Exercises on finite generation of the mapping class groups

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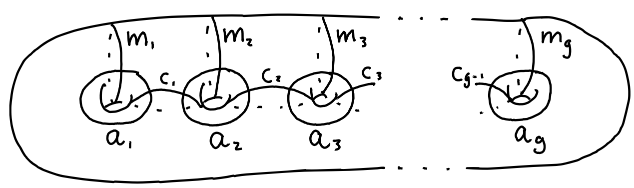
October 29, 2020

Lickorish generators for MCG(S_g), following Farb–Margalit.



Lemma 1. *Suppose*

- *X connected simplicial graph (no loops or double edges)*
- *G acts on X by simplicial automorphisms*
- *G acts transitively on ordered pairs of adjacent vertices of X*

Let v, w be adjacent vertices of X and let $h \in G$ so that $h(w) = v$. Then $G = \langle h, \mathrm{Stab}_G(v)\rangle$, where $\mathrm{Stab}_G(v) = \{g \in G \mid g(v) = v\}$.

Let $\mathcal{N}(S_g)$ be the graph with:

- a vertex for every isotopy class of non-separating curves in S_g
- an edge between two vertices if the curves intersect exactly once

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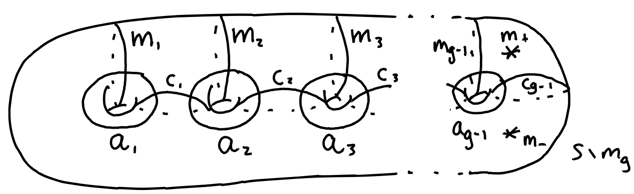
Exercise 1. Prove that $\mathcal{N}(S_g)$ is connected, using a similar surgery argument to that for the curve graph given in the lectures. (Hint: for the case where we don't have two consecutive intersections with the same orientation, try looking at three consecutive intersections instead of just two.)

Exercise 2. Prove that $G = \mathrm{MCG}(S_g)$ and $X = \mathcal{N}(S_g)$ satisfy the hypotheses of Lemma 1.

Exercise 3. Let a, m be adjacent vertices of $\mathcal{N}(S_g)$, i.e. non-separating curves intersecting exactly once. Prove that $T_a T_m(a) = m$. Deduce that $\mathrm{MCG}(S_g)$ is generated by T_{a_g}, T_{m_g} and $\mathrm{Stab}_{\mathrm{MCG}(S_g)}(m_g)$.

Exercise 4. Let $\mathrm{Stab}_{\mathrm{MCG}(S_g)}(\vec{m}_g)$ be the subgroup of $\mathrm{MCG}(S_g)$ that stabilises m_g with orientation. Prove that $\mathrm{Stab}_{\mathrm{MCG}(S_g)}(m_g)$ is generated by T_{a_g}, T_{m_g} and $\mathrm{Stab}_{\mathrm{MCG}(S_g)}$. (Hint: check that $T_{a_g} T_{m_g}^2 T_{a_g}$ preserves m_g but reverses its orientation.)

Lemma 2. $\mathrm{Stab}_{\mathrm{MCG}(S_g)}(\vec{m}_g) = \mathrm{PMCG}(S - m_g) \times \langle T_{m_g} \rangle$



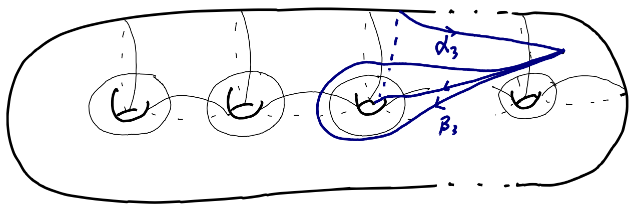
$$S - m_g \cong S_{g-1,2}$$

The Birman exact sequence gives us:

$$1 \to \pi_1(S_{g-1,1}, m_-) \xrightarrow{\text{Push}} \mathrm{PMCG}(S_{g-1,2}) \xrightarrow{F_*} \mathrm{PMCG}(S_{g-1,1}) \to 1 \qquad (1)$$

$$1 \to \pi_1(S_{g-1}, m_+) \xrightarrow{\text{Push}} \mathrm{PMCG}(S_{g-1,1}) \xrightarrow{F_*} \mathrm{PMCG}(S_{g-1}) \to 1 \qquad (2)$$

Lemma 3. *$\pi_1(S_{g-1}, m_+)$ is generated by $g-1$ pairs α_k, β_k, which we can arrange as in the example.*



Exercise 5 (Using the sequence (2))**.**

1. $\text{Push}(\alpha_{g-1}) = T_{c_{g-1}} T_{m_{g-1}}^{-1}$.
2. $T_{m_k} T_{a_k}(\alpha_k) = \beta_k$.
3. More generally, every α_i, β_i is related to α_{g-1} by a product of Lickorish twists.
4. If $\gamma \in \pi_1(S_{g-1}, m_+)$, $f \in \text{MCG}(S_{g-1} - m_+)$ and f_* is the image of f in $\text{MCG}(S_{g-1})$ then $\text{Push}(f_*(\gamma)) = f \,\text{Push}(\gamma)\, f^{-1}$.
5. Conclude that $\text{PMCG}(S_{g-1,1})$ is generated by Lickorish twists.

Exercise 6. Complete the proof that the Lickorish twists generate $\text{MCG}(S_g)$ using the sequence (1). You might want to use the chain relation (Lemma 4 below) to prove that the "mirror image" curves of the m_i in the lower half of the surface can be written in terms of the Lickorish twists.

Lemma 4 (Chain relation (odd number of curves))**.** *Let $c_1, c_2, \ldots, c_k$ be a sequence of curves with k odd so that:*

- *c_i and c_{i+1} intersect exactly once for every $1 \leq i \leq k-1$*
- *c_i and c_j do not intersect for $|i-j| \geq 2$.*

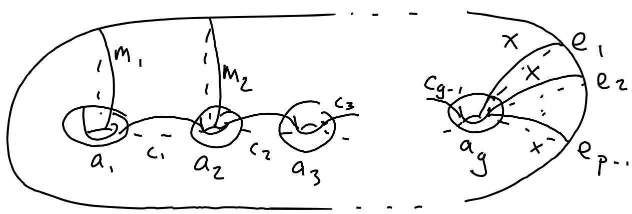
Let d_1, d_2 be the two boundary components of a small regular neighbourhood of $\bigcup_i c_i$. Then $(T_{c_1} \ldots T_{c_k})^{k+1} = T_{d_1} T_{d_2}$.

Exercise 7. Prove the chain relation using the Alexander method (or otherwise).

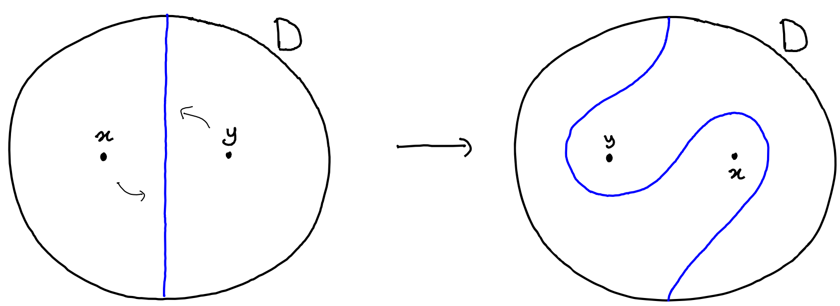
Generators for MCG(S) when S has punctures.

Theorem 5 (Humphries)**.** *If we remove the curves $m_3, \dots, m_g$ from the Lickorish generating set for then the remaining twists still generate* MCG(S_g)*.*

Exercise 8. Assuming Humphries' theorem, prove that Dehn twists about the following set of curves generate PMCG($S_{g,p}$), for $g \geq 2$.



If our surface has at least two punctures we also need **half twists**. Take two punctures x and y and choose a twice punctured disc $D \subset S$ containing x and y (the mapping class obtained will depend on the choice of isotopy class of D).



There are two directions of half twists and they are inverses of each other. Every half twist induces a permutation of the punctures of S.

Exercise 9. Prove that MCG(S) is generated by PMCG(S) plus a set of half twists which generate the permutation group of the punctures of S.

Quick proof that MCG(S) is finitely generated.

Consider a graph $\mathcal{G}(S)$ which has:

- a vertex for every pair $\{a, b\}$ of curves in S so that
  - $a \cup b$ cuts S into discs and once punctured discs
  - the maximal number of intersections between a and b is at most K
- an edge of length 1 between two vertices if they intersect at most L times

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Exercise 10 ($\mathcal{G}(S)$ is a proper geodesic metric space)**.**

1. Prove that for appropriate K and L, $\mathcal{G}(S)$ is connected.
2. Prove that $\mathcal{G}(S)$ is locally finite.

Exercise 11. Use the Schwarz-Milnor lemma and the action of MCG(S) on $\mathcal{G}(S)$ to prove that MCG(S) is finitely generated. Why do we need the bounds K and L?