] on M, then the manifolds $M|H_1$ and $M|H_2$ are isospectral, but not

[23] Every result we shall quote is also valid for the Neumann condition $((\partial f)_\perp = 0$ on the boundary) instead of the Dirichlet condition.

[24] That is with a unique stratum corresponding to the trivial little group; in that case

isometric [25]. P. Bérard (to appear) has generalized the Sunada theorem to the case when the subgroups H_i do not act freely (then the orbit spaces are orbifolds; their boundaries have singularities). That new theorem has lead to an example in two dimensions published in a research note of the Bulletin of A.M.S. last year [24].

Those mathematical results extend to a weaker equivalence of subgroups: that of almost conjugate subgroups, explained in appendix.

So, when you have established the list of strata in your symmetry problem, look for the strata with quasi-conjugate (and/or almost conjugate) subgroups and try to tell which physical phenomena they still distinguish and which ones they do not!

5. Appendix

We denote respectively by $[g]_G$, $[H]_G$ the conjugacy class in G of the element $g \in G$, of the subgroup $H < G$. Let G be finite, let D be a finite dimensional linear representation of H, whose character is χ_H^D. Then the character χ_G^Δ of the induced representation of G is given by

$$\Delta = \mathrm{Ind}_H^G\, D; \qquad \chi_G^\Delta(g) = |H|^{-1}|C_G(g)| \sum_{g' \in H \cap [g]_G} \chi_H^D(g'). \tag{18}$$

Here $C_G(g)$, the centraliser of g in G, is the subgroup of the elements of G commuting with g.

When D is the trivial representation (i.e. $h \in H$, $\chi_H^D(h) = 1$) the induced representation Δ is called a permutation representation since its matrices are permutation matrices (all their elements vanish except one per line and per column which is 1).

Definition: two subgroups H_1, H_2 of the finite group G are *almost conjugate* if for every $g \in G$, $H_1 \cap [g]_G = H_2 \cap [g]_G$.

Notice that this implies $|H_1| = |H_2|$ and these two groups have same number of elements of a given order.

Definition: two isomorphic almost conjugate subgroups are said *quasi conjugate*.

Eq. (18) shows that the representations of a finite group, induced from equivalent finite representations of two quasi-conjugate subgroups, are equivalent. It also shows that the permutation representations induced from almost conjugate subgroups are equivalent.

6. References

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the orbit space is a manifold.

[25] In the recent book of Buser [23], chapter 11 is a good review of the problem and in chapter 12 many examples are constructed.

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