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ON THE ENERGETICS OF REISSNER NORDSTRØM GEOMETRIES*

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ABSTRACT

We point out the existence of a generalized ergosphere in the Reissner Nordstrøm geometry and we give an explicit formula to determine its range. These results are compared and contrasted with the ones obtained in the case of the Kerr solution. An explicit process of energy extraction from a Reissner Nordstrøm black hole is given.

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The recent discovery of binary x-ray sources[1] and their identification as collapsed objects[2] has given raise to interest both in the problem of particles acceleration[3] and energy extraction[4] from black holes. That, indeed, energy can be extracted from black holes has been explicitly pointed out by Floyd and Penrose[5] and Ruffini and Wheeler[6], in the case of a rotating black hole. Christodoulou and Ruffini[7] have pointed out that not only rotational energy, but also electromagnetic energy can be extracted and they established the general formula governing the energetics of magnetic black holes:

$$(1) \quad M^2 = E^2 = (m_{ir} + Q^2/4m_{ir})^2 + L^2/4m^2_{ir}$$

with $a^2 = L^2/M^2 \leq Q^2 + M^2$, m_{ir} being the irreducible mass L the angular momentum, Q the charge, M the mass of the collapsed object. Here and in the following we use geometrized units with $G = c = 1$. This formula clearly shows that as much as 50% of the total mass energy of a black hole can be stored in electromagnetic energy and at least in principle, extracted by reversible transformations.

The region in which this extraction can occur is, here, analyzed in the case of a Reissner-Nordstrøm geometry and by analogy with the Kerr-Newman[7] analysis we call this region the "generalized ergosphere". The aim of this letter is mainly to focus on the major features of this process of energy extraction in the case in which the collapsed object is endowed with a net charge. No reference is, here, made to the possible direct application of this analysis to realistic astrophysical process. However, Ruffini and

Treves[8] have recently pointed out that the most likely solution for a collapsed object endowed with rotation and a magnetic field should be expected to possess a net charge.

The lagrangian for a particle μ and charge q moving in a given background metric is

$$(2) \quad \mathcal{L} = \frac{1}{2}\mu g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + q A_\alpha \dot{x}^\alpha$$

here A_α is the electromagnetic four potential and the dots represent differentiation with respect to the proper time τ. Greek indices run from 0 to 3, latin indices from 1–3. In our specific case we have

$$(3) \quad ds^2 = e^{-\nu} dt^2 + e^{\nu} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

with

$$(4.1) \quad e^{-\nu} = (1 - 2M/r + Q^2/r^2)$$

and

$$(4.2) \quad A_i = 0 \quad i = 1,3 \qquad (4.3) \quad A_0 = -Q/r$$

Since we are dealing with a problem endowed with spherical symmetry we can limit ourselves to analyze the orbits in the equatorial plane. We have for the equation of motion of the test particle (not geodesic!)

$$(5)\quad \frac{d^2x^\gamma}{ds^2} + \Gamma^\gamma_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = (q/\mu)\, F^\gamma_{\ \alpha}\frac{dx^\alpha}{ds}$$

The Lagrangian being explicitly independent from the time coordinate t and from the azimuthal coordinate ϕ, we have the following two conserved quantities:

$$(6.1)\quad p_o = -E = \frac{\partial \mathcal{L}}{\partial \dot t} = -\mu\,(1 - 2M/r + Q^2/r^2)\dot t - q\,Q/r$$

and

$$(6.2)\quad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \mu\, r^2 \sin^2\theta\, \dot\phi$$

E being the energy of the particle as measured by the observer at rest at infinity. Since our background metric is static we have a timelike Killing vector field $\xi^\mu(t) = (1,0,0,0)$ and (6.1) is clearly equivalent to $-E = p_\mu\, \xi^\mu(t)$. The fact that E is, indeed, a conserved quantity is an immediate consequence of the equation

$$(7)\quad p^\mu \nabla_\mu (p_\alpha \xi^\alpha) = p^\mu p_{\alpha;\mu}\xi^\alpha + p^\mu p_\alpha \xi^\alpha{}_{;\mu} = 0$$

That the first term in the sum is identically zero follows from Eq.(5) and the second term from the equations defining the Killing vector field $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$. We have also for the Hamiltonian

$$(8)\quad H = \frac{\mu}{2}\,[e^{-\nu}\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\,\dot\phi^2 - \dot t^2 e^{-\nu}] = -\frac{\mu}{2}$$

then from Eq.(6.1) and (6.2) we have

$$(9)\quad e^{-\nu}\dot r^2 + r^2\dot\theta^2 + p_\phi^2/(\mu^2 r^2 \sin^2\theta) - (p_o/\mu + Qq/\mu r)^2 e^{-\nu} = -1$$

By assuming $\theta = \frac{\pi}{2}$ and $\dot r = 0$ we obtain the equation governing the "effective potential" of the orbits in the equatorial plane of the Reissner-Nordstrøm geometry, see Fig. 1. We have

$$(10)\quad E_{eff} = -p_o = Qq/r \pm \mu((1 + p_\phi^2/\mu^2 r^2)e^\nu)^{1/2}$$

Here the positive sign corresponds to the positive roots state and the negative sign to the negative roots state. The separation between the positive and negative roots goes to 2μ for $r \to +\infty$. At the horizon $r_+ = M + (M^2 - Q^2)^{1/2}$ this separation goes to zero and we have

$$(11)\quad \lim_{r \to r_+} E^+ = E^- = Qq/r_+$$

Most important for us here is the existence of the negative energy states for the positive energy solutions which, indeed, allow us to introduce the concept of "effective ergosphere". We also have the following important relation

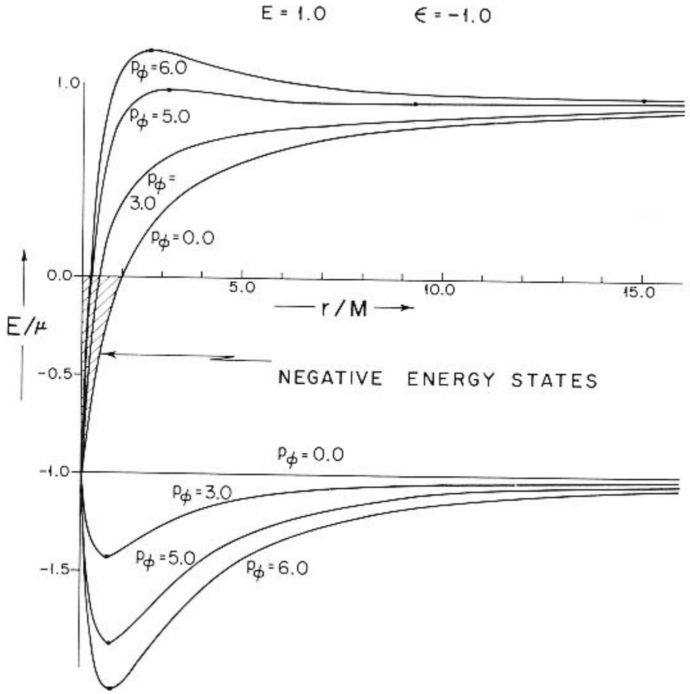
$$(12)\quad E^+(r,Q,q) = -E^-(r,Q,-q)$$

which interchanges the negative and positive roots state through the transformation

$$(13)\quad t \to -t \qquad q \to -q$$

Fig. 1

Positive and negative root states for a charge $\varepsilon = q/\mu = -1.0$ in the field of an extreme Reissner-Nordstrøm black hole $(Q/M = 1)$. The negative energy states of positive roots solution are, here, dashed. The circular orbits for $\varepsilon = +1$ obtained by the inversion of the E^+ and E^- roots (see text) are further analyzed in Ref. ([illegible]). As usual minima in the effective potential indicate stable circular orbits and maxima unstable.



The negative energy states of positive roots are markedly different in the Kerr and in the Reissner-Nordstrøm geometries. In the Kerr case the energy states for $r \to r_+ = M + (M^2 - a^2)^{1/2}$ are given by

$$(14) \quad \lim_{r \to r_+} E^+ = \frac{a\,p_\phi}{r_+^2 + a^2}$$

Therefore, only counterstating orbits $(p_\phi < 0)$ can reach negative energy states and their value is simply proportional to the angular momentum p_ϕ.

In the case here under consideration for $r \to r_+ = M + (M^2 - Q^2)^{1/2}$ the negative states are totally independent from the value of the angular momentum - a big difference also exists in the extension of the ergosphere. In the Kerr case, the ergosphere does not depend upon any detail of the test particle and extends from the horizon up to the infinite red-shift surface

$$(15) \quad M + (M^2 - a^2)^{1/2} \leq r \leq M + (M^2 - a^2\cos^2\theta)^{1/2}$$

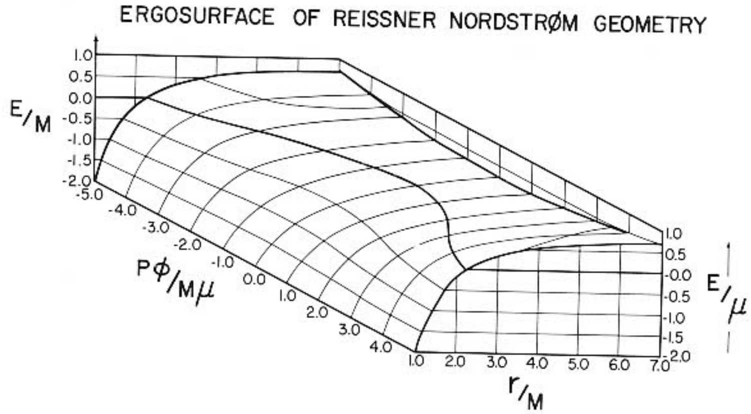
moreover, negative energy states can be reached in the entire range of coordinate given by Eq. 14 only in the limit $p_\phi \to -\infty$. No extraction of energy is possible in the Kerr case for $p_\phi = 0$ (see e.g. Fig. 18 in Ref. 6). In the Reissner-Nordstrøm case the effective ergosphere extends in the range

$$(16) \quad M + (M^2 - Q^2)^{1/2} \leq r \leq M + (M^2 - Q^2(1 - q^2/\mu^2))^{1/2}$$

The effective ergosphere does, indeed, depend upon the value of the test charge Moreover, see Fig. 2, energy extraction is now possible in the entire range of coordinates given by Eq.(15) if and only if $p_\phi = 0$. No extraction is possible in the limit $|p_\phi| \to +\infty$. The entire extraction process is

Fig. 2

Effective potential for a charge $\epsilon = q/\mu = -2.0$ in the field of an extreme Reissner-Nordstrøm black hole $(Q/m = 1)$. The effective potential is unchanged by the transformation $p_\phi \to -p_\phi$. The ergosurface, surface of $E \leq 0$ in the r, p_ϕ plane, extends up to $r = M + (M^2 + Q^2(1 - q^2/\mu^2))^{1/2}$ in the limiting case of $p_\phi = 0$.



completely symmetric with respect to the interchange $p_\phi \to -p_\phi$.

From the Eq. (1) we have for infinitesimal transformations

(16.1) $dM = Q\, dQ/(M + (M^2 - Q^2)^{1/2})$

The integration of Eq. 16 gives immediately the Christodoulou-Ruffini formula controlling the energetics of black holes in the limit $a \to 0$.

(16.2) $M = m_{ir} + Q^2/4m_{ir}$

Finally, let us give an example of a process of energy extraction. We can consider the process envisaged in Fig. 3. A particle P_o comes in from infinity and infringes the ergosphere of an extreme Reissner-Nordstrøm black hole $(Q = M)$. The parameters of the particle P_o are

(17) $\mu_o = 2.197$ $E_o = 4.5$ $p_{\phi o} = 2.0$ $q_o = 6.586$

and the radius of the ergosphere

(18) $r/M = 1 + q/\mu = 3.997$

When the particle P_o attains its turning point at $r = 2M$ decays in two particles:

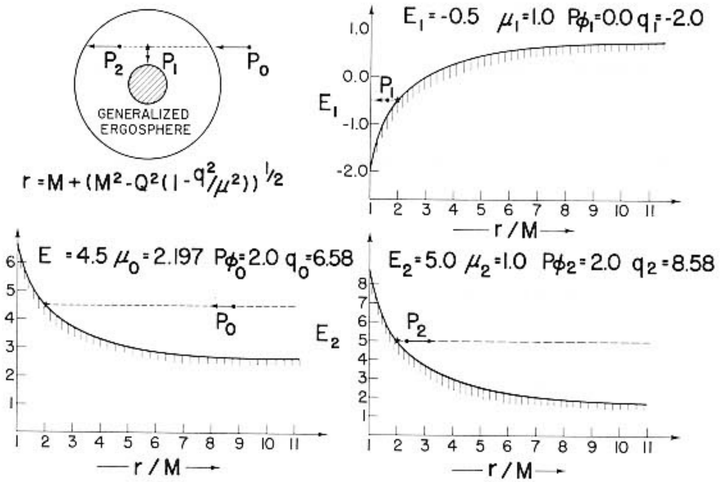
one P_1 goes inside the black hole

(19) $\mu_1 = 1.0$ $E_1 = -0.5$ $p_{\phi 1} = 0$ $\frac{r}{M} = 2.0$ $q_1 = -2.0$

the other P_2

Fig. 3.

A process of energy extraction from an extreme Reissner-Nordstrøm black hole. A particle P_o comes in and splits into two debris the particle P_2 coming out has more energy of the initial particle P_o. The process is most favorable for $P_{\phi 1} = 0$, see Fig. 2.



$$(20) \quad \mu_2 = 1.0 \quad E_2 = 5.0 \quad P_{\phi 2} = 2.0 \quad \frac{r}{M} = 2.0 \quad q_2 = 8.586$$

goes back to infinity. The particle P_2 has more energy of the initial particle P_o. The energy gain is due to the loss of electromagnetic energy (Coulomb energy) from the black hole.

It is important, here, to remark that, exactly like in the Kerr case, since the separation between the positive and negative roots states goes to zero at the horizon we can have processes of energy extraction that can approach as much as we like reversibility. Let us finally remark that the energy extraction process in the Reissner-Nordstrøm geometry appears to be much more promising than the one in the Kerr case for two different and equally important reasons: (a) the energy gain per event appear to be very much larger and (b) the reduction in rest mass necessary to make any energy gain process possible is, here, much less stringent than in the Kerr case[9]. Details of this work are going to appear elsewhere[10].

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Electromagnetic Field of a Particle Moving in a Spherically Symmetric Black-Hole Background (*).

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Very much has been speculated recently on the possible existence of collapsed objects, usually called « black holes », and on their possible observation. Christodoulou and Ruffini (1) have shown that « black holes », which had been considered for a long time to be merely sinks for radiation and other forms of energy, could in fact yield up to 50% of their total energy under favourable circumstances. For these reasons it was justified to think of black holes as the strongest storehouses of energy in the Universe (2). The search for these objects should therefore be directed toward a careful examination of strongly energetic events. From the theoretical point of view a detailed analysis of accretion processes has to be undertaken. Some aspects of this analysis (in the « one-particle approximation »!) have been examined for the emission of gravitational radiation by Zerilli (3), by Davis and Ruffini (4), by Davis, Ruffini, Press and Price (5), and by Davis, Ruffini and Tiomno (6). Some related aspects concerning the scattering of gravitational waves from black holes have been given by Vishveshwara (7). In this paper the theoretical basis for the analysis of the electromagnetic radiation emitted by a charge moving in the gravitational field of spherically symmetric black holes is presented.

(*) Work partially supported by NSF GP-30799X and NSF GU-3186.
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ISRAEL ([8]) has shown that the most general spherically symmetric collapsed objects with closed simply connected horizon is the one given by the Reissner Nordström metric, which in Schwarzschild-like co-ordinate assumes the form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{1}$$

here $\mu, \nu = 0, 3$ and $G = c = 1$, M is the mass and Q the charge of the given background metric.

We consider here a black hole with $Q < M$. We analyze the electromagnetic field generated by a particle of mass m and charge q moving in the above metric (1) under the assumptions that the charge $q \ll M$ and the mass $m \ll M$. The electromagnetic vector potential associated with the moving particle can be expanded in terms of four-dimensional vector spherical harmonics obtained from scalar spherical harmonics by means of the following operations:

$$\frac{\boldsymbol{r}}{r}\, Y^{lm}(\theta, \varphi)\,, \tag{2a}$$

$$\boldsymbol{\nabla} Y^{lm}(\theta, \varphi)\,, \tag{2b}$$

$$\boldsymbol{L}\, Y^{lm}(\theta, \varphi)\,, \tag{2c}$$

$$e_t\, Y^{lm}(\theta, \varphi) = (Y^{lm}, 0, 0, 0)\,. \tag{2d}$$

Here as usual $\boldsymbol{L}$ is the angular-momentum operator and $\boldsymbol{\nabla}$ the gradient. The parity of eq. (2.a), (2.b), (2.d) is $(-1)^l$ (electric) and the parity of eq. (2.c) is $(-1)^{l+1}$ (magnetic). We obtain then

$$A_\mu(r, \theta, \varphi, t) = \sum_{lm}\left(\begin{bmatrix} 0 \\ 0 \\ \dfrac{a^{lm}(r,t)}{\sin\theta}\dfrac{\partial Y^{lm}}{\partial\varphi} \\ -a^{lm}(r,t)\sin\theta\dfrac{\partial Y^{lm}}{\partial\theta}\end{bmatrix} + \begin{bmatrix} f^{lm}(r,t)\, Y^{lm} \\ h^{lm}(r,t)\, Y^{lm} \\ k^{lm}(r,t)\dfrac{\partial Y^{lm}}{\partial\theta} \\ k^{lm}(r,t)\dfrac{\partial Y^{lm}}{\partial\varphi}\end{bmatrix}\right). \tag{3}$$

The covariant Maxwell equations to be fulfilled in the given background are given by

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu \qquad \text{or} \qquad (\sqrt{-g}\, F^{\mu\nu})_{,\nu} = \sqrt{-g}\, 4\pi J^\mu\,. \tag{4}$$

In the above equation; (,) indicates covariant (ordinary) derivative and $g = \det g_{\alpha\beta}$. Further we have

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}\,. \tag{5}$$

([8]) W. ISRAEL: *Phys. Rev.*, **164**, 1776 (1967).

The four-current j^μ can itself be expanded in terms of vector harmonics

$$4\pi J_\mu = \sum_{l,m}\left(\begin{bmatrix} 0 \\ 0 \\ \dfrac{\alpha^{lm}(r,t)}{\sin\theta}\dfrac{\partial Y^{lm}}{\partial\varphi} \\ -\alpha^{lm}(r,t)\sin\theta\dfrac{\partial Y^{lm}}{\partial\theta}\end{bmatrix} + \begin{bmatrix} \Psi^{lm}(r,t)\, Y^{lm} \\ \eta^{lm}(r,t)\, Y^{lm} \\ \chi^{lm}(r,t)\dfrac{\partial Y^{lm}}{\partial\theta} \\ \chi^{lm}(r,t)\dfrac{\partial Y^{lm}}{\partial\varphi}\end{bmatrix}\right). \tag{6}$$

The nonvanishing components of the electromagnetic-field tensor $F^{\mu\nu}$ of parity $(-1)^{l+1}$ are given by

$$\begin{cases} F^{0\theta}_{lm} = g^{00}a^{lm}{}_{,0}\, Y^{lm}{}_{,\varphi}/(r^2\sin\theta)\,;\; F^{0\varphi}_{lm} = -g^{00}a^{lm}{}_{,0}\, Y^{lm}{}_{,\theta}/(r^2\sin\theta)\,, \\ F^{r\theta}_{lm} = g^{rr}a^{lm}_{,r}\, Y^{lm}_{,\varphi}/(r^2\sin\theta)\,;\; F^{r\varphi}_{lm} = -g^{rr}a^{lm}_{,r}\, Y^{lm}_{,\theta}/(r^2\sin\theta)\,, \\ F^{\theta\varphi}_{lm} = l(l+1)\,a^{lm}/(r^4\sin\theta)\,. \end{cases} \tag{7}$$

Equation (4) gives rise to a single differential equation for the radial function

$$(g^{rr}a^{lm}_{,r})_{,r} - g_{rr}\frac{\partial^2 a^{lm}}{\partial t^2} - \frac{l(l+1)}{r^2}\, a^{lm} = \alpha^{lm}\,. \tag{8}$$

The other equations obtained from eqs. (4) are either identically satisfied or equivalent to (8). Similarly, the nonvanishing components of $F^{\mu\nu}$ with parity $(-1)^l$ are

$$\begin{cases} F^{0r}_{lm} = (f^{lm}_{,r} - h^{lm}_{,0})\, Y^{lm}\,; \qquad F^{0\theta}_{lm} = g^{00}(k^{lm}_{,0} - f^{lm})\, Y^{lm}_{,\theta}/r^2\,, \\ F^{0\varphi}_{lm} = g^{00}(k^{lm}_{,0} - f^{lm})\, Y^{lm}_{,\varphi}/(r^2\sin\theta)\,, \\ F^{r\theta}_{lm} = g^{rr}(k^{lm}_{,r} - h^{lm})\, Y^{lm}_{,\theta}/r^2\,; \qquad F^{r\varphi}_{lm} = g^{rr}(k^{lm}_{,r} - h^{lm})\, Y^{lm}_{,\varphi}/(r^2\sin\theta)\,, \end{cases} \tag{9}$$

and from eqs. (4) we obtain now the following set of differential equations for the radial parts:

$$g_{00}[r^2(f^{lm}_{,r} - h^{lm}_{,0})]_{,r} - l(l+1)\,(k^{lm}_{,0} - f^{lm}) = \Psi^{lm} r^2\,, \tag{10a}$$

$$g_{rr}(h^{lm}_{,0} - f^{lm}_{,r})_{,0} - \frac{l(l+1)}{r^2}\,(k^{lm}_{,r} - h^{lm}) = \eta^{lm}\,, \tag{10b}$$

$$[(h^{lm} - k^{lm}_{,r})g^{rr}]_{,r} - (k^{lm}_{,0} - f^{lm})_{,0}\, g^{00} = \chi^{lm}\,. \tag{10c}$$

The remaining equations obtained from (4) are either identically satisfied or equivalent to eqs. (10). The equation of conservation of current to be identically fulfilled is

$$J^\mu{}_{;\mu} = 0$$

This gives the subsidiary equation

$$\frac{1}{r^2}\,(r^2 g^{rr}\eta^{lm})_{,r} + \Psi^{lm}{}_{,0}\, g^{00} = \frac{l(l+1)}{r^2}\,\chi^{lm}\,. \tag{11}$$

We introduce a new function $b^{lm}(r, t)$ defined by the following equation:

$$h^{lm}_{,0} - f^{lm}_{,r} = \frac{l(l+1)}{r^2}\, b^{lm}\,, \tag{12}$$

and substituting in the expressions (10a) and (10b) we get

$$g^{rr} b^{lm}_{\ \ ,0} = k^{lm}_{,0} - f^{lm} + \frac{r^2 \Psi^{lm}}{l(l+1)}\,, \tag{12a}$$

$$g^{00} b^{lm}_{\ \ ,0} = h^{lm} - k^{lm}_{,r} - \frac{r^2 \eta^{lm}}{l(l+1)}\,. \tag{12b}$$

With the help of eqs. (12) we find that the eq. (10c) is identically satisfied in view of (11). Further, integrability condition of these equations is summarized in the compact and simple differential equation

$$(g^{rr} b^{lm}_{\ \ ,r})_{,r} + g^{00} b^{lm}_{,00} - \frac{l(l+1)}{r^2}\, b^{lm} = \frac{1}{l(l+1)}\left[(r^2 \Psi^{lm})_{,r} - \eta^{lm}_{,0} r^2\right]. \tag{13}$$

Equations (13) and (8) take the same form in vacuum, in which case they were first given by WHEELER [9]. Equation (13) solves completely the problem of the determination of the electric multipole expansion as $F^{\mu\nu}_{lm}$ given by (9) is expressed in terms of b^{lm}, $b^{lm}_{\ \ ,0}$, $b^{lm}_{,r}$, Ψ^{lm} and η^{lm}. Both eqs. (13) and (8) have to be solved for any given J^μ by assuming as boundary conditions purely ingoing waves at the surface of the black hole.

The four-current J^μ in the case of a point charge moving in the given background. is given by

$$J^\mu = \frac{q}{\sqrt{-g}}\,\frac{\mathrm{d}z^\mu}{\mathrm{d}t}\,\delta(\boldsymbol{x} - \boldsymbol{z}(t))\,, \tag{14}$$

$z^\mu = (t, \boldsymbol{z}) = (t, R(t), \Theta(t), \varphi(t))$ describes the trajectory of the particle as given by the equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}z^\mu}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}s}\right) = F^\mu{}_\alpha\,\frac{\mathrm{d}z^\alpha}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}s} + \Gamma^\mu_{\alpha\beta}\,\frac{\mathrm{d}z^\alpha}{\mathrm{d}t}\,\frac{\mathrm{d}z^\beta}{\mathrm{d}t}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2.$$

Both $F^\mu{}_\alpha$ and $\Gamma^\mu_{\alpha\beta}$ are given by the background geometry. From (4) and (6) we obtain

$$\Psi^{lm}(r, t) = \frac{q}{r^2 g^{00}}\,\delta(r - R)\, Y^{lm*}(\Theta, \Phi)\,, \tag{15a}$$

$$\eta^{lm}(r, t) = \frac{q}{r^2 g^{rr}}\,\frac{\mathrm{d}R}{\mathrm{d}t}\,\delta(r - R)\, Y^{lm*}(\Theta, \Phi)\,, \tag{15b}$$

$$\alpha^{lm}(r, t) = \frac{q}{l(l+1)}\left[-\frac{\mathrm{d}\Phi}{\mathrm{d}t}\sin\Theta\, Y^{lm*}{}_{,\Theta} + \frac{1}{\sin\Theta}\,\frac{\mathrm{d}\Theta}{\mathrm{d}t}\, Y^{lm*}{}_{,\Phi}\right]\delta(r - R)\,. \tag{15c}$$

These are the relevant source terms to be used when solving eqs. (8) and (13).

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[9] J. A. WHEELER: *Geometrodynamics* (New York, 1962), p. 203. Similar results for the sourceless case were also obtained by L. FAVELLA - *Tesi di Laurea*, 1957, Torino University. We thank Prof. T. REGGE to have pointed out this interesting work.

Finally we have to give the expression for the energy radiated. The tensor momentum energy for the electromagnetic field is

$$T^\mu{}_\nu = (F^{\mu\varrho} F_{\varrho\nu} + \tfrac{1}{4}\delta^\mu{}_\nu F^{\varrho\sigma} F_{\varrho\sigma})/4\pi\,.$$

The flux of energy through a surface S is given by the expression

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_S = \int_S T^r_0 r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,. \tag{16a}$$

Here

$$\begin{aligned} T^r_0 r^2 \sin\theta = {} & -g^{rr} \sum_{lm}\sum_{l'm'} [a^{lm}{}_{,r} a^{l'm'}{}_{,0} + (k^{lm}_{,r} - h^{lm})(k^{l'm'}{}_{,0} - f^{l'm'})]\,(Y^{lm}_{,\theta} Y^{l'm'}_{,\theta} \sin\theta + \\ & + Y^{lm}_{,\varphi} Y^{l'm'}_{,\varphi}/\sin\theta) + g^{rr} \sum_{lm}\sum_{l'm'} [a^{lm}_{,r}(f^{l'm'} - k^{l'm'}_{,0}) + (k^{lm}_{,r} - h^{lm})\, a_{l'm',0}](Y^{lm}_{,\varphi} Y^{l'm'}{}_{,\theta} - Y^{lm}{}_{,\theta} Y^{l'm'}{}_{,\varphi})\,. \end{aligned} \tag{16b}$$

The spectral distribution of the radiation is immediately obtained by substituting the Fourier transform of eqs. (3) into (16) and taking the usual time average. Detailed computations of the energy spectrum and amount of radiation emitted by a charged particle falling into a black hole have been done by two of us (R.R. and J.T.) and will be published elsewhere.

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